

Formulae for null curves deriving from elliptic curves

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Abstract

Any elliptic curve can be realised in the tangent bundle of the complex projective line as a double cover branched at four distinct points on the zero section. Such a curve generates, via classical osculation duality, a null curve in \mathbb{C}^3 and thus an algebraic minimal surface in \mathbb{R}^3 . We derive simple formulae for the coordinate functions of such a null curve.

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1. Introduction

Suppose that (g, f) is a pair of holomorphic functions on a Riemann surface M . The following are versions of the classical Weierstrass formulae in free form [1,3,5,9,10], and give the coordinate functions of a *null* holomorphic curve $\psi : M^* \rightarrow \mathbb{C}^3$; which is to say that ψ satisfies $(\psi'_1)^2 + (\psi'_2)^2 + (\psi'_3)^2 = 0$. Here M^* is obtained from M by deleting a finite number of points, and $df/dg = f'/g'$, $d^2f/dg^2 = (df/dg)'/g'$, etc.

$$\psi_1 = -\frac{1}{2} \left\{ \frac{1}{2}(1 - g^2) \frac{d^2f}{dg^2} + g \frac{df}{dg} - f \right\} \quad (1)$$

$$\psi_2 = -\frac{i}{2} \left\{ \frac{1}{2}(1 + g^2) \frac{d^2f}{dg^2} - g \frac{df}{dg} + f \right\} \quad (2)$$

$$\psi_3 = \frac{1}{2} \left\{ g \frac{d^2f}{dg^2} - \frac{df}{dg} \right\}. \quad (3)$$

The normalisation agrees with [5]. Every non-planar null curve in \mathbb{C}^3 may be described in this way; this is important in the study of minimal surfaces in \mathbb{R}^3 , because any minimal surface in \mathbb{R}^3 can be described as the real part of a null curve in \mathbb{C}^3 .

Recall that charge 2 monopole spectral curves are real elliptic curves, satisfying a certain transcendental constraint, and are explicitly exhibited as double covers of \mathbb{P}_1 [6]. In [11], formulae for the null curves that derive, via (1)–(3),

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from these elliptic curves are given. It turns out that their surprisingly simple form is not a result of the special nature of spectral curves; similar formulae hold in general, for double covers of the complex projective line \mathbb{P}_1 , that are branched over four distinct points. This can be described quite simply in terms of elliptic functions, as follows in Section 2. In Section 3 we make some comments about the geometry that underlies these formulae.

2. Formulae

To fix notation, suppose that Λ is a lattice in \mathbb{C} , with a pair of basic periods $\{\omega_1, \omega_2\}$; let $E = \mathbb{C}/\Lambda$ and $\wp : E \rightarrow \mathbb{P}_1$ be the auxiliary Weierstrass \wp -function. Furthermore let g_2 and g_3 be the invariants of \wp ; $\wp(\omega_j/2) = e_j$, $j = 1, 2, 3$, and let $\varepsilon_1 = \sqrt{e_3 - e_2}$, $\varepsilon_2 = \sqrt{e_1 - e_3}$, $\varepsilon_3 = \sqrt{e_2 - e_1}$.

Consider the pair of meromorphic functions

$$g(u) = -\frac{1}{\varepsilon_1\varepsilon_2}(\wp(u) - e_3), \tag{4}$$

$$f(u) = i\varepsilon_3\wp'(u). \tag{5}$$

Let $E^* = E \setminus \{\frac{1}{2} - \text{periods}\}$. Substituting (4) and (5) into (1)–(3) yields the null meromorphic curve $\Psi : E^* \rightarrow \mathbb{C}^3$, with coordinate functions

$$\Psi_1(u) = -i\varepsilon_3 \left\{ \frac{4\wp(u)^3 - 24e_3\wp(u)^2 - 2(g_2 + 24e_1e_2)\wp(u) + 17g_3 - 2e_3g_2}{16\wp'(u)} \right\}$$

$$\Psi_2(u) = -\varepsilon_3 \left\{ \frac{4\wp(u)^3 - 24e_3\wp(u)^2 + 2(5g_2 + 12e_1e_2)\wp(u) + 17g_3 - 2e_3g_2}{16\wp'(u)} \right\}$$

$$\Psi_3(u) = i\varepsilon_1\varepsilon_2\varepsilon_3 \left\{ \frac{12\wp(u)^2 + 12e_3\wp(u) + g_2}{8\wp'(u)} \right\}.$$

Now let $f_j(u)$ denote the square root of $\wp(u) - e_j$, $j = 1, 2, 3$, whose residue at the origin is 1. Let $\Omega : E^* \rightarrow \mathbb{C}^3$ be the null meromorphic curve whose coordinate functions are given by

$$\Omega(u) = (\varepsilon_1 f_1^3(2u), \varepsilon_2 f_2^3(2u), \varepsilon_3 f_3^3(2u)). \tag{6}$$

Theorem 1. $\Psi = A\Omega$, where $A \in SO(3, \mathbb{C})$, is given by

$$A = \frac{1}{i\varepsilon_3} \begin{pmatrix} \varepsilon_1 & \varepsilon_2 & 0 \\ 0 & 0 & i\varepsilon_3 \\ \varepsilon_2 & -\varepsilon_1 & 0 \end{pmatrix}. \tag{7}$$

Proof. Recall the following classical identities:

$$f_1(2u) = \frac{(e_1 - e_2)(e_1 - e_3) - (\wp(u) - e_1)^2}{\wp'(u)} \tag{8}$$

$$f_2(2u) = \frac{(e_2 - e_1)(e_2 - e_3) - (\wp(u) - e_2)^2}{\wp'(u)} \tag{9}$$

$$f_3(2u) = \frac{(e_3 - e_1)(e_3 - e_2) - (\wp(u) - e_3)^2}{\wp'(u)}. \tag{10}$$

Now substitute into (6) and apply A. \square

Remark. (10) appears (with a typo) in Section 333 of [12].

3. Comments on the geometry

Recall that any complex torus \mathbb{C}/Λ may be realised as a non-singular plane cubic, or alternatively, as the intersection of two quadrics in \mathbb{P}_3 . The former description is related to the Weierstrass \wp -function, whilst in the latter the torus is embedded via theta functions [8]. We will now briefly explain how the geometry underlying the Weierstrass formulae (1)–(3) is closely connected to the latter construction and thus perhaps the existence of Ω in (6), together with its ubiquity (cf. Corollary 1) and the transparency of its description are not perhaps surprising after all. See [3,10] for the basics. In Corollary 1 we reformulate Theorem 1 in more geometric language.

Let \mathbb{T} be the total space of the holomorphic tangent bundle of \mathbb{P}_1 . \mathbb{T} embeds into \mathbb{P}_3 where, through the addition of a single point at infinity, it is compactified to a quadric cone $\mathcal{C}(Q_1)$, with vertex v_∞ say.

Recall that classical osculation duality gives a correspondence between non-planar curves in \mathbb{P}_3 and \mathbb{P}_3^* . $\mathcal{C}(Q_1)$ determines a ‘dual’ quadric cone $\mathcal{C}(Q_1^*) \subset \mathbb{P}_3^*$, where Q_1^* lies on v^* , which parameterises all the hyperplanes of \mathbb{P}_3 lying tangent to Q_1 .

Osculation of a non-planar curve $S \subset \mathbb{P}_3$, that happens to lie on $\mathcal{C}(Q_1)$, gives a curve $S^\natural \subset \mathbb{P}_3^*$, which is such that every hyperplane in \mathbb{P}_3^* osculating S^\natural lies tangent to the ‘quadric at infinity’ Q_1^* . Writing $\mathbb{C}^3 = \mathbb{P}_3^* \setminus v^*$, this means that the affine part of S^\natural in \mathbb{C}^3 is a null curve relative to the affine cone $\mathcal{C}(Q_1^*) \subset \mathbb{C}^3$. Every non-planar null curve in \mathbb{C}^3 arises in this way. This was discovered by Lie [3]; but see also [5], where it is described in the following way.

Hyperplanes of \mathbb{P}_3 , which do not pass through v_∞ , cut out global sections on \mathbb{T} when embedded as above. The 2-jet of a germ of a local holomorphic section of \mathbb{T} determines a global section; thus a holomorphic curve S lying in \mathbb{T} generates, via ‘osculation’, a holomorphic curve in $H^0(\mathbb{P}_1, \mathcal{O}(\mathbb{T})) \cong \mathbb{C}^3$. Those global sections that have a double root on \mathbb{P}_1 comprise an affine null cone in $H^0(\mathbb{P}_1, \mathcal{O}(\mathbb{T}))$, which in suitable coordinates is given by $(z_1^2 + z_2^2 + z_3^2 = 0)$. If $\psi : S^* \rightarrow \mathbb{C}^3$ is given by osculation then it is null.

