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Simultaneous Diophantine approximation in the real, complex and p–adic fields.

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Abstract

In this paper it is shown that if the volume sum $\sum_{r=1}^{\infty} \Psi(r)$ converges for a monotonic function Ψ then the set of points $(x, z, w) \in \mathbb{R} \times \mathbb{C} \times \mathbb{Q}_p$ which simultaneously satisfy the inequalities $|P(x)| \leqslant H^{-v_1}\Psi^{\lambda_1}(H)$, $|P(z)| \leqslant H^{-v_2}\Psi^{\lambda_2}(H)$ and $|P(w)|_p \leqslant H^{-v_3}\Psi^{\lambda_3}(H)$ with $v_1 + 2v_2 + v_3 = n - 3$ and $\lambda_1 + 2\lambda_2 + \lambda_3 = 1$ for infinitely many integer polynomials P has measure zero.

1. *Introduction*

Throughout, let

$$P(f) = a_n f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0$$

be an integer polynomial with $a_n \neq 0$. The degree of P is $\deg P = n$ and the height of P is $H = H(P) = \max_{1 \leqslant j \leqslant n} |a_j|$. Let $\mathcal{P}_n$ be the set of integer polynomials of degree at most n. This paper concerns Diophantine approximation on such polynomials in the real, complex and p–adic fields simultaneously. That is, we will study the set of points $(x, z, w) \in \mathbb{R} \times \mathbb{C} \times \mathbb{Q}_p$ for which the values of $|P(x)|$, $|P(z)|$ and $|P(w)|_p$ are simultaneously small. Similar problems have been studied for the spaces $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Q}_p$ individually and these results are discussed below. Before we proceed, some notation is needed. Let $\mu_1(A_1)$ be the Lebesgue measure of a measurable set $A_1 \subset \mathbb{R}$; let $\mu_2(A_2)$ denote the Lebesgue measure of a measurable set $A_2 \subset \mathbb{C}$; and finally, let $\mu_3(A_3)$ denote the Haar measure of a measurable set $A_3 \subset \mathbb{Q}_p$. Using these definitions, define the product measure μ on $\mathbb{R} \times \mathbb{C} \times \mathbb{Q}_p$ by setting $\mu(A) = \mu_1(A_1)\mu_2(A_2)\mu_3(A_3)$ for a set $A = A_1 \times A_2 \times A_3$ with $A_1 \in \mathbb{R}$, $A_2 \in \mathbb{C}$ and $A_3 \in \mathbb{Q}_p$.

Let $L_n(v)$ denote the set of $x \in \mathbb{R}$ for which the inequality

$$|P(x)| < H^{-v}$$

has infinitely many solutions $P \in \mathcal{P}_n$. Using either Dirichlet's box principle or Minkowski's linear forms theorem it is not difficult to show that if $v \leqslant n$ then $L_n(v)$ has full Lebesgue measure. It was shown in [**10**] that $\mu_1(L_n(v)) = 0$ for $v > 4n$ and this was improved by Sprindzuk in [**11**] who solved Mahler's conjecture of 1932 by proving that

$$\mu_1(L_n(v)) = 0$$

for $v > n$. Now consider the set $L_n(\Psi)$ of points $x \in \mathbb{R}$ for which the inequality

$$|P(x)| < H^{-n+1}\Psi(H)$$

has infinitely many solutions $P \in \mathcal{P}_n$. In [**1**] Baker strengthened Sprindzuk's theorem and proved that if Ψ is a monotonically decreasing positive function then $\mu_1(L_n(\Psi)) = 0$ when $\sum_{H=1}^{\infty} \Psi(H) < \infty$. It is clear that for $\Psi(H) = H^{-1-\varepsilon}$ with $\varepsilon > 0$ Sprindzuk's result follows directly from Baker's theorem. If $n = 1$ then for $x \in I = [a, b] \subset \mathbb{R}$ the stronger classical Khintchine theorem [**8**] holds:

$$\mu_1(L_1(\Psi) \cap I) = \begin{cases} 0 & \text{if } \sum_{H=1}^{\infty} \Psi(H) < \infty, \\ \mu_1(I) & \text{if } \sum_{H=1}^{\infty} \Psi(H) = \infty. \end{cases}$$

In [**2**] and [**5**] it was proved that for any n:

$$\mu_1(L_n(\Psi) \cap I) = \begin{cases} 0 & \text{if } \sum_{H=1}^{\infty} \Psi(H) < \infty, \\ \mu_1(I) & \text{if } \sum_{H=1}^{\infty} \Psi(H) = \infty \end{cases} \tag{1}$$

for any interval $I \subset \mathbb{R}$.

These results have been further generalized to the fields of complex [**7**] and p–adic [**3, 9**] numbers. Sprindzuk's theorem has also been generalized to simultaneous approximation on $S = \mathbb{R} \times \mathbb{C} \times \mathbb{Q}_p$ [**12**, lemma 3].

In this paper an analogue of the convergence result in (1) will be proved for $S = \mathbb{R} \times \mathbb{C} \times \mathbb{Q}_p$. To that end more notation is needed. Fix a parallelepiped $\mathbf{T} = I \times K \times D$, where I is an interval in $\mathbb{R}$, K is a disc in $\mathbb{C}$ and D is a cylinder in $\mathbb{Q}_p$. Let $\mathbf{v} = (v_1, v_2, v_3)$ and $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ be real vectors with $\lambda_i > 0$ and $v_i \geqslant 0$ such that $v_1 + 2v_2 + v_3 = n - 3$ and $\lambda_1 + 2\lambda_2 + \lambda_3 = 1$. Finally, let $L_n(\mathbf{v}, \lambda, \Psi)$ denote the set of points $(x, z, w) \in \mathbf{T}$ for which the system of inequalities

$$\begin{aligned} |P(x)| &\leqslant H^{-v_1}\Psi^{\lambda_1}(H), \\ |P(z)| &\leqslant H^{-v_2}\Psi^{\lambda_2}(H), \\ |P(w)|_p &\leqslant H^{-v_3}\Psi^{\lambda_3}(H), \end{aligned} \tag{2}$$

holds for infinitely many $P \in \mathcal{P}_n$. The main result of this paper is the following theorem.

THEOREM 1. *If $n \geqslant 3$, Ψ is a real, positive, monotonically decreasing function such that $\sum_{H=1}^{\infty} \Psi(H) < \infty$ then*

$$\mu(L_n(\mathbf{v}, \lambda, \Psi)) = 0.$$

The next section contains some preliminary results and auxilliary lemmas. The theorem is then proved in Section 3.

2. *Preliminary results*

As $\Psi^{\lambda}(H)$ is monotonic and the series $\sum_{H=1}^{\infty}\Psi(H)$ converges it is easy to show that $\Psi(H) < c_1 H^{-1}$, where c_1 is independent of H. Therefore, instead of (2) the weaker system

$$\begin{aligned} |P(x)| &\ll H^{-v_1-\lambda_1},\\ |P(z)| &\ll H^{-v_2-\lambda_2},\\ |P(w)|_p &\ll H^{-v_3-\lambda_3}, \end{aligned} \tag{3}$$

will be considered at some stages for simplicity. Here and throughout $A \ll B$ means that there exists a constant $C > 0$ such that $A \leqslant CB$; $A \asymp B$ is equivalent to $A \ll B \ll A$.

In the main, positive constants which depend only on n will be denoted by $c(n)$; the usual formal rules apply so that $c(n) + c(n) = c(n)$ and $c(n)c(n) = c(n)$. Where necessary these constants will be numbered $c_j(n)$, $j = 1, 2, \ldots$.

2·1. *Reduction to irreducible, leading polynomials*

In this subsection, it will first be shown that only irreducible polynomials $P \in \mathbb{Z}[x]$ need to be considered. This follows from the lemma below which is proved in [**12**].

LEMMA 1. *Let* $G(v)$ *be the set of points* (x, z, w) *for which the inequality*

$$|P(x)||P(z)|^2|P(w)|_p < H^{-v}, \quad n = \deg P \geqslant 2, \quad H = H(P),$$

has infinitely many solutions $P \in \mathbb{Z}[x]$. *Then, for* $v > n - 2$

$$\mu(G(v)) = 0.$$

Assume that $P = P_1 P_2$ is reducible and satisfies (2). Let $\deg P_1 = d \leqslant n - 1$. Then, without loss of generality we may assume that

$$|P_1(x)||P_1(z)|^2|P_1(\omega)|_p \ll H(P_1)^{-n+3}\psi(H(P_1)) \ll H(P_1)^{-d+1}.$$

Thus, from Lemma 1, the set of (x, z, ω) which satisfy (2) for infinitely many reducible polynomials P has measure zero. From now on we will assume that P is irreducible.

A polynomial P will be called *leading* if it satisfies

$$\begin{aligned} H(P) < c(n)|a_n|, \quad c(n) \geqslant 1,\\ |a_n|_p > c(n). \end{aligned} \tag{4}$$

In the next lemma it will be demonstrated that by taking translations and reciprocals (if necessary) each polynomial P can be transformed into a polynomial T satisfying (4). Since there are only a finite number of possible translations, any point x which satisfies (2) infinitely often must also satisfy it for infinitely many leading polynomials for one particular translation. Similar reductions were made in [**11**] for the metrics considered separately. As this reduction to leading polynomials has not been previously published in the simultaneous case we will prove it here.

LEMMA 2. *Let* $p_1, p_2, \ldots, p_k$ *be a set of distinct prime numbers and* $P \in \mathbb{Z}[x]$ *be a primitive, irreducible polynomial. Let* $C = C(n, p_1, \ldots, p_k)$ *be a constant. There exists a natural number* $m \leqslant C$ *with the following property: let* $Q(x) = P(x + m)$ *and* $T(x) = x^n Q(1/x)$, *then the polynomial* $T(x) = b_n x^n + \cdots + b_1 x + b_0 \in \mathbb{Z}[x]$ *satisfies*

$$|b_n| \gg H(T), \quad |b_n|_{p_i} \gg 1, \quad i = 1, \cdots, k.$$

Proof. Assume that for some d the system of inequalities

$$\max_{1\leqslant k\leqslant n+1} |P(k)|_{p_1} < p_1^{-d}. \tag{5}$$

holds. Thus, for each $i = 1, \ldots, n+1$

$$i^n a_n + i^{n-1} a_{n-1} + \cdots + i a_1 + a_0 = p_1^d |\theta_i|_{p_1} \tag{6}$$

where $\theta_i = p_1^{d_i}\theta_i'$, with $d_i \geqslant 1$, $\theta_i \in \mathbb{N}$ and $(p_1, \theta_i') = 1$. Since P is primitive there exists j_0, $0 \leqslant j_0 \leqslant n$, such that $|a_{j_0}|_{p_1} = 1$. System (6) will now be solved for a_{j_0} to obtain

$$a_{j_0} = \frac{\Delta_{j_0}}{\Delta},$$

where Δ is the determinant of the $(n+1)\times(n+1)$ matrix (b_{ij}) with $b_{ij} = i^{j-1}$, $1 \leqslant i, j \leqslant n+1$. It is readily verified that $\Delta = \prod_{k=0}^{n-1}(n-k)!$.

If p_1^r divides $k!$ then

$$r \leqslant \left[\frac{k}{p_1}\right] + \left[\frac{k}{p_1^2}\right] + \cdots \leqslant k\sum_{j=1}^{\infty} p_1^{-j} \leqslant k.$$

Hence, p_1 divides Δ to the power at most n^n. It can also be readily verified that p^d divides Δ_{j_0} and hence that p^{d-n^n} divides a_{j_0}. If $d > n^n$ this contradicts the fact that $|a_{j_0}|_{p_1} = 1$ and therefore provides a contadiction to (5). Thus, there exists $m_0 \in \{1, \ldots, n+1\}$ such that $|P(m_0)|_{p_1} \geqslant 1$.

Define the integer l_1 by $|P(m_0)|_{p_1} = p_1^{-l_1}$ and choose $l_1' > l_1$. Consider the numbers $r_1(m_1) = m_1 p_1^{l_1'} + m_0$, $1 \leqslant m_1 \leqslant n+1$. Clearly, $|P(r_1(m_1))|_{p_1} = |P(m_0)|_{p_1} \geqslant 1$. The above argument from (5) onwards is now repeated for the numbers $r_1(m_1)$, $1 \leqslant m_1 \leqslant n+1$. Assume that there exists d such that $|P(r_1(m_1))|_{p_2} < p_2^{-d}$. Let Δ' be the determinant of the matrix (b_{ij}) with $b_{ij} = (ip_1^{l_1'} + m_0)^{j-1}$, $1 \leqslant i, j \leqslant n+1$. Then

$$\Delta' = \left(p_1^{l_1'}\right)^{\frac{n(n+1)}{2}} \prod_{k=0}^{n-1}(n-k)!.$$

Hence, there exists a number m_1' in $\{1, \ldots, n+1\}$ such that $|P(r_1(m_1'))|_{p_2} \geqslant 1$; i.e. there exists l_2 such that $|P(r_1(m_1'))|_{p_2} = p_2{}^{-l_2}$.

Repeat again; so for $l_2' > l_2$ consider the numbers $r_2(m_2) = m_2 p_1^{l_1'} p_2^{l_2'} + m_1' p_1^{l_1'} + m_0$, $1 \leqslant m_2 \leqslant n+1$. Clearly by construction, $|P(r_2(m_2))|_{p_1} \geqslant 1$ and $|P(r_2(m_2))|_{p_2} \geqslant 1$. Following the previous argument also yields that $|P(r_2(m_2)|_{p_3} \geqslant 1$. Continue this process to obtain finally that there exists a number m_{k-1}', $1 \leqslant m_{k-1}' \leqslant n+1$, such that $|P(r_{k-1}(m_{k-1}'))|_{p_i} \geqslant 1$ for $i = 1, \ldots, k$.

Similarly for the Archimedean metric consider the numbers

$$r_k(m_k) = m_k p_1^{l_1'} \cdots p_k^{l_k'} + \cdots + m_2' p_1^{l_1'} p_2^{l_2'} + m_1' p_1^{l_1'} + m_0$$

for $m_k = 1, \ldots, n+1$. It will now be demonstrated that among these $n+1$ numbers it is possible to find m_k' such that $|P(r_k(m_k'))| \geqslant H$. Assume that the system of the inequalities

$$\max_{1\leqslant m_k\leqslant n+1} |P(r_k(m_k))| \leqslant c_1 H \tag{7}$$

holds for some $c_1 > 0$ (to be chosen). Clearly, if $H(P) = H$ then there exists i_0, $0 \leqslant i_0 \leqslant n$, such that $|a_{i_0}| = H$. Solve system (6) for a_{i_0} to obtain that $P(r_k(m_k)) = \xi_k c_1 H$, where

$|\xi_j| \leqslant 1$, $1 \leqslant j \leqslant n+1$, and

$$a_{i_0} = \frac{\Delta''_{i_0}}{\Delta''}.$$

Here, Δ'' is the determinant of the matrix (b_{ij}) where

$$b_{ij} = (ip^{l'_1} \cdots p^{l'_k} + \cdots + m'_2 p_1^{l'_1} p_2^{l'_2} + m'_1 p_1^{l'_1} + m_0)^{j-1}, 1 \leqslant i, j \leqslant n+1,$$

so that

$$\Delta'' = \left(p_1^{l'_1} \cdots p_k^{l'_k}\right)^{\frac{n(n+1)}{2}} \prod_{k=0}^{n-1} (n-k)!.$$

Hence there exists a constant $c_2 > 0$ such that $\Delta''_{i_0} = c_1 c_2 H$ holds. Now choose c_1 such that $c_1 c_2 < 1$. Since $|a_{i_0}| = H$ this contradicts (7). Hence, there exists m'_k such that $|P(r_k(m'_k))| \geqslant H$.

Define the polynomial Q by $Q(x) = P(x + r_k(m'_k)) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$, where $b_0 = P(r_k(m'_k))$; and define the polynomial T as $T(x) = x^n Q(1/x) = g_n x^n + g_{n-1}x^{n-1} + \cdots + g_1 x + g_0$, where $g_n = P(r_k(m'_k))$. Then, $H(T) \asymp H(Q) \asymp H(P)$ and T has the properties required in the statement.

2·2. *Preliminary setup and auxilliary lemmas*

From now on we will assume that P is a leading, irreducible polynomial. To this end let $\mathcal{P}_n(H)$ denote the set of polynomials $P \in \mathcal{P}_n$ satisfying (4) for which $H(P) = H$. Let $P \in \mathcal{P}_n(H)$ have roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $\mathbb{C}$ and roots $\gamma_1, \gamma_2, \ldots, \gamma_n$ in $\mathbb{Q}_p^*$, where $\mathbb{Q}_p^*$ is the smallest field containing $\mathbb{Q}_p$ and all algebraic numbers. Then, from (4) it is not difficult to show that

$$|\alpha_i| \ll 1, \quad |\gamma_i|_p \ll 1, \quad i = 1, \ldots, n;$$

i.e. the roots are bounded. Define the sets

$$S_1(\alpha_j) = \{x \in \mathbb{R} : |x - \alpha_j| = \min_{1 \leqslant i \leqslant n} |x - \alpha_i|\},$$
$$S_2(\alpha_s) = \{z \in \mathbb{C} : |z - \alpha_s| = \min_{1 \leqslant i \leqslant n} |z - \alpha_i|\},$$
$$S_3(\gamma_k) = \{w \in \mathbb{Q}_p : |w - \gamma_k|_p = \min_{1 \leqslant i \leqslant n} |w - \gamma_i|_p\}.$$

We will consider the sets $S_1(\alpha_j)$, $S_2(\alpha_s)$, $S_3(\gamma_k)$ for a fixed set j, s, k and for simplicity we will assume that $j = 1$, $\alpha_s = \beta_1$ and $k = 1$, where the set of roots $\beta_1, \beta_2, \ldots, \beta_n$ is a permutation of the roots $\alpha_1, \alpha_2, \ldots, \alpha_n$. Reorder the other roots of P so that

$$|\alpha_1 - \alpha_2| \leqslant |\alpha_1 - \alpha_3| \leqslant \ldots \leqslant |\alpha_1 - \alpha_n|,$$
$$|\beta_1 - \beta_2| \leqslant |\beta_1 - \beta_3| \leqslant \ldots \leqslant |\beta_1 - \beta_n|,$$
$$|\gamma_1 - \gamma_2|_p \leqslant |\gamma_1 - \gamma_3|_p \leqslant \ldots \leqslant |\gamma_1 - \gamma_n|_p.$$

Also, for the polynomial $P \in \mathcal{P}_n(H)$ define the real numbers ρ_{ij} $(i = 1, 2, 3)$ by

$$|\alpha_1 - \alpha_j| = H^{-\rho_{1j}}, \quad 2 \leqslant j \leqslant n, \quad \rho_{12} \geqslant \rho_{13} \ldots \geqslant \rho_{1n},$$
$$|\beta_1 - \beta_j| = H^{-\rho_{2j}}, \quad 2 \leqslant j \leqslant n, \quad \rho_{22} \geqslant \rho_{23} \ldots \geqslant \rho_{2n},$$
$$|\gamma_1 - \gamma_j|_p = H^{-\rho_{3j}}, \quad 2 \leqslant j \leqslant n, \quad \rho_{32} \geqslant \rho_{33} \ldots \geqslant \rho_{3n}.$$

Since the roots $|\alpha_j|$, $|\beta_s|$, $|\gamma_k|_p$ are bounded there exists $\varepsilon_1 > 1$ such that $\rho_{ij} \geqslant -\epsilon_1/2$ for $i = 1, 2, 3$ and $2 \leqslant j \leqslant n$. Choose $\varepsilon > 0$ such that $\varepsilon_1 = \varepsilon N^{-1}$ for some sufficiently large

N and let $T = [\varepsilon_1^{-1}]$. Also, define the integers k_j, l_j and m_j, $2 \leqslant j \leqslant n$, by the relations

$$\frac{k_j - 1}{T} \leqslant \rho_{1j} < \frac{k_j}{T}, \quad \frac{l_j - 1}{T} \leqslant \rho_{2j} < \frac{l_j}{T}, \quad \frac{m_j - 1}{T} \leqslant \rho_{3j} < \frac{m_j}{T},$$
$$k_2 \geqslant k_3 \geqslant \cdots \geqslant k_n \geqslant 0, \quad l_2 \geqslant l_3 \geqslant \cdots \geqslant l_n \geqslant 0 \quad m_2 \geqslant m_3 \geqslant \cdots \geqslant m_n \geqslant 0.$$

Finally, define the numbers q_i, r_i and s_i by

$$\begin{aligned} q_i &= \frac{k_{i+1} + \cdots + k_n}{T}, \quad (1 \leqslant i \leqslant n-1), \\ r_i &= \frac{l_{i+1} + \cdots + l_n}{T}, \quad (1 \leqslant i \leqslant n-1), \\ s_i &= \frac{m_{i+1} + \cdots + m_n}{T}, \quad (1 \leqslant i \leqslant n-1). \end{aligned} \tag{8}$$

Each polynomial $P \in \mathcal{P}_n(H)$ is now associated with three integer vectors $\mathbf{q} = (k_2, \ldots, k_n)$, $\mathbf{r} = (l_2, \ldots, l_n)$ and $\mathbf{s} = (m_2, \ldots, m_n)$ and the number of these vectors is finite (and depends only on n, p and T), see [**11**, lemma 24, p46 and lemma 12, p99]. Let $\mathcal{P}_n(H, \mathbf{q}, \mathbf{r}, \mathbf{s})$ denote the set of polynomials $P \in \mathcal{P}_n(H)$ with the same triple of vectors $(\mathbf{q}, \mathbf{r}, \mathbf{s})$.

From now on it will be assumed without loss of generality that $x \in S_1(\alpha_1)$, $z \in S_2(\beta_1)$ and $w \in S_3(\gamma_1)$. In many places in the proof of the theorem the values of the polynomials will be estimated by means of a Taylor series. To obtain an upper bound on the terms in the Taylor series (and for other purposes) the following two lemmas (proved in [**4**] and [**9**]) will be used.

LEMMA 3. *If $P \in \mathcal{P}_n$ then*

$$|u - \alpha| \leqslant 2^n |P(u)| |P'(\alpha)|^{-1},$$
$$|w - \gamma_1|_p \leqslant |P(w)|_p |P'(\gamma_1)|_p^{-1},$$
$$|u - \alpha| \leqslant \min_{2 \leqslant j \leqslant n} \left(2^{n-j} |P(u)| |P'(\alpha)|^{-1} \prod_{k=2}^{j} |\alpha - \alpha_k| \right)^{\frac{1}{j}},$$
$$|w - \gamma_1|_p \leqslant \min_{2 \leqslant j \leqslant n} \left(|P(w)|_p |P'(\gamma_1)|_p^{-1} \prod_{k=2}^{j} |\gamma_1 - \gamma_k|_p \right)^{\frac{1}{j}}$$

where u represents x or z and α is α_1 or β_1 as required.

Fix $\delta_1 > 0$. As δ_1 is arbitrary we may assume without loss of generality that any complex number z lying in the parallelepiped $\mathbf{T}$ satisfies $|\mathrm{Im}\, z| \geqslant \delta_1$. From Lemma 3, when $j = n$ we obtain that $|z - \beta| < H(P)^{-\nu}$ with $\nu > 0$; as the RHS tends to zero it will follow that there exists a root β such that $|\mathrm{Im}\, \beta| > \delta/2$. In this case there is also a conjugate root $\bar{\beta}$ of P such that $|\beta - \bar{\beta}| > \delta_1$, and for any real root α of P the inequalities $|\beta - \alpha| = |\bar{\beta} - \alpha| > \delta/2$ hold. Collecting this information, we have

$$|\mathrm{Im}\, \beta| > \frac{1}{2}\delta_1, \quad |\mathrm{Im}\, z| \geqslant \delta_1, \quad |\beta - \bar{\beta}| > \delta_1, \quad |\beta - \alpha| > \frac{1}{2}\delta_1. \tag{9}$$

LEMMA 4. *Let $P \in \mathcal{P}_n(H, \mathbf{q}, \mathbf{r}, \mathbf{s})$. Then*

$$|P^{(l)}(\alpha_1)| < c(n) H^{1 - q_l + (n-l)\varepsilon_1},$$
$$|P^{(l)}(\beta_1)| < c(n) H^{1 - r_l + (n-l)\varepsilon_1},$$
$$|P^{(l)}(\gamma_1)|_p < c(n) H^{-s_l + (n-l)\varepsilon_1},$$

for $1 \leqslant l \leqslant n-1$.

At several points in the proof of the theorem there are various cases (of different types of polynomial) to consider; usually the existence of one case is disproved by finding a contradiction to the final inequality in the next lemma which is proved in [**6**].

