Indag. Mathem., N.S., 10 (3), 337–347 September 27, 1999

Hausdorff dimension and p-adic diophantine approximation

by H. Dickinson[1], M.M. Dodson[2] and J. Yuan[3]

[1]*Department of Mathematics, University of York, Heslington, York, YO1 5DD, England, e-mail: hd3@york.ac.uk*
[2]*Department of Mathematics, University of York, Heslington, York, YO1 5DD, England, e-mail: mmd1@york.ac.uk*
[3]*Department of Mathematics, Northwest University, 710069 Xi'an, Shaanxi, P.R. China, e-mail: ec@nwu.edu.cn*

Communicated by Prof R. Tijdeman at the meeting of September 21, 1998

1 INTRODUCTION

Analogues of the Jarník–Besicovitch theorem and its generalisations have been proved for the p-adic field $\mathbb{Q}_p$ and for $\mathbb{Q}_p^n$ (details are in [1] and [9]). The p-adic norm of a point $\xi = (\xi_1, \ldots, \xi_n)$ in $\mathbb{Q}_p^n$ will be written $|\xi|_p = \max_{1 \le j \le n} |\xi_j|_p$ and for any rational integer vector $\mathbf{q} = (q_1, \ldots, q_n)$, the usual supremum metric will be denoted by $|\mathbf{q}| = \max_i |q_i|$. We identify the set of $m \times n$ matrices on $\mathbb{Z}_p$ with $\mathbb{Z}_p^{mn}$ and consider the set

$$W(m, n; \tau) = \{X \in \mathbb{Z}_p^{mn} : |\mathbf{q}X|_p < |\mathbf{q}|^{-\tau} \text{ for infinitely many } \mathbf{q} \in \mathbb{Z}^m\},$$

where $\mathbf{q}X$ represents a system of n linear forms over $\mathbb{Z}_p$ in m variables. This set is not a direct analogue of the set in the general Jarník–Besicovitch theorem [4] which involves the distance from $\mathbb{Z}^n$ rather than the distance from the origin. Indeed $W(1, 1; \tau)$ reduces to $\{0\}$. If the p-adic distance from $\mathbb{Z}^n$ is taken, then the corresponding set has full measure since for each X, $|\mathbf{q}X - \mathbf{r}|$ can be made arbitrarