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On the stability of convex sums of rank-1 perturbed matrices

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Abstract

A compact condition is obtained that guarantees the stability of a single convex sum of a pair of related matrices. It is anticipated that the proof presented for a single pencil can be modified to yield conditions for a pair of related pencils.

1 Introduction

We are interested in the problem of determining appropriate conditions on the matrices A_1 and A_2 that are necessary and sufficient for the stability of the convex sums (pencils)

$$\alpha A_1 + (1-\alpha)A_2, \quad \alpha \in [0,1], \tag{1}$$
$$\alpha A_1^{-1} + (1-\alpha)A_2, \quad \alpha \in [0,1], \tag{2}$$

where A_1, A_2 are real $n \times n$ matrices and $A_2 = A_1 + B$ where rank $B = 1$. Problems of this nature can be readily found in the literature on robust stability of linear systems [1]. More recently, research on the stability of switching systems constructed by switching between two linear systems [2,3], has illustrated the importance of convex sums of the form of equations (1) and (2). In this paper we begin the study of establishing compact conditions that are necessary and sufficient for the stability of both convex sums simultaneously by presenting a new sufficient condition for the stability of a single convex sum. We note that the proof of our condition is based on the original matrices A_1 and A_2, and hence differs considerably from treatments of similar problems in the literature [1,4].

2 Main result

Due to space considerations some of the proofs are omitted. Full details can be found in [5].

Lemma 1. Suppose A_1 and $A_2 = A_1 + B$ are real $n \times n$ stable matrices and that B has rank one. Then $A_1^{-1}A_2$ has no negative eigenvalues.

The main result of this paper is the following theorem which gives a sufficient condition for the stability of a convex sum of two matrices.

Theorem 1. Suppose A_1 and $A_2 = A_1 + B$ are real $n \times n$ stable matrices and that B has rank one. Suppose that $A_1 A_2$ has no negative eigenvalues. Then $\alpha A_1 + (1-\alpha)A_2$ is stable for all α with $0 \leq \alpha \leq 1$.

Proof : We will prove the result by contradiction. Suppose that for some α with $0 < \alpha < 1$, we have that $\alpha A_1 + (1-\alpha)A_2$ is unstable. Thus, $A_1 + \delta B$ is unstable and has a purely imaginary eigenvalue $i\gamma$ (γ real) for some δ with $0 < \delta < 1$. Note that Lemma 1 implies that $\gamma \neq 0$. Let v_1, v_2 be eigenvectors of $A_1 + \delta B$ corresponding to $i\gamma, -i\gamma$ respectively. We may assume that $v_1 = u_1 + iu_2$ and $v_2 = u_1 - iu_2$ where u_1, u_2 are real vectors. We get that u_1 and u_2 are linearly independent over the reals. Let U be the real subspace spanned by u_1 and u_2. Note that

$$((A_1 + \delta B)^2 + \gamma^2 I)u = 0, \quad \forall\, u \in U \tag{3}$$

Let $K = \{w \in \mathbb{R}^n : Bw = 0\}$. Then, $\dim K = n-1$ and $\dim U = 2$ and so $\dim(K \cap U) \geq 1$. Let u be a non-zero element in $K \cap U$. Then $Bu = 0$ and so (3) implies that $((A_1 + \delta B)A_1 + \gamma^2 I)u = 0$. Thus, $(A_1 + \delta B)A_1$ (and hence, $A_1(A_1 + \delta B)$) has $-\gamma^2$ as an eigenvalue. So,

$$\det(A_1(A_1 + \delta B) + \gamma^2 I) = 0 \tag{4}$$

For $x \geq 0$ and $y > 0$ we now define

$$f(x,y) = \det(A_1(A_1 + yB) + xI)$$

$I + xA_1^{-2}$ is invertible for $x \geq 0$ and so $f(x,y) = (\det A_1)^2 \det(I + xA_1^{-2})g(x,y)$ where $g(x,y) = \det(I + yA_1^{-1}B(I + xA_1^{-2})^{-1})$.

Note that $C = A_1^{-1}B(I+xA_1^{-2})^{-1}$ has rank one and therefore has at most one non-zero eigenvalue and that eigenvalue (if it exists) has algebraic multiplicity one. Thus, tr C is either the unique non-zero eigenvalue or else it is zero. So, $g(x,y) = 1 + y\mathrm{tr}(A_1^{-1}B(I + xA_1^{-2})^{-1})$.

We define

$$h(x) = \mathrm{tr}(A_1^{-1}B(I + xA_1^{-2})^{-1})$$

so that $g(x,y) = 1 + yh(x)$. Note that $g(\gamma^2, \delta) = 0$ and so $h(\gamma^2) = -\frac{1}{\delta} < -1$.

Now, $h(0) = \mathrm{tr}(A_1^{-1}B)$. Note that $A_1^{-1}A_2 = I + A_1^{-1}B$, where $A_1^{-1}B$ has rank one. Thus, $A_1^{-1}B$ has at most one non-zero eigenvalue and that eigenvalue (if it exists) has algebraic multipicity one. So, $\mathrm{tr}(A_1^{-1}B)$ is either the unique non-zero eigenvalue or else it is zero. Thus, $\mathrm{tr}(A_1^{-1}B) \geq -1$. Thus, $h(0) \geq -1$. In fact, $h(0) > -1$ since otherwise $\mathrm{tr}(A_1^{-1}B) = -1$, and so $\det(I + A_1^{-1}B) = 0$ which implies that $\det(A_1 + B) = 0$, which is false.

Now, h is a continuous function for $x \geq 0$ and so by the Intermediate Value Theorem we have that there exists a z with $0 < z < \gamma^2$ such that $h(z) = -1$. Thus, $g(z,1) = 0$ and so $f(z,1) = 0$. Therefore,

$$\det(A_1(A_1 + B) + zI) = 0$$

and so $A_1A_2 = A_1(A_1 + B)$ has $-z$ as an eigenvalue, which is a contradiction. Q.E.D.

The Corollary below shows a surprising connection between the convex sums (1) and (2).

Corollary 1. Suppose A_1 and $A_2 = A_1 + B$ are real $n \times n$ stable matrices and that B has rank one. Suppose that $\alpha A_1 + (1-\alpha)A_2$ is invertible $\forall\, \alpha \in [0,1]$ and that there exists a β with $0 < \beta < 1$ such that $\beta A_1 + (1-\beta)A_2$ has a non-zero purely imaginary eigenvalue, i.e. the convex sum $\alpha A_1 + (1-\alpha)A_2$ is unstable by eigenvalues passing through the imaginary axis (excluding the origin). Then there exists a τ with $0 < \tau < 1$ such that $\tau A_1^{-1} + (1-\tau)A_2$ has a zero eigenvalue, i.e. the convex sum $\alpha A_1^{-1} + (1-\alpha)A_2$ is unstable with an eigenvalue passing through the origin in the complex plane.

3 Conclusions

The main result of this paper gives a sufficient condition for when the convex sum of two matrices is stable. We also prove a surprising connection between the two convex sums (1) and (2).

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