LEMMA 5. *Let* P_1 *and* P_2 *be two integer polynomials of degree at most n with no common roots and* $\max(H(P_1), H(P_2)) \leqslant H$. *Let* $\delta > 0$ *and* $\eta_i > 0$ *for* $i = 1, 2, 3$. *Let* $I \subset \mathbb{R}$ *be an interval,* $K \subset \mathbb{C}$ *be a disk and* $D \subset \mathbb{Q}_p$ *be a cylinder with* $\mu_1(I) = H^{-\eta_1}$, $\operatorname{diam} K = H^{-\eta_2}$ *and* $\mu_3(D) = H^{-\eta_3}$. *If there exist* $\tau_1 > -1$, $\tau_2 > -1$ *and* $\tau_3 > 0$ *such that for all* $(x, z, w) \in I \times K \times D$

$$|P_j(x)| < H^{-\tau_1},$$
$$|P_j(z)| < H^{-\tau_2},$$
$$|P_j(w)|_p < H^{-\tau_3},$$

for $j = 1, 2$, *then*

$$\tau_1 + 2\tau_2 + \tau_3 + 3 + 2\max(\tau_1 + 1 - \eta_1, 0) + 4\max(\tau_2 + 1 - \eta_2, 0) + 2\max(\tau_3 - \eta_3, 0) < 2n + \delta.$$

Finally, we state two classical results. The first is proved in [**2**] and is an adaptation of Cauchy's Condensation Test. The second is the convergence half of the Borel–Cantelli Lemma which will be used throughout the proof of the theorem.

LEMMA 6. *Let* $\Psi(H)$, $H = 1, 2, \ldots,$ *be a monotonically decreasing sequence of positive numbers. If the series* $\sum_{H=1}^{\infty} \Psi(H)$ *converges, then for any number* $c > 0$ *the series* $\sum_{k=0}^{\infty} 2^k \Psi(c2^k)$ *also converges.*

LEMMA 7 (Borel–Cantelli). *Let* (Ω, μ) *be a measure space with* $\mu(\Omega)$ *finite and let* A_i, $i \in n$ *be a family of measurable sets. Let*

$$A = \{\omega \in \Omega : \omega \in A_i \textit{ for infinitely many } i \in n\}$$

and suppose that the sum $\sum_{i=1}^{\infty} \mu(A_i) < \infty$. *Then* $\mu(A) = 0$.

3. *Proof of the Theorem*

Since $|\alpha_i| \ll 1$ and $|\gamma_i|_p \ll 1$ for $1 \leqslant i \leqslant n$ it follows from Lemma 3 (using $j = n$ and $H \leqslant H_0$) that the set of points (x, z, w), for which (2) is satisfied, is a subset of the set $\mathbf{T} = I \times K \times D$, where $I = [-c(n), c(n)]$, $K = \{z : |z| \leqslant c(n)\}$ and $D = \{w : |w|_p \ll 1\}$.

Remember that the polynomials $P \in \mathcal{P}_n(H)$ are irreducible and satisfy (4). A polynomial $P \in \mathcal{P}_n(H, \mathbf{q}, \mathbf{r}, \mathbf{s})$ will be called (i_1, i_2, i_3)–linear if for $i_j = 0$, $j = 1, 2, 3$, the system of inequalities

$$\begin{aligned} q_1 + k_2 T^{-1} &< v_1 + \lambda_1 + 1, \\ r_1 + l_2 T^{-1} &< v_2 + \lambda_2 + 1, \\ s_1 + m_2 T^{-1} &< v_3 + \lambda_3, \end{aligned} \tag{10}$$

holds, and for $i_j = 1$, $j = 1, 2, 3$, the inequality signs in (10) are reversed. For example, (0, 1, 1)–linearity means that in (10) the first inequality has $<$ and the second and third have $\geqslant$. Denote by $\mathcal{P}_n^{(i_1,i_2,i_3)}$, $i_j = 0, 1$, $j = 1, 2, 3$, the class of (i_1, i_2, i_3)–linear polynomials $P \in \mathcal{P}_n$. If $(x, z, w) \in L_n(\mathbf{v}, \lambda, \Psi)$ then there exist infinitely many polynomials satisfying at least one of these eight kinds of linearity. Let $L_n^{(i_1,i_2,i_3)}(v, \lambda, \Psi)$ denote the set of points $(x, z, w) \in \mathbf{T}$ for which the system of inequalities (2) holds for infinitely many $P \in \mathcal{P}_n^{(i_1,i_2,i_3)}$.

It should be clear that $L_n(v, \lambda, \Psi) = \bigcup_{i_1,i_2,i_3=0,1} L_n^{(i_1,i_2,i_3)}(v, \lambda, \Psi)$. Therefore, the theorem will be proved by showing that each of $L_n^{(i_1,i_2,i_3)}(v, \lambda, \Psi)$ has measure zero.

The constants

$$d_1 = q_1 + 2r_1 + s_1 \quad \text{and} \quad d_2 = (k_2 + 2l_2 + m_2)T^{-1}$$

will be used heavily in the rest of the proof which consists of a series of propositions with different linearity conditions and different ranges of $d_1 + d_2$ considered separately.

Throughout the proof the facts that

$$|P'(\alpha_1)| = H|\alpha_1 - \alpha_2| \ldots |\alpha_1 - \alpha_n| = H^{1-q_1}, \ |P'(\beta_1)| = H^{1-r_1}, \ \ |P'(\gamma_1)|_p = H^{-s_1} \quad (11)$$

will be used; these follow directly from (8).

First, the polynomials which are (0, 0, 0)—linear are considered.

Case 1. (0, 0, 0)—linearity

To prove that $\mu(L_n^{(0,0,0)}(\mathbf{v}, \lambda, \Psi)) = 0$ four propositions, each dealing with a different range of $d_1 + d_2$, will be proved. If $(x, z, w) \in L_n^{(0,0,0)}(\mathbf{v}, \lambda, \Psi)$ then there exist infinitely many polynomials $P \in \mathcal{P}_n^{(0,0,0)}$ satisfying one of these conditions on $d_1 + d_2$ for which (2) holds. Thus if we can prove that the set of points for which there exist infinitely many polynomials $P \in \mathcal{P}_n^{(0,0,0)}$ which satisfy (2) with $d_1 + d_2$ in each of these ranges is of measure zero we will have proved that $\mu(L_n^{(0,0,0)}(\mathbf{v}, \lambda, \Psi)) = 0$ as required.

PROPOSITION 1. *Assume that* $\sum_{H=1}^{\infty} \Psi(H) < \infty$. *The set of points* $(x, z, w) \in \mathbf{T}$ *for which the system of inequalities* (2) *is satisfied for infinitely many polynomials* $P \in \mathcal{P}_n^{(0,0,0)}$ *with* $d_1 + d_2 > n + \varepsilon$ *has measure zero.*

Proof. Assume that $P \in \mathcal{P}_n^{(0,0,0)}$ with $2^t \leqslant H(P) < 2^{t+1}$ and $d_1 + d_2 > n + \varepsilon$. We denote the set of such P by $\mathcal{P}_1^t$. Let $\sigma(P)$ be the set of points $\mathbf{u} = (x, z, w) \in \mathbf{T} \bigcap S_1(\alpha_1) \times S_2(\beta_1) \times S_3(\gamma_1)$ which satisfy (3). By Lemma 3 and (11) each $\mathbf{u} \in \sigma(P)$ satisfies the inequalities

$$\begin{aligned} |x - \alpha_1| &\ll 2^{-t(v_1+\lambda_1+1-q_1)}, \\ |z - \beta_1| &\ll 2^{-t(v_2+\lambda_2+1-r_1)}, \\ |w - \gamma_1|_p &\ll 2^{-t(v_3+\lambda_3-s_1)}. \end{aligned} \quad (12)$$

Let $A_t = \bigcup_{P \in \mathcal{P}_1^t} \sigma(P)$. Then, the set of points satisfying the conditions in the proposition is the set of points lying in infinitely many A_t. In order to use the Borel–Cantelli Lemma we aim to prove that $\sum_{t=1}^{\infty} \mu(A_t) < \infty$.

The initial parallelepiped $\mathbf{T}$ is divided into smaller parallelepipeds $M = I_M \times K_M \times D_M$ such that

$$\mu_1(I_M) = 2^{-tk_2T^{-1}}, \quad \operatorname{diam}(K_M) = 2^{-tl_2T^{-1}}, \quad \mu_3(D_M) = 2^{-tm_2T^{-1}}. \quad (13)$$

It will be said that the polynomial P *belongs* to the parallelepiped M if there exists $\mathbf{u} \in M$ such that (3) holds. Assuming that P belongs to M we now develop $P \in \mathcal{P}_1^t$ as a Taylor series at each coordinate of $\mathbf{u}$. Note that $P(\alpha_1) = P(\beta_1) = P(\gamma_1) = 0$. Obviously,

$$P(t) = \sum_{j=1}^{n} (j!)^{-1} P^{(j)}(\zeta_1)(t - \zeta_1)^j$$

for $t = x, z, w$ and $\zeta_1 = \alpha_1, \beta_1, \gamma_1$ respectively. The upper bound for $|P(z)|$ is now obtained by using the following inequalities, which come directly from (8),

$$r_j + jl_2T^{-1} = r_j + l_2T^{-1} + (j-1)l_2T^{-1} \geqslant r_j + l_2T^{-1} + (l_2 + \cdots + l_j)T^{-1} = r_1 + l_2T^{-1}.$$

These imply, by using (13) and Lemma 4, that

$$|P'(\beta_1)||z - \beta_1| \ll 2^{t(1-r_1+(n-1)\varepsilon_1 - l_2T^{-1})} \ll 2^{-t(r_1+l_2T^{-1}-1-(n-1)\varepsilon_1)}$$

and

$$|P^{(j)}(\beta_1)||z - \beta_1|^j \ll 2^{t(1-r_j+(n-j)\varepsilon_1 - jl_2T^{-1})} \ll 2^{-t(r_1+l_2T^{-1}-1-(n-1)\varepsilon_1)}, \quad 2 < j \leqslant n.$$

Clearly these further imply that $|P(z)| \ll 2^{-t(r_1+l_2T^{-1}-1-(n-1)\varepsilon_1)}$ for any $z \in K_M$. It is not difficult to acquire similar estimates for $|P(x)|$ and $|P(w)|_p$ so that

$$\begin{aligned} |P(x)| &\ll 2^{-t(q_1+k_2T^{-1}-1-(n-1)\varepsilon_1)}, \\ |P(z)| &\ll 2^{-t(r_1+l_2T^{-1}-1-(n-1)\varepsilon_1)}, \\ |P(w)|_p &\ll 2^{-t(s_1+m_2T^{-1}-(n-1)\varepsilon_1)} \end{aligned} \tag{14}$$

hold for any $(x, z, w) \in M$.

First, assume that at least two polynomials P_1 and P_2 belong to a parallelepiped M. These polynomials are irreducible, with degree at most n and height at most 2^{t+1}. For these the system of inequalities (14) holds. Using Lemma 5, with

$$\begin{aligned} \tau_1 &= q_1 + k_2T^{-1} - 1 - (n-1)\varepsilon_1, \\ \tau_2 &= r_1 + l_2T^{-1} - 1 - (n-1)\varepsilon_1, \\ \tau_3 &= s_1 + m_2T^{-1} - (n-1)\varepsilon_1, \\ \eta_1 &= k_2T^{-1}, \\ \eta_2 &= l_2T^{-1}, \\ \eta_3 &= m_2T^{-1}, \end{aligned}$$

we obtain that

$$3q_1 + k_2T^{-1} + 6r_1 + 2l_2T^{-1} + 3s_1 + m_2T^{-1} - 12(n-1)\varepsilon_1 < 2n + \delta.$$

Since $q_1 \geqslant k_2T^{-1}$, $2r_1 \geqslant 2l_2T^{-1}$ and $s_1 \geqslant m_2T^{-1}$ this further implies that

$$2(d_1 + d_2) - 12(n-1)\varepsilon_1 < 2n + \delta,$$

which for δ sufficiently small contradicts the condition on $d_1 + d_2$ in the statement of the proposition.

From above it may be assumed that at most one polynomial $P \in \mathcal{P}_1^t$ belongs to each parallelepiped M. The number of parallelepipeds and therefore the number of such polynomials is at most $c(n)2^{t(k_2+2l_2+m_2)T^{-1}} = c(n)2^{td_2}$. Hence, from (12)

$$\mu(A_t) \ll 2^{-t(v_1+2v_2+v_3+\lambda_1+2\lambda_2+\lambda_3+3-d_1-d_2)} \ll 2^{-t(n+1-d_1-d_2)}.$$

From (10) we have $d_1 + d_2 < n + 1$ so that $\sum_{t=1}^{\infty} \mu(A_t) \leqslant \sum_{t=1}^{\infty} 2^{-t(n+1-d_1-d_2)} < \infty$ and the proposition follows from the Borel–Cantelli Lemma.