Let ζ be an affine coordinate on \mathbb{P}_1 , giving the local coordinates $(\zeta, \eta) \rightarrow (\zeta, \eta d/d\zeta)$ on \mathbb{T} . If the affine part of a curve S in \mathbb{T} is described in these coordinates by a pair of meromorphic functions (g, f) on a Riemann surface M , then, with respect to a certain choice of basis for $H^0(\mathbb{P}_1, \mathcal{O}(\mathbb{T}))$ [11], the coordinate functions of the auxiliary null curve $\psi : M^* \rightarrow \mathbb{C}^3$ are given by the Weierstrass formulae (1)–(3).

Now, $Q_0 = \mathcal{C}(Q_1)$ fixes a point in \mathbb{P}_9 , the space parameterising all quadrics in \mathbb{P}_3 . Any other quadric Q_1 intersects Q_0 in a curve which completes a curve on \mathbb{T} of the form

$$a\eta^2 + b(\zeta)\eta + c(\zeta) = 0, \tag{11}$$

where a is constant, $\deg(b) \leq 2$ and $\deg(c) \leq 4$. Thus the \mathbb{P}_8 of pencils which contain Q_0 may be identified with the complete linear system $|2E_0|$ of such curves on \mathbb{T} .

Let $\Delta(\zeta) = b^2(\zeta) - 4ac(\zeta)$. Suppose that $a \neq 0$, and the branch locus $(\Delta(\zeta) = 0)$ comprises four distinct points (one of which may be at $\zeta = \infty$). In this generic case of transverse intersection, (11) is a smooth elliptic curve S on \mathbb{T} , double covering \mathbb{P}_1 , where the projection map has four branch points. Completing the square, and letting $\mu = \eta + b(\zeta)/2a$, (11) becomes $\mu^2 = \Delta(\zeta)/4a^2$. The point being that the branch locus lies on the global section $\eta = -b(\zeta)/2a$. Of course, changing the choice of zero section merely translates the origin in $H^0(\mathbb{P}_1, \mathcal{O}(\mathbb{T}))$.

By applying a fractional linear transformation we can move the four branch points on \mathbb{P}_1 to any other 4-tuple with the same cross-ratio. Thus the curve S may be brought to any one of the standard forms; cf. [2,7]. The corresponding bundle automorphism of \mathbb{T} induces via the adjoint representation an $SO(3, \mathbb{C})$ -rotation of $H^0(\mathbb{P}_1, \mathcal{O}(\mathbb{T}))$ [10].

Remark. This can also be understood in terms of the four singular elements in the pencil of quadrics with base locus S , and projective equivalence of pencils; cf. Proposition 22.38 in [4].

This means we can reformulate Theorem 1 as follows:

Corollary 1. *Suppose that S is an elliptic curve in $|2E_0|$ as above. If $\Phi : S^* \rightarrow \mathbb{C}^3$ is the auxiliary null curve induced by osculation then S admits a parameterisation $p : \mathbb{C}/\Lambda \rightarrow S$, so that $\Phi \circ p = z + \lambda B\Omega$, for some $z \in \mathbb{C}^3$, $\lambda \in \mathbb{C}^*$ and $B \in SO(3, \mathbb{C})$, where Ω is as in (6).*

Remarks. 1. Theorem 1 shows that Ω ’s dual curve on \mathbb{T} is an elliptic double cover of \mathbb{P}_1 , branched at four distinct points on \mathbb{P}_1 . This can be deduced by direct calculation of Ω ’s dual.

2. Much of the geometry of the null curve and the associated minimal surface can be read directly off S , without recourse to explicit formulae. See [9,10] for further details.

3. For real curves with rectangular lattices, $e_1 > e_2 > e_3$, and hence A in [Theorem 1](#) lies in $SO(3, \mathbb{R})$. In this case we obtain branched minimal immersions of twice-punctured Klein bottles into \mathbb{R}^3 . This is worked out in [11] for the special case of charge 2 monopole spectral curves.

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