PROPOSITION 2. *Assume that* $\sum_{H=1}^{\infty} \Psi(H) < \infty$. *The set of points* $(x, z, w) \in \mathbf{T}$ *for which the system of inequalities* (2) *is satisfied for infinitely many polynomials* $P \in \mathcal{P}_n^{(0,0,0)}$ *with* $d_1 + d_2 < \varepsilon$ *has measure zero.*

Proof. Assume that $P \in \mathcal{P}_n^{(0,0,0)}$ with $2^t \leqslant H(P) < 2^{t+1}$ and $d_1 + d_2 < \varepsilon$. We denote the set of such P by $\mathcal{P}_2^t$. If $d_1 + d_2 < \varepsilon$ then clearly $q_1 < \varepsilon$, $r_1 < \varepsilon$ and $s_1 < \varepsilon$. Let $\sigma_2(P)$ be the set of points $(x, z, w) \in \mathbf{T} \bigcap S_1(\alpha_1) \times S_2(\beta_1) \times S_3(\gamma_1)$ satisfying (2) for a polynomial P. By Lemma 3, every point in $\sigma_2(P)$ satisfies

$$\begin{aligned} |x-\alpha_1| &\ll 2^{-tv_1}\Psi^{\lambda_1}(2^t)|P'(\alpha_1)|^{-1}, \\ |z-\beta_1| &\ll 2^{-tv_2}\Psi^{\lambda_2}(2^t)|P'(\beta_1)|^{-1}, \\ |w-\gamma_1|_p &\ll 2^{-tv_3}\Psi^{\lambda_3}(2^t)|P'(\gamma_1)|_p^{-1}. \end{aligned} \tag{15}$$

Let $A_t = \bigcup_{P\in\mathcal{P}_2^t}\sigma_2(P)$. Then, the set of points satisfying the conditions in the proposition is the set of points lying in infinitely many A_t. As in the previous proposition we aim to prove that $\sum_{t=1}^{\infty}\mu(A_t) < \infty$ and then use the Borel–Cantelli Lemma.

For t sufficiently large, the parallelepiped $\sigma_3(P)$ defined by the inequalities

$$\begin{aligned} |x-\alpha_1| &< c_1(n)|P'(\alpha_1)|^{-1}, \\ |z-\beta_1| &< c_1(n)|P'(\beta_1)|^{-1}, \\ |w-\gamma_1|_p &< c_1(n)|P'(\gamma_1)|_p^{-1}, \end{aligned}$$

contains $\sigma_2(P)$. The value of $c_1(n)$ is determined later.

Fix the vector $\mathbf{b} = (a_3, a_4, \ldots, a_n)$ where a_j is the jth coefficient of $P \in \mathcal{P}_2^t$. The subclass of polynomials P with the same vector $\mathbf{b}$ is denoted by $\mathcal{P}_{2,\mathbf{b}}^t$. As before, develop the polynomials in $\mathcal{P}_{2,\mathbf{b}}^t$ as Taylor series in $\sigma_3(P)$ to obtain an upper bound for $|P(x)|$, $|P(z)|$, and $|P(w)|_p$. The real case will be demonstrated. From Lemma 4, (11) and since $q_j \leqslant q_1 < \varepsilon$ for $j \geqslant 2$

$$|P'(\alpha_1)||x-\alpha_1| < c_1(n)c(n),$$

and

$$|P^{(j)}(\alpha_1)||x-\alpha_1|^j < 2^{t(1-q_j+(n-j)\varepsilon_1-j+jq_1)}c_1(n)c(n) < c_1(n)c(n), \; 2 \leqslant j \leqslant n.$$

Using exactly the same arguments for $|P(z)|$ and $|P(w)|_p$ the system of inequalities

$$\begin{aligned} |P(x)| &< c_1(n)c(n), \\ |P(z)| &< c_1(n)c(n), \\ |P(w)|_p &< c_1(n)c(n) \end{aligned}$$

therefore holds. It will now be shown that if $P_1, P_2 \in \mathcal{P}_{2,\mathbf{b}}^t$ then the parallelepipeds $\sigma_3(P_1)$ and $\sigma_3(P_2)$ are disjoint for sufficiently small $c_1(n)$. Assume that this is not the case so that

$$\sigma_3(P_1, P_2) = \sigma_3(P_1) \bigcap \sigma_3(P_2) \neq \varnothing.$$

Let $R(f) = P_1(f) - P_2(f)$ so that R is of the form $R(f) = b_2 f^2 + b_1 f + b_0$ with $|b_i| \leqslant 2^{t+2}$, for $i = 0, 1, 2$. It should be clear that

$$\max(|R(x)|, |R(z)|) < c_1(n)c(n).$$

Using the previous equation we have

$$\begin{aligned} b_2x^2 + b_1x + b_0 &= \theta_1(x)c_1(n)c(n), \\ b_2z^2 + b_1z + b_0 &= \theta_2(z)c_1(n)c(n), \\ b_2\bar{z}^2 + b_1\bar{z} + b_0 &= \overline{\theta_2(z)}c_1(n)c(n), \end{aligned} \tag{16}$$

where $|\theta_k| \leqslant 1$ for $k = 1, 2$. If Δ is the determinant of this system of equations then $\Delta = 2z_2(z_2^2 + (x-z_1)^2)i$ where $z = z_1 + iz_2$ and $\bar{z} = z_1 - iz_2$. From (9) we have $|\Delta| > 2\delta_1^3$.

The system of equations (16) is now solved with respect to one of the coefficients $b_j \neq 0$, $0 \leqslant j \leqslant 2$ to obtain that $1 \leqslant |b_j| < c_1(n)c(n)\delta_1^{-3}$. (There must exist at least one $j = 0, 1, 2$ for which $|b_j| \geqslant 1$.) This is a contradiction for sufficiently small $c_1(n) = c_1(n, \delta_1)$. Hence, the parallelepipeds $\sigma_3(P_1)$ and $\sigma_3(P_2)$ are disjoint and

$$\sum_{P\in P_{2,\mathbf{b}}^t} \mu(\sigma_3(P)) \leqslant \mu(T).$$

The definitions of $\sigma_2(P)$ and $\sigma_3(P)$ further imply that

$$\mu(\sigma_2(P)) < c_1(n)^{-4}c(n)^4\mu(\sigma_3(P))2^{-t(v_1+2v_2+v_3)}\Psi^{\lambda_1+2\lambda_2+\lambda_3}(2^t) \leqslant \mu(\sigma_3(P))2^{-t(n-3)}\Psi(2^t).$$

Since the number of classes $\mathcal{P}_{2,\mathbf{b}}^t$ is at most $c(n)2^{t(n-2)}$ we obtain from the above two displayed inequalities that

$$\sum_{t=0}^{\infty}\mu(A_t) \leqslant \sum_{t=0}^{\infty}\sum_{\mathbf{b}}\sum_{P\in P_{2,\mathbf{b}}^t}\mu(\sigma_2(P)) < \sum_{t=0}^{\infty}\sum_{\mathbf{b}}\sum_{P\in P_{2,\mathbf{b}}^t}\mu(\sigma_3(P))2^{-t(n-3)}\Psi(2^t)$$
$$\leqslant \sum_{t=0}^{\infty}2^t\Psi(2^t)\mu(T) < \infty$$

by Lemma 6. Hence, by the Borel–Cantelli Lemma the result follows.

PROPOSITION 3. *Assume that* $\sum_{H=1}^{\infty}\Psi(H) < \infty$. *The set of points* $(x, z, w) \in \mathbf{T}$ *for which the system of inequalities* (2) *is satisfied for infinitely many polynomials* $P \in \mathcal{P}_n^{(0,0,0)}$ *with* $\varepsilon \leqslant d_1 + d_2 < 4 - \varepsilon$ *has measure zero.*

Proof. Assume that $P \in \mathcal{P}_n^{(0,0,0)}$ with $2^t \leqslant H(P) < 2^{t+1}$ and $\varepsilon \leqslant d_1 + d_2 < 4 - \varepsilon$. We denote the set of such P by $\mathcal{P}_3^t$. Let $\sigma_2(P)$ be defined as in Proposition 2 and let $A_t = \bigcup_{P\in\mathcal{P}_3^t}\sigma_2(P)$. As before the set of points satisfying the conditions in the proposition is the set of points lying in infinitely many A_t and again we aim to prove that $\sum_{t=1}^{\infty}\mu(A_t) < \infty$ and use the Borel–Cantelli Lemma.

Choose numbers V_1, V_2 and V_3 such that $V_1 + 2V_2 + V_3 = 1$ and

$$\begin{aligned} q_1 + k_2T^{-1} + (n-1)\varepsilon_1 &< V_1 + 1 < v_1 + \lambda_1 + 1,\\ r_1 + l_2T^{-1} + (n-1)\varepsilon_1 &< V_2 + 1 < v_2 + \lambda_2 + 1,\\ s_1 + m_2T^{-1} + (n-1)\varepsilon_1 &< V_3 < v_3 + \lambda_3. \end{aligned} \tag{17}$$

This is possible as follows. The inequalities above define a parallelpiped. Consider, the intersection of the parallelepiped with the planes given by the equations $V_1 + 2V_2 + V_3 = k$ as k varies. At the "top right" vertex $V_1 + 2V_2 + V_3 = n - 2 > 1$. At the "bottom left" vertex

$$\begin{aligned} V_1 + 2V_2 + V_3 &= q_1 + 2r_1 + s_2 + (k_2 + 2l_2 + m_2)T^{-1} + 4(n-1)\varepsilon_1 - 3\\ &= d_1 + d_2 + 4(n-1)\varepsilon_1 - 3 < 1 - \varepsilon/2 \end{aligned}$$

as $d_1 + d_2 < 4 - \varepsilon$. Thus, by continuity, the plane $V_1 + 2V_2 + V_3 = 1$ intersects the interior of the parallelepiped and we can choose the numbers V_1, V_2, V_3 from any of the points in this intersection.

Define another parallelepiped $\sigma_4(P)$ to be the set of points $(x, z, w) \in \mathbf{T} \cap S_1(\alpha_1) \times S_2(\beta_1) \times S_3(\gamma_1)$ satisfying the inequalities

$$\begin{aligned} |x-\alpha_1| &< 2^{-tV_1}|P'(\alpha_1)|^{-1}, \\ |z-\beta_1| &< 2^{-tV_2}|P'(\beta_1)|^{-1}, \\ |w-\gamma_1|_p &< 2^{-tV_3}|P'(\gamma_1)|_p^{-1}. \end{aligned} \tag{18}$$

Clearly, $\sigma_2(P) \subset \sigma_4(P)$. The polynomial P is now developed as a Taylor series in $\sigma_4(P)$ and each term estimated from above. This will be demonstrated for the complex coordinate. From (17), (18), (11), (8), Lemma 3 and Lemma 4

$$\begin{aligned} |P'(\beta_1)||z-\beta_1| &\ll 2^{-tV_2}, \\ |P''(\beta_1)||z-\beta_1|^2 &\ll 2^{t(1-r_2+(n-2)\varepsilon_1-2V_2-2+2r_1)} \ll 2^{t(r_1+l_2T^{-1}+(n-2)\varepsilon_1-2V_2-1)} \ll 2^{-tV_2}, \\ |P^{(j)}(\beta_1)||z-\beta_1|^{(j)} &\ll 2^{t(1-r_j+(n-j)\varepsilon_1-jV_2-j+jr_1)} \ll 2^{-tV_2},\ 3 \leqslant j \leqslant n. \end{aligned}$$

It is easy to do the same for $|P(x)|$ and $|P(w)|_p$ so that

$$\begin{aligned} |P(x)| &\ll 2^{-tV_1}, \\ |P(z)| &\ll 2^{-tV_2}, \\ |P(w)|_p &\ll 2^{-tV_3}. \end{aligned} \tag{19}$$

We similarly estimate $P'(x) = \sum_{i=1}^{n}(i!)^{-1}P^{(i)}(\alpha_1)(x-\alpha_1)^{i-1}$ on $\sigma_4(P)$. As before, each term is considered separately using Lemmas 3 and 4, (8) and (17) to obtain

$$\begin{aligned} |P'(\alpha_1)| &\ll 2^{-t(1-q_1+(n-1)\varepsilon_1)}, \\ |P^{(i)}(\alpha_1)||x-\alpha_1|^{i-1} &\ll 2^{t(1-q_i+(n-i)\varepsilon_1-(i-1)V_1-(i-1)(1-q_1))} \\ &\ll 2^{t(1-q_1+(n-1)\varepsilon_1)},\ 2 \leqslant i \leqslant n. \end{aligned}$$

From this and similar inequalities for $P'(z)$ the following inequalities hold on $\sigma_4(P)$.

$$\begin{aligned} |P'(x)| &\ll 2^{t(1-q_1+(n-1)\varepsilon_1)}, \\ |P'(z)| &\ll 2^{t(1-r_1+(n-1)\varepsilon_1)}. \end{aligned} \tag{20}$$

If both $q_1 < \varepsilon/2$ and $r_1 < \varepsilon/2$ then the proof is as in Proposition 2. Therefore, we will assume that $\max(q_1, r_1) \geqslant \varepsilon/2$. Let this maximum be q_1 so that from now on it is assumed that $q_1 \geqslant \varepsilon/2$. Fix the vector $\mathbf{d} = (a_4, a_5, \ldots, a_n)$, $|a_j| \leqslant 2^{t+1}$ and let $\mathcal{P}^t_{3,\mathbf{d}}$ denote the set of polynomials $P \in \mathcal{P}^t_3$ with the same vector $\mathbf{d}$. Now, Sprindzuk's method of essential and inessential domains is used, see [**11**] for details. The parallelepiped $\sigma_4(P_1)$ is called *essential* if for all polynomials $P_2 \in \mathcal{P}^t_{3,\mathbf{d}}$, $P_2 \neq P_1$,

$$\mu(\sigma_4(P_1) \cap \sigma_4(P_2)) < \frac{1}{2}\mu(\sigma_4(P_1)).$$

If, on the other hand, there exists $P_2 \in \mathcal{P}^t_{3,\mathbf{d}}$, $P_2 \neq P_1$, such that

$$\mu(\sigma_4(P_1) \cap \sigma_4(P_2)) \geqslant \frac{1}{2}\mu(\sigma_4(P_1)),$$

then the parallelepiped $\sigma_4(P_1)$ is called *inessential*. If $\mathbf{u}$ lies in infinitely many parallelpipeds $\sigma_2(P)$ then it lies in infinitely many essential or inessential parallelepipeds $\sigma_4(P)$. Denote the set $P \in \mathcal{P}^t_{3,\mathbf{d}}$ such that $\sigma_4(P)$ is essential by $\mathcal{E}^t_{3,\mathbf{d}}$ and the set of $P \in \mathcal{P}^t_{3,\mathbf{d}}$ for which $\sigma_4(P)$ is inessential by $\mathcal{I}^t_{3,\mathbf{d}}$.

First, assume that $P \in \mathcal{E}^t_{3,\mathbf{d}}$. Then, $\sum_{P_1 \in \mathcal{E}^t_{3,\mathbf{d}}} \mu(\sigma_4(P_1)) \ll \mu(T)$. Also, from (15) and (18),

$$\mu(\sigma_2(P_1)) \ll \mu(\sigma_4(P_1))2^{t(-v_1-2v_2-v_3+V_1+2V_2+V_3)}\Psi(2^t) = \mu(\sigma_4(P_1))2^{t(-n+4)}\Psi(2^t).$$

From this and the fact that the number of classes $\mathcal{P}^t_{3,\mathbf{d}}$ is at most $c(n)2^{t(n-3)}$ we have

$$\sum_{t=1}^{\infty}\sum_{\mathbf{d}}\sum_{P_1 \in \mathcal{E}^t_{3,\mathbf{d}}} \mu(\sigma_2(P_1)) \ll \sum_{t=1}^{\infty} 2^t\Psi(2^t)\mu(T) < \infty$$

by Lemma 6. Thus, by the Borel–Cantelli Lemma the set of points lying in infinitely many $\sigma_2(P)$ with $P \in \mathcal{E}^t_{3,\mathbf{d}}$ has measure zero.

Now, assume that $P_1 \in \mathcal{I}^t_{3,\mathbf{d}}$ so that there exists $P_2 \in \mathcal{P}^t_{3,\mathbf{d}}$ such that

$$\sigma_4(P_1, P_2) = \sigma_4(P_1) \bigcap \sigma_4(P_2), \quad \text{and} \quad \mu(\sigma_4(P_1, P_2)) \geqslant \frac{1}{2}\mu(\sigma_4(P_1)).$$

The systems of inequalities (19) and (20) hold simultaneously on $\sigma_4(P_1, P_2)$ for both P_1 and P_2. Hence, if $R(f) = P_2(f) - P_1(f) = b_3 f^3 + b_2 f^2 + b_1 f + b_0$ then R satisfies

$$\begin{aligned} |R(x)| &\ll 2^{-tV_1}, \\ |R(z)| &\ll 2^{-tV_2}, \\ |R(w)|_p &\ll 2^{-tV_3}, \\ |R'(x)| &\ll 2^{t(1-q_1+(n-1)\varepsilon_1)}, \\ |R'(z)| &\ll 2^{t(1-r_1+(n-1)\varepsilon_1)}, \end{aligned} \tag{21}$$

with $q_1 \geqslant \varepsilon/2$. If θ_1, θ_2 and θ_3 are the complex roots of R then

$$R(f) = b_3(f - \theta_1)(f - \theta_2)(f - \theta_3),$$

and

$$R'(\theta_1) = b_3(\theta_1 - \theta_2)(\theta_1 - \theta_3).$$

From (21) it follows that one root is real, and the other two are complex conjugates. Let $\theta_1 \in \mathbb{R}$, $\theta_3 = \bar{\theta}_2$ and assume that $|b_3| \asymp H(R)$ (by making the reduction to leading polynomials as in Section 2·1 if necessary). By (9) the value of $|\theta_1 - \theta_2|$ cannot get close to zero. Thus, the roots θ_1, θ_2, and $\bar{\theta}_2$ satisfy the inequality $|\theta_1 - \theta_2| = |\theta_1 - \bar{\theta}_2| > c_2(\delta_1)$ for some constant $c_2(\delta_1)$, and

$$|R'(\theta_1)| > c_2(\delta_1)H(R).$$

This, together with (21) and Lemma 3 implies that

$$|x - \theta_1| \ll 2^{-tV_1}H^{-1}(R)$$

for $x \in \sigma_4(P_1, P_2)$. From (18) the inequality $|R(x)| \ll 2^{-tV_1}$ holds on an interval of length $c(n)2^{-tV_1}|P'(\alpha_1)|^{-1}$. From this and (11) it follows that $2^{-tV_1}H^{-1}(R) \gg 2^{-t(V_1+1-q_1)}$ which further implies that $H(R) < 2^{t(1-q_1)}$. Passing from 2^t to $H(R)$ in (21) gives that $|R(x)||R(z)|^2|R(w)|_p \ll H(R)^{-1/(1-q_1)} \ll H(R)^{-v}$ with $v > 1$ since $q_1 > \varepsilon/2$. Thus, Lemma 5 can be used to show that the set of points $\mathbf{u}$ lying in infinitely many inessential parallelepipeds has zero measure. Together with the result for the essential parallelepipeds this is enough to prove the proposition.

PROPOSITION 4. *Assume that $\sum_{H=1}^{\infty}\Psi(H) < \infty$. The set of points $(x, z, w) \in \mathbf{T}$ for which the system of inequalities* (2) *is satisfied for infinitely many polynomials $P \in \mathcal{P}_n^{(0,0,0)}$ with*

$$4-\varepsilon \leqslant d_1+d_2 \leqslant n+\varepsilon \tag{22}$$

has measure zero.

Proof. This is the longest of the propositions and many of the results in the other linearity cases use methods from this proposition.

Instead of system (2) we use system (3) and we follow the proof of Propostion 1 until system (14). Assume that $P \in \mathcal{P}_n^{(0,0,0)}$ with $2^t \leqslant H < 2^{t+1}$ and $4-\varepsilon \leqslant d_1+d_2 \leqslant n+\varepsilon$. Denote this set by $\mathcal{P}_4^t$. Let $A_t = \bigcup_{P\in\mathcal{P}_4^t}\sigma(P)$ where $\sigma(P)$ is as defined in (12).

Let $u = n+1-d_1-d_2$ and fix $\theta = u-\varepsilon_2$ with $\varepsilon_2 > 0$ sufficiently small. Assume that there are at most $2^{t\theta}$ polynomials belonging to each parallelepiped M. Then, by Lemma 3, the measure of A_t is at most the measure of the parallelepiped $\sigma(P)$ multiplied by the number of parallelepipeds M and $2^{t\theta}$, that is

$$\mu(A_t) \ll 2^{-t(v_1+2v_2+v_3+\lambda_1+2\lambda_2+\lambda_3+3-d_1-d_2-\theta)} \ll 2^{-t(n+1-d_1-d_2-\theta)} \ll 2^{-t\varepsilon_2}.$$

Then, $\sum_{t=1}^{\infty}\mu(A_t) \ll \sum_{t=1}^{\infty}2^{-t(n+1-d_1-d_2-\theta)} < \infty$. Therefore the measure of the set of points lying in infinitely many sets A_t is zero by the Borel–Cantelli Lemma.

From now on, we assume that there exists a parallelpiped M with at least $2^{t\theta}$ polynomials belonging to it. From (22), $1-\varepsilon \leqslant u \leqslant n-3+\varepsilon$. Let $u_1 = u-d$ where $d = 0.23$. Writing u_1 as a sum of integer and fractional parts $[u_1]+\{u_1\}$ calculate

$$n-[u_1] = d_1+d_2-1+\{u_1\}+d > 3. \tag{23}$$

According to the Dirichlet box principle, there are $k \geqslant c(n)2^{t(d+\{u_1\}-\varepsilon_2)}$ polynomials $P_1, \ldots, P_k$ among these $2^{t\theta}$ polynomials whose first $[u_1]$ highest coefficients are the same. Consider the $k-1$ polynomials $R_j(f) = P_j(f) - P_1(f)$ for $2 \leqslant j \leqslant k$. It can be readily verified from (14) that

$$\begin{aligned} |R_j(x)| &\ll 2^{t(1-q_1-k_2T^{-1}+(n-1)\varepsilon_1)},\\ |R_j(z)| &\ll 2^{t(1-r_1-l_2T^{-1}+(n-1)\varepsilon_1)},\\ |R_j(w)|_p &\ll 2^{t(-s_1-m_2T^{-1}+(n-1)\varepsilon_1)}, \end{aligned} \tag{24}$$

with $2 \leqslant j \leqslant k$, $\deg R_j \leqslant n-[u_1]$ and $H(R) \leqslant 2^{t+2}$. The polynomials $R_j(f) = b_{n-[u_1]}f^{n-[u_1]} + \cdots + b_1 f + b_0$ are now divided into sets. In each set the values of the coefficients $b_{n-[u_1]}, \ldots, b_1$ lie in an interval of length $2^{t(1-h_1)}$ where $h_1 = \{u_1\}(n-[u_1])^{-1}$, obviously there are 2^{th_1} intervals for each coefficient. Again apply Dirichlet's box principle to obtain that there are $m \geqslant 2^{t(d-\varepsilon_2)}$ polynomials R_j in one such set. These will be renumbered $R_1, \ldots, R_m$. Again, consider the differences of these polynomials and define $S_i(f) = R_{i+1}(f) - R_1(f)$, which satisfy

$$\begin{aligned} |S_i(x)| &\ll 2^{t(1-q_1-k_2T^{-1}+(n-1)\varepsilon_1)},\\ |S_i(z)| &\ll 2^{t(1-r_1-l_2T^{-1}+(n-1)\varepsilon_1)},\\ |S_i(w)|_p &\ll 2^{t(-s_1-m_2T^{-1}+(n-1)\varepsilon_1)}, \end{aligned} \tag{25}$$

with $1 \leqslant i \leqslant m-1$, $\deg S_i \leqslant n-[u_1]$, and $H(S_i) \ll 2^{t(1-h_1)}$. It follows automatically from (25) that the constant coefficient of each S_i will take values $\ll 2^{t(1-h_1)}$.

The polynomials S_i are now examined closely. There are three possibilities to consider. These three possibilities will also appear further on in the proof of this proposition and again in Propositions 6 and 7. In each case the arguments will be the same.

Case A. All the polynomials S_i have the form $i_1 S_0, i_2 S_0, \ldots, i_{m-1} S_0$ for some fixed polynomial S_0. Then, in this case, $i' = \max_{1 \leqslant j \leqslant m-1} |i_j| \gg 2^{t(d-\varepsilon_2)}$ and (25) holds for $i' S_0$ with $H(S_0) \ll 2^{t(1-h_1-d+\varepsilon_2)}$. By (25),

$$|S_0(x)||S_0(z)|^2|S_0(w)|_p \ll 2^{t(3-d_1-d_2-3d+4(n-1)\varepsilon_1)}. \tag{26}$$

From (22) and (23) we have that

$$d_1 + d_2 - 3 + 3d - 4(n-1)\varepsilon_1 > (n - [u_1] - 2)(1 - h_1 - d + \varepsilon_2).$$

Thus, by Lemma 1, the set of points $\mathbf{u}$ which satisfy (25) for infinitely many such polynomials S has measure zero.

Case B. One of the polynomials S_i, $1 \leqslant i \leqslant m-1$ (say, S_1), is reducible, i.e. $S_1 = S_1^{(1)} S_1^{(2)}$. From system (25) we obtain that

$$|S_1(x)||S_1(z)|^2|S_1(w)|_p \ll 2^{t(3-d_1-d_2+4(n-1)\varepsilon_1)}.$$

Note that $H(S_1) \asymp H(S_1^{(1)})H(S_1^{(2)})$. Then, for either $S_1^{(1)}$ or $S_2^{(2)}$ the inequality

$$|S_1^{(i)}(x)||S_1^{(i)}(z)|^2|S_1^{(i)}(w)|_p \ll H(S_1^{(i)})^{3-d_1-d_2+4(n-1)\varepsilon_1}$$

holds and $\deg S_1^{(i)}(f) \leqslant n - [u_1] - 1$. It is not difficult to show that

$$d_1 + d_2 - 3 - 4(n-1)\varepsilon_1 > (n - [u_1] - 3)(1 - h_1) \tag{27}$$

holds for $d = 0.23$ and ε_2, ε_1 sufficiently small. So, again by Lemma 1, the set of points which satisfy (25) for infinitely many such polynomials S has measure zero.

Case C. All of the S_i are irreducible and there are at least two polynomials, S_1 and S_2 say, which have no common roots. The aim here is to obtain a contradiction to Lemma 5. To this end let $h = 1 - h_1$, pass to the height of the polynomials S_i in (25) and (13) and define

$$\tau_1 = (q_1 + k_2 T^{-1} - 1 - (n-1)\varepsilon_1)h^{-1},\ \eta_1 = k_2 T^{-1} h^{-1},$$
$$\tau_2 = (r_1 + l_2 T^{-1} - 1 - (n-1)\varepsilon_1)h^{-1},\ \eta_2 = l_2 T^{-1} h^{-1},$$
$$\tau_3 = (s_1 + m_2 T^{-1} - (n-1)\varepsilon_1)h^{-1},\ \eta_3 = m_2 T^{-1} h^{-1}.$$

By Lemma 5, the inequality

$$3q_1 + k_2 T^{-1} + 6r_1 + 2l_2 T^{-1} + 3s_1 + m_2 T^{-1} - 12(n-1)\varepsilon_1 - 9h_1 < 2(n - [u_1])h + \delta$$

must hold. As $q_1 \geqslant k_2 T^{-1}$, $2r_1 \geqslant 2l_2 T^{-1}$ and $s_1 \geqslant m_2 T^{-1}$ this implies, using (23) that

$$2(d_1 + d_2) - 12(n-1)\varepsilon_1 - \frac{9\{u_1\}}{n - [u_1]} < 2(d_1 + d_2) - 2 + 2d + \delta.$$

This is a contradiction when $d = 0.23$, $n - [u_1] \geqslant 6$ and δ and ε_1 are sufficiently small. Hence, the set of (x, z, w) for which the inequalities hold for infinitely many such polynomials S_i with $n - [u_1] \geqslant 6$ is empty.

It remains to prove the result when $n - [u_1] = 4$ or 5. Let $p = n - [u_1]$. We return to the polynomials R_j satisfying (24). The first inequality of system (24) holds for any polynomial R_j on the interval I_M where $M = I_M \times K_M \times D_M$. As $R_j = P_j - P_1$ we develop the derivatives $P_j^{(i)}(x)$, for each of the polynomials P_j, $j = 1, \ldots, k$, as Taylor series on I_M.

Let α_{1j} denote an appropriate root of P_j. We have,

$$P_j^{(i)}(x) = P_j^{(i)}(\alpha_{1j}) + P_j^{(i+1)}(\alpha_{1j})(x-\alpha_{1j}) + \frac{1}{2}P_j^{(i+2)}(\alpha_{1j})(x-\alpha_{1j})^2 + \cdots$$

and, by Lemma 4

$$|P_j^{(i)}(\alpha_{1j})| \ll 2^{t(1-q_i+(n-i)\varepsilon_1)},$$
$$|P_j^{(i+i_1)}(\alpha_{1j})||x-\alpha_{1j}|^{i_1} \ll 2^{t(1-q_{i+i_1}+(n-i-i_1)\varepsilon_1-i_1k_2T^{-1})} \ll 2^{t(1-q_i+(n-i-1)\varepsilon_1)},$$

for $2 \leqslant i_1 \leqslant p-i$, which implies that

$$|P_j^{(i)}(x)| \ll 2^{t(1-q_i+(n-1)\varepsilon_1)},\ 1 \leqslant i \leqslant p, 1 \leqslant j \leqslant k$$

on I_M. Clearly, this also implies that

$$|R_j^{(i)}(x)| \ll 2^{t(1-q_i+(n-1)\varepsilon_1)},\ 1 \leqslant i \leqslant p,$$

on I_M.

Let x_0 denote the centre of I_M. Each of the ranges of R_j and its derivatives at the point x_0 are divided into 2^{tv} intervals with $v = \{u_1\}(p+1)^{-1}$. This means that, from (24) the interval $[-c(n)2^{t(1-q_1-k_2T^{-1}+(n-1)\varepsilon_1)}, c(n)2^{t(1-q_1-k_2T^{-1}+(n-1)\varepsilon_1)}]$ is divided into 2^{tv} intervals of equal length $c(n)2^{t(1-q_1-k_2T^{-1}+(n-1)\varepsilon_1-v)}$, and the range of the lth derivative $(1 \leqslant l \leqslant p)$ namely $[-c(n)2^{t(1-q_l+(n-1)\varepsilon_1)}, c(n)2^{t(1-q_l+(n-1)\varepsilon_1)}]$ is divided into intervals of length $c(n)2^{t(1-q_l+(n-1)\varepsilon_1-v)}$. As a result there are at most $c(n)2^{t(p+1)v}$ different combinations of smaller intervals and, using Dirichlet's box principle (since $(p+1)v = \{u_1\}$) there exist at least $2^{t(d-\varepsilon_2)}$ polynomials R_j, belonging to some fixed combination of intervals.

It is clear that for any point $x \in I_M$, the polynomials $T_j(x) = R_{j+1}(x) - R_1(x)$ with R_{j+1} and R_1 from the same combination of intervals satisfy the inequalities

$$T_j(x_0) = |R_{j+1}(x_0) - R_1(x_0)| \ll 2^{t(1-q_1-k_2T^{-1}+(n-1)\varepsilon_1-\{u_1\}(p+1)^{-1})}$$
$$T_j^{(i)}(x_0) = |R_{j+1}^{(i)}(x_0) - R_1^{(i)}(x_0)| \ll 2^{t(1-q_i+(n-1)\varepsilon_1-\{u_1\}(p+1)^{-1})},$$

for $1 \leqslant i \leqslant p$. Develop the polynomials T_j as Taylor series on I_M at the point x_0 so that

$$T_j(x) = \sum_{i=0}^{p}(i!)^{-1}T_j^{(i)}(x_0)(x-x_0)^i.$$

Using the above estimates

$$|T_j^{(i)}(x)||x-x_0|^i \ll 2^{t(1-q_i-ik_2T^{-1}+(n-1)\varepsilon_1-\{u_1\}(p+1)^{-1})}$$
$$\ll 2^{t(1-q_1-k_2T^{-1}+(n-1)\varepsilon_1-\{u_1\}(p+1)^{-1})},$$

from (8). This further implies that

$$|T_j(x)| \ll 2^{t(1-q_1-k_2T^{-1}+(n-1)\varepsilon_1-\{u_1\}(p+1)^{-1})} \tag{28}$$

for $1 \leqslant j \leqslant m-1$, and $x \in I_M$.

As earlier in this proposition there are the same three cases to consider (exactly as *Cases A, B* and *C*). Some of the details below are therefore omitted.

Case A. All the polynomials T_j have the form sT_0 for some T_0. Therefore, there exists s such that $|s| \gg 2^{t(d-\varepsilon_2)}$ (since there are $2^{t(d-\varepsilon_2)}$ polynomials T_j) so that $H(T_0) \leqslant 2^{t(1-d+\varepsilon_2)}$

and the system of inequalities

$$\begin{aligned}|T_0(x)| &\ll H(T_0)^{(1-q_1-k_2T^{-1}-d+(n-1)\varepsilon_1-\{u_1\}(p+1)^{-1})(1-d+\varepsilon_2)^{-1}}\\ |T_0(z)| &\ll H(T_0)^{(1-r_1-l_2T^{-1}-d+(n-1)\varepsilon_1)(1-d+\varepsilon_2)^{-1}}\\ |T_0(w)|_p &\ll H(T_0)^{(-s_1-m_2T^{-1}+(n-1)\varepsilon_1)(1-d+\varepsilon_2)^{-1}}\end{aligned}$$

hold. The first one comes from (28) and the other two from (24). The inequality

$$\begin{aligned}d_1+d_2-3+\{u_1\}(p+1)^{-1}+3d-4(n-1)\varepsilon_1 &> (n-[u_1]-2)(1-d)\\ &= (d_1+d_2-3+d+\{u_1\})(1-d)\end{aligned}$$

holds for $n-[u_1] \leqslant 5$, $d = 0.23$, and ε, ε_1 sufficiently small. Therefore, from Lemma 1, the set of points which satisfy the above system for infinitely many such polynomials T has measure zero.

Case B. All the polynomials T_j are reducible. If there exists a factor $T_j^{(k)}$ of each T_j with degree $\leqslant n-[u_1]-2$ satisfying (by (24) and (28))

$$|T_j^{(k)}(x)||T_j^{(k)}(z)|^2|T_j^{(k)}(w)|_p \ll 2^{t(1-q_1-k_2T^{-1}-\{u_1\}(p+1)^{-1}+2(1-r_1-l_2T^{-1})-s_1-m_2T^{-1}+4(n-1)\varepsilon_1)}$$

then, as above, Lemma 1 can be applied immediately to $T_j^{(k)}$.

If, on the other hand, each of the T_j consist of a linear factor and a factor of degree $n-[u_1]-1$ proceed as follows. First note that if the linear factors are the same for two polynomials so that $T_1 = T_0T_1'$ and $T_2 = T_0T_2'$ then the polynomials T_1' and T_2' have no common roots and a contradiction to Lemma 5 may be obtained. Hence, we assume that all the linear factors are different so that there exists T_j with a linear factor of height at least $2^{t(\frac{d-\varepsilon_2}{2})}$, since the number of different polynomials T_j is greater than $2^{t(d-\varepsilon_2)}$. Note that since $|\mathrm{Im}\, z| > \delta_1$ we have $|az+b|^2 \gg a^2$. By splitting the range for the approximating index in the real variable into intervals of length ε and using a simple counting and covering argument to estimate the measures of the sets satisfying the appropriate approximations it can be readily verified that the set of (x, z, ω) which satisfy $|ax+b||az+b|^2|aw+b|_p \ll 2^{-t\varepsilon_1}$ has measure zero. Therefore, we may assume that for the linear polynomial $T_0(f) = af+b$ with $|a| > 2^{t(d-\varepsilon_2)/2}$, the inequality

$$|ax+b||az+b|^2|aw+b|_p \gg 2^{-t\epsilon_1}$$

holds for any ϵ_1. Let $T_j = T_0t_j$. Then the height of the polynomial t_j is at most $2^{t(1-(d-\varepsilon_2)/2)}$ and satisfies, by (24), (28) and the previous inequality,

$$|t_j(x)||t_j(z)|^2|t_j(w)|_p \ll H(t_j)^{(1-q_1-k_2T^{-1}-\{u_1\}(p+1)^{-1}+2(1-r_1-l_2T^{-1})-s_1-m_2T^{-1}+4n\varepsilon_1)(1-(d-\varepsilon_2)/2)^{-1}}.$$

For $p \leqslant 5$ and ε_1, ε_2 and ε sufficiently small we have that

$$d_1+d_2-3+\{u_1\}(p+1)^{-1}-4n\varepsilon_1 > (d_1+d_2-4+d+\{u_1\})(1-(d-\varepsilon_2)/2).$$

Thus, again by Lemma 1, the set of points for which infinitely many such T exist has measure zero.

Case C. There exists a pair of polynomials T_1 and T_2 with no common roots. The second and third inequalities of (24) remain the same and the first is replaced by (28). Define, $\tau_1 = q_1+k_2T^{-1}-1-(n-1)\varepsilon_1+\{u_1\}(p+1)^{-1}$, $\tau_2 = r_1+l_2T^{-1}-1-(n-1)\varepsilon_1$ and $\tau_3 = s_1+m_2T^{-1}-(n-1)\varepsilon_1$. Then, by Lemma 5, the inequality

$$3q_1+k_2T^{-1}+6r_1+2l_2T^{-1}+3s_1+m_2T^{-1}-12(n-1)\varepsilon_1+\frac{3\{u_1\}}{p+1} < 2(n-[u_1])+\delta$$

must hold. However, since $q_1 \geqslant k_2T^{-1}$, $r_1 \geqslant l_2T^{-1}$ and $s_1 \geqslant m_2T^{-1}$, there is a contradiction when $d = 0.23$, $p \leqslant 5$ and ε_1 and δ are sufficiently small. The proof of the proposition is complete.

These four propositions together imply that $\mu(L_n^{(0,0,0)}(\mathbf{v}, \lambda, \Psi)) = 0$.

Case 2. (1, 1, 1)–linearity. We assume that the system

$$\begin{aligned} q_1 + k_2T^{-1} &\geqslant 1 + v_1 + \lambda_1, \\ r_1 + l_2T^{-1} &\geqslant 1 + v_2 + \lambda_2, \\ s_1 + m_2T^{-1} &\geqslant v_3 + \lambda_3, \end{aligned} \tag{29}$$

holds together with system (3).

PROPOSITION 5. *If* $\sum_{H=1}^{\infty} \Psi(H) < \infty$ *then* $\mu(L_n^{(1,1,1)}(\mathbf{v}, \lambda, \Psi)) = 0$.

Proof. Using (3) and Lemma 3 we obtain

$$\begin{aligned} |x - \alpha_1| &\ll \min_{2\leqslant j\leqslant n} 2^{-t\left(\frac{v_1+\lambda_1+1-q_j}{j}\right)} = 2^{-t\mu_1}, \\ |z - \beta_1| &\ll \min_{2\leqslant j\leqslant n} 2^{-t\left(\frac{v_2+\lambda_2+1-r_j}{j}\right)} = 2^{-t\mu_2}, \\ |w - \gamma_1|_p &\ll \min_{2\leqslant j\leqslant n} 2^{-t\left(\frac{v_3+\lambda_3-s_j}{j}\right)} = 2^{-t\mu_3}. \end{aligned} \tag{30}$$

Note that from (29) it can be shown that $\mu_1 > v_1 + \lambda_1 + 1 - q_1$. Assume that the minimums in (30) are at j_1, j_2 and j_3 in the first, second and third inequality respectively and let $\sigma_5(P)$ be the parallelepiped defined by these inequalities. Define $\mathcal{P}_5^t$ to be the set of $P \in \mathcal{P}_n^{(1,1,1)}$ with $2^t \leqslant H(P) < 2^{t+1}$ and let $A_t = \bigcup_{P\in\mathcal{P}_5^t} \sigma_5(P)$.

Divide the parallelepiped $\mathbf{T}$ into smaller parallelepipeds M with sidelengths $2^{-t(\mu_1-\gamma)}$, $2^{-t\mu_2}$ and $2^{-t\mu_3}$ where $\gamma = (10n)^{-1}$. Assume that P belongs to M and develop it as a Taylor series on M. As before, obtain an upper bound for all the terms in the series. The estimates for the real coordinate are presented below. As usual we use Lemma 4.

$$\begin{aligned} |P'(\alpha_1)||x - \alpha_1| &\ll 2^{t\gamma}|P'(\alpha_1)2^{-t\mu_1}| \ll 2^{t(\gamma+1-q_1+(n-1)\varepsilon_1-v_1-\lambda_1-1+q_1)} \\ &\ll 2^{t(-v_1-\lambda_1+n\gamma+(n-1)\varepsilon_1)}, \\ |P^{(j)}(\alpha_1)||x - \alpha_1|^{(j)} &\ll 2^{jt\gamma}|P^{(j)}(\alpha_1)2^{-jt\mu_1}| \ll 2^{t(j\gamma+1-q_j+(n-j)\varepsilon_1-v_1-\lambda_1-1+q_j)} \\ &\ll 2^{t(-v_1-\lambda_1+n\gamma+(n-1)\varepsilon_1)}, \end{aligned}$$

for $2 \leqslant j \leqslant n$.

In exactly the same way estimate $|P(z)|$ and $|P(w)|_p$ to obtain

$$\begin{aligned} |P(x)| &\ll 2^{-t(v_1+\lambda_1-0.1-(n-1)\varepsilon_1)}, \\ |P(z)| &\ll 2^{-t(v_2+\lambda_2-(n-1)\varepsilon_1)}, \\ |P(w)|_p &\ll 2^{-t(v_3+\lambda_3-(n-1)\varepsilon_1)}. \end{aligned} \tag{31}$$

First assume that there exists a parallelepiped M to which at least two polynomials P_1 and P_2 belong (remember that we may assume P_1 and P_2 are irreducible). For these polynomials the system of inequalities (31) holds and they have no common roots. We intend to find a contradiction to Lemma 5. To this end define

$$\tau_1 = v_1+\lambda_1-0.1-(n-1)\varepsilon_1,\ \ \eta_1=\frac{v_1+\lambda_1+1-q_{j_1}}{j_1}-\gamma,$$
$$\tau_2 = v_2+\lambda_2-(n-1)\varepsilon_1,\ \ \eta_2=\frac{v_2+\lambda_2+1-r_{j_2}}{j_2},$$
$$\tau_3 = v_3+\lambda_3-(n-1)\varepsilon_1,\ \ \eta_3=\frac{v_3+\lambda_3-s_{j_3}}{j_3}.$$

Then, by Lemma 5, putting the denominators of η_i to be 2, which is the worst case,

$$2v_1+2\lambda_1-0.3+2\gamma+4v_2+4\lambda_1+2v_3+2\lambda_3-12(n-1)\varepsilon_1+6+(q_{j_1}+2r_{j_2}+s_{j_3})-3<2n+\delta$$

so that

$$\delta>2\gamma+0.7-12(n-1)\varepsilon_1+(q_{j_1}+2r_{j_2}+s_{j_3}).$$

Clearly for small δ and sufficiently small ε_1 this is untrue. Thus, there exists no parallelepiped M to which at least two irreducible polynomials belong.

Hence, we may assume that at most one polynomial $P\in\mathcal{P}_5^t$ belongs to each parallelepiped M. The number of such parallelepipeds is $c(n)2^{t(\mu_1+2\mu_2+\mu_3-\gamma)}$. Then, using (30),

$$\mu(A_t)\ll 2^{-t(\mu_1+2\mu_2+\mu_3-\mu_1-2\mu_2-\mu_3+\gamma)}\ll 2^{-t\gamma}.$$

Since $L_n^{(1,1,1)}(\mathbf{v},\lambda,\Psi)$ is the set of points lying in infinitely many A_t and $\sum_{t=0}^{\infty}\mu(A_t)\ll\sum_{t=1}^{\infty}2^{-t\gamma}<\infty$ the Borel–Cantelli Lemma may again be invoked and is enough to complete the proof.

Case 3. $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$–linearity.

Only the $(1,0,0)$–linearity case will be proved. The other two cases are exactly the same.

PROPOSITION 6. *If* $\sum_{H=1}^{\infty}\Psi(H)<\infty$ *then* $\mu(L_n^{(1,0,0)}(\mathbf{v},\lambda,\Psi))=0$.

Proof. We assume that (3) and the system of inequalities (from $(1,0,0)$–linearity)

$$\begin{aligned} q_1+k_2T^{-1} &\geqslant 1+v_1+\lambda_1,\\ r_1+l_2T^{-1} &< 1+v_2+\lambda_2,\\ s_1+m_2T^{-1} &< v_3+\lambda_3,\end{aligned}\tag{32}$$

hold.

First assume that we can replace the last two inequalities in (32) by

$$\begin{aligned} 0.9+v_2+\lambda_2<r_1+l_2T^{-1}<1+v_2+\lambda_2,\\ -0.1+v_3+\lambda_3<s_1+m_2T^{-1}<v_3+\lambda_3.\end{aligned}\tag{33}$$

Now follow Propostion 5; thus, as usual divide the parallelepiped $\mathbf{T}$ into smaller parallelepipeds M with sidelengths $2^{-t\mu_1}$, $2^{-t(l_2T^{-1}-\varepsilon_1)}$ and $2^{-t(m_2T^{-1}-\varepsilon_1)}$, where $\mu_1=\max_{2\leqslant j\leqslant n}(v_1+\lambda_1+1-q_j)j^{-1}$ and assume that this maximum is reached at $j=j_1$.

Assume that there exists at least one parallelepiped to which at least two polynomials belong, develop these polynomials as Taylor series on M, and estimate from above all the terms in the decomposition. Since the polynomials are irreducible and they do not have common roots we can apply Lemma 5. By (33) a contradiction is obtained exactly as in Propostion 5.

Thus, only the case when at most one polynomial belongs to each parallelepiped M needs to be considered. Let the set of $P\in\mathcal{P}_n^{(1,0,0)}$ with $2^t\leqslant H(P)<2^{t+1}$ which satisfy (32) and (33) be denoted by $\mathcal{P}_6^t$ and denote by $\sigma(P)$ the set of $\mathbf{u}$ for which (3) holds. Define $A_t = \bigcup_{P\in\mathcal{P}_6^t}\sigma(P)$. For a fixed P, by Lemmas 3 and 4, the measure of the set of points which satisfy (3) is at most $c(n)2^{t(-\mu_1-(2v_2+2\lambda_2+2-2r_1)-(v_3+\lambda_3-s_1))}$. The number of parallelepipeds M is at most $2^{t(\mu_1+(2l_2+m_2)T^{-1}-3\varepsilon_1)}$. Hence, from this and (32)

$$\mu(A_t) \ll 2^{-t(2v_2+2\lambda_2+2-2(r_1+l_2T^{-1})+v_3+\lambda_3-(s_1+m_2T^{-1})+3\varepsilon_1)} \ll 2^{-3\varepsilon_1 t}.$$

Thus, as the series $\sum_{t=1}^{\infty}\mu(A_t) \ll \sum_{t=1}^{\infty} 2^{-3\varepsilon_1 t} < \infty$, the set of points which satisfy (3), (32) and (33) infinitely often has measure zero by the Borel–Cantelli Lemma.

Now we will investigate the case where either both or one of the following inequalities hold:

$$\begin{aligned} r_1 + l_2T^{-1} &\leqslant 0.9 + v_2 + \lambda_2,\\ s_1 + m_2T^{-1} &\leqslant -0.1 + v_3 + \lambda_3. \end{aligned} \tag{34}$$

The two cases are similar so only the case where both of the inequalities above hold will be demonstrated. Let $\mathcal{P}_7^t$ denote the set of polynomials in $\mathcal{P}_n^{(1,0,0)}$ with $2^t \leqslant H(P) < 2^{t+1}$ for which (32) and (34) hold. Divide the parallelepiped $\mathbf{T}$ into smaller parallelepipeds M with sidelengths $2^{-t\mu_1}$, $2^{-tl_2T^{-1}}$ and $2^{-tm_2T^{-1}}$. Fix $u = n - v_1 - \lambda_1 - (2r_1 + s_1) - (2l_2 + m_2)T^{-1}$ and let $\theta = u - \varepsilon_2$ for some ε_2 sufficiently small. Assume that at most $2^{t\theta}$ polynomials belong to each M. Let $A_t = \bigcup_{P\in\mathcal{P}_7^t}\sigma(P)$. Then

$$\begin{aligned} \mu(A_t) &\ll 2^{-t(\mu_1+2(v_2+\lambda_2+1-r_1)+(v_3+\lambda_3-s_1)-\mu_1-2l_2T^{-1}-m_2T^{-1}-\theta)}\\ &\ll 2^{-t(u-\theta)} \ll 2^{-t\varepsilon_2}. \end{aligned}$$

Clearly, the series $\sum_{t=1}^{\infty} 2^{-t\varepsilon_2}$ converges and as usual the proof may be completed using the Borel–Cantelli Lemma.

Thus, we now assume that there exists a parallelepiped M to which at least $2^{t\theta}$ polynomials belong. Let $u = u_1 + d$ with $0 < d < 1$ so that

$$n - [u_1] = n - u + \{u_1\} + d = v_1 + \lambda_1 + d_1' + d_2' + \{u_1\} + d$$

and

$$n - u_1 = v_1 + \lambda_1 + d_1' + d_2' + d,$$

where $d_1' = 2r_1 + s_1$ and $d_2' = (2l_2 + m_2)T^{-1}$. (We used the fact that $n - 2 = v_1 + 2v_2 + v_3 + \lambda_1 + 2\lambda_2 + \lambda_3$.) Using Taylor's formula and (32)

$$|P(x)| \ll 2^{-t(v_1+\lambda_1-(n-1)\varepsilon_1)}.$$

Replacing the first inequality in (14) by this we have

$$\begin{aligned} |P(x)| &\ll 2^{-t(v_1+\lambda_1-(n-1)\varepsilon_1)},\\ |P(z)| &\ll 2^{-t(r_1+l_2T^{-1}-1-(n-1)\varepsilon_1)},\\ |P(w)|_p &\ll 2^{-t(s_1+m_2T^{-1}-(n-1)\varepsilon_1)}. \end{aligned}$$

The rest of the proof exactly follows that of Propostion 4 with (14) replaced by this system. This is done briefly below. Consider the polynomials $R_j(f) = P_j(f) - P_1(f)$ for $2 \leqslant j \leqslant k$, $k \geqslant c(n)2^{t(\{u_1\}+d-\varepsilon_2)}$, whose first $[u_1]$ highest coefficients are the same. These R_j are then renumbered and the polynomials $S_i = R_{i+1} - R_1$ considered where each of the coefficients of R_i lies in an interval of length $2^{t(1-h_1)}$ where $h = \{u_1\}(n - [u_1])^{-1}$. Pass to the height of the polynomials S_i from the height of P and for $1 \leqslant i \leqslant m-1$, $m \geqslant 2^{t(d-\varepsilon_2)}$, the inequalities

$$\begin{aligned} |S_i(x)| &\ll 2^{-t(v_1+\lambda_1-(n-1)\varepsilon_1)}, \\ |S_i(z)| &\ll 2^{-t(r_1+l_2T^{-1}-1-(n-1)\varepsilon_1)}, \\ |S_i(w)|_p &\ll 2^{-t(s_1+m_2T^{-1}-(n-1)\varepsilon_1)}, \end{aligned} \tag{35}$$

hold on M with $\deg S_i \leqslant n-[u_1]$, $H(S_i) \ll 2^{t(1-h_1)}$. Exactly, as in Proposition 4, there are three possibilities.

Case A. Instead of inequality (26) we obtain

$$\begin{aligned} |S_0(x)||S_0(z)|^2|S_0(w)|_p &\ll 2^{t(-v_1-\lambda_1-2r_1-2l_2T^{-1}+2-s_1-m_2T^{-1}-3d+4(n-1)\varepsilon_1)} \\ &\ll 2^{-t(v_1+\lambda_1+d_1'+d_2'-2-4(n-1)\varepsilon_1+3d)}. \end{aligned}$$

Lemma 1 can be applied if the inequality

$$v_1+\lambda_1+d_1'+d_2'-2-4(n-1)\varepsilon_1+3d > (n-[u_1]-2)(1-d-h_1+\varepsilon_2)$$

holds. It is not difficult to show that for $n-[u_1] \geqslant 3$, $d = 0.23$ and ε_1, ε_2 sufficiently small that this is indeed the case. The fact that $n-[u_1] \geqslant 3$ follows from (35) since any polynomial satsifying (35) must have one real root and two complex roots.

Case B. If there exist reducible polynomials among the S_i then Lemma 1 can be applied if the inequality

$$v_1+\lambda_1+d_1'+d_2'-2-4(n-1)\varepsilon_1 > (n-[u_1]-3)(1-h_1)$$

holds. (This is similar to (27).) By Lemma 5, $n-[u_1]-1 \geqslant 3$ and the inequality above holds for $d = 0.23$ and ε_1 sufficiently small.

Case C. Finally, if there exist two polynomials S_1 and S_2 which have no common roots Lemma 5 can be applied with

$$\begin{aligned} \tau_1 &= (v_1+\lambda_1-(n-1)\varepsilon_1)h^{-1}, \quad \eta_1 = \mu_1 h^{-1}, \\ \tau_2 &= (r_1+l_2T^{-1}-(n-1)\varepsilon_1-1)h^{-1}, \quad \eta_2 = l_2T^{-1}h^{-1}, \\ \tau_3 &= (s_1+m_2T^{-1}-(n-1)\varepsilon_1)h^{-1}, \quad \eta_3 = m_2T^{-1}h^{-1}. \end{aligned}$$

These imply that the inequality

$$\begin{aligned} &2v_1+2\lambda_1+2+6r_1+2l_2T^{-1}+3s_1+m_2T^{-1}-12(n-1)\varepsilon_1+q_2(S)-\frac{9\{u_1\}}{n-[u_1]} \\ &\quad < 2(n-[u_1])\left(1-\frac{\{u_1\}}{n-[u_1]}\right)+\delta = 2(v_1+\lambda_1+d_1'+d_2'+d)+\delta \end{aligned}$$

holds (the worst case $j_1 = 2$ has been assumed). Exactly as in Proposition 4 we obtain the proof of the inequality for the case $n-[u_1] \geqslant 6$. When $n-[u_1] = 4$ or $n-[u_1] = 5$ the approximation is again strengthened for x and the proof is completed as in Proposition 4.

Case 4. $(1,1,0)$, $(1,0,1)$ and $(0,1,1)$–linearity.

These cases are all the same so only the case $(1,0,1)$–linearity will be demonstrated.

PROPOSITION 7. *If* $\sum_{H=1}^{\infty}\Psi(H) < \infty$ *then* $\mu(L_n^{(1,0,1)}(\mathbf{v},\lambda,\Psi)) = 0$.

Proof. If (1, 0, 1)–linearity holds then (3) and

$$\begin{aligned} q_1 + k_2T^{-1} &\geqslant 1 + v_1 + \lambda_1, \\ r_1 + l_2T^{-1} &< 1 + v_2 + \lambda_2, \\ s_1 + m_2T^{-1} &\geqslant v_3 + \lambda_3, \end{aligned} \tag{36}$$

also hold.

For now also assume the restriction

$$0.7 + v_2 + \lambda_2 < r_1 + l_2T^{-1}. \tag{37}$$

Define

$$\mu_1 = \max_{2\leqslant j\leqslant n} ((1 + v_1 + \lambda_1 - q_j)j^{-1})^{1/j}$$

and

$$\mu_3 = \max_{2\leqslant j\leqslant n} ((v_3 + \lambda_3 - s_j)j^{-1})^{1/j},$$

and assume that these maxima are reached at j_1 and j_3 respectively.

The proof of this propostion now follows that of Proposition 4, 5 or 6 with the appropriate changes. Let $\mathcal{P}_8^t$ be the set of $P \in \mathcal{P}_n^{(1,0,1)}$ with $2^t \leqslant H(P) < 2^{t+1}$ for which (36) and (37) hold. Let $A_t = \bigcup_{P\in\mathcal{P}_8^t}\sigma(P)$. Divide the parallelepiped $\mathbf{T}$ into smaller parallelepipeds M with sidelengths $2^{-t\mu_1}$, $2^{-t(l_2T^{-1}-\varepsilon_1)}$ and $2^{-t\mu_3}$.

First, following Proposition 5, assume that there exists a parallelepiped M to which at least two polynomials belong and develop these polynomials as Taylor series. Obtain an upper bound for each term in the decomposition. As the polynomials are irreducible and have no common roots we can apply Lemma 5 and by (37), a contradiction is obtained. Thus, we may assume that at most one polynomial belongs to each parallelepiped M. Then,

$$\mu(A_t) \ll 2^{-t(\mu_1+2v_2+2\lambda_2+2-2r_1+\mu_3-\mu_1-\mu_3-2l_2T^{-1}-2\varepsilon_1)} \ll 2^{-2t\varepsilon_1}.$$

Again, $\sum_{t=1}^{\infty} \mu(A_t) < \infty$ and the proof may be completed using the Borel–Cantelli Lemma.

To complete the proof we need to consider the case

$$r_1 + l_2T^{-1} \leqslant 0.7 + v_2 + \lambda_2. \tag{38}$$

Let $\mathcal{P}_9^t$ be the set of $P \in \mathcal{P}_n^{(1,0,1)}$ with $2^t \leqslant H(P) < 2^{t+1}$ which satisfy (36) and (38). Let $A_t = \bigcup_{P\in\mathcal{P}_9^t}\sigma(P)$.

Divide the parallelepiped $\mathbf{T}$ into smaller parallelepipeds M with sidelengths $2^{-t\mu_1}$, $2^{-tl_2T^{-1}}$ and $2^{-t\mu_3}$. Fix $u = 2(v_2 + \lambda_2 + 1 - r_1 - l_2T^{-1})$ and $\theta = u - \varepsilon_2$ with $\varepsilon_2 > 0$ sufficiently small. Assume that at most $2^{t\theta}$ polynomials belong to each parallelepiped M. Then,

$$\mu(A_t) \ll 2^{-t(\mu_1+\mu_3+2v_2+2\lambda_2+2-2r_1-\mu_1-\mu_3-2l_2T^{-1}+\theta)} \ll 2^{-t(u-\theta)} \ll 2^{-t\varepsilon_2},$$

The series $\sum_{t=1}^{\infty} \mu(A_t) < \infty$ so the set of points lying in infinitely many A_t has measure zero by the Borel–Cantelli Lemma.

Thus from now on we assume that there exists a parallelepiped M to which at least $2^{t\theta}$ polynomials belong. Let $u = u_1 + d$ with $0 < d < 1$ and assume that P belongs to M. Develop P as a Taylor series on M and estimate from above all the terms in the decomposition to obtain

$$|P(x)| \ll 2^{-t(v_1+\lambda_1-(n-1)\varepsilon_1)},$$
$$|P(z)| \ll 2^{-t(r_1+l_2T^{-1}-1-(n-1)\varepsilon_1)},$$
$$|P(w)|_p \ll 2^{-t(v_3+\lambda_3-(n-1)\varepsilon_1)}.$$

Again, we follow the proof of Propostion 6 using the above system instead of (14). From the polynomials P we shall pass to the polynomials $R_j = P_j - P_1$ for $2 \leqslant j \leqslant k$, $k \geqslant 2^{t(d+\{u_1\}-\varepsilon_2)}$, then renumber these R_j and further pass to polynomials $S_i = R_{i+1} - R_1$, with $1 \leqslant i \leqslant m-1$, $m \geqslant 2^{t(d-\varepsilon_2)}$, exactly as in (24) and (25). We obtain

$$|S_i(x)| \ll 2^{-t(v_1+\lambda_1-(n-1)\varepsilon_1)},$$
$$|S_i(z)| \ll 2^{-t(r_1+l_2T^{-1}-1-(n-1)\varepsilon_1)},$$
$$|S_i(w)|_p \ll 2^{-t(v_3+\lambda_3-(n-1)\varepsilon_1)},$$

where $\deg S_i \leqslant n-[u_1]$ and $H(S_i) \ll 2^{t(1-h_1)}$.

The usual three possibilities are considered.

Case A. First, we obtain the inequality

$$|S_0(x)||S_0(z)|^2|S_0(w)|_p \ll 2^{-t(v_1+v_3+\lambda_1+\lambda_3+2r_1+2l_2T^{-1})-2-4(n-1)\varepsilon_1)}$$

in the same way as (26) was obtained. As (27) was shown to hold it can similarly be shown that

$$v_1+v_3+\lambda_1+\lambda_3+2r_1+2l_2T^{-1}-2-4(n-1)\varepsilon_1+3d > (n-[u_1]-2)(1-d-h_1)$$

also holds for $d=0.23$ and sufficiently small ε_1, ε_2. Thus, Lemma 1 may be applied.

Case B. Now assume that there exist reducible polynomials among the S_i. Then,

$$v_1+v_3+\lambda_1+\lambda_3+2r_1+2l_2T^{-1}-2-4(n-1)\varepsilon_1 > (n-[u_1]-3)(1-h_1)$$

is true if $d=0.23$ and ε_1 is sufficiently small. Again, this is similar to showing that (27) holds; and again we apply Lemma 1.

Case C. Finally, apply Lemma 5 to two polynomials S_1 and S_2 with no common roots. Let

$$\tau_1 = (v_1+\lambda_1-(n-1)\varepsilon_1)h^{-1},\ \ \eta_1=\mu_1h^{-1},$$
$$\tau_2 = (r_1+l_2T^{-1}-(n-1)\varepsilon_1-1)h^{-1},\ \ \eta_2=l_2T^{-1}h^{-1},$$
$$\tau_3 = (v_3+\lambda_3-(n-1)\varepsilon_1)h^{-1},\ \ \eta_3=\mu_3h^{-1}.$$

Using Lemma 5, the inequality

$$2v_1+2\lambda_1+2v_3+2\lambda_3+2+6r_1+2l_2T^{-1}+q_2+s_2-12(n-1)\varepsilon_1-\frac{9\{u_1\}}{n-[u_1]}$$
$$< 2(v_1+\lambda_1+v_3+\lambda_3+2r_1+2l_2T^{-1}+d)+\delta$$

must hold and is weakest when $j_1=j_3=2$. This is a contradiction for $d=0.23$, $n-[u_1]\geqslant 6$ and sufficiently small δ and ε_1. As in Proposition 4 we obtain the proof of the inequality for $n-[u_1]=4$ and $n-[u_1]=5$ separately and in exactly the same manner. Proposition 7 is proved. Putting all the propositions together completes the proof of the theorem.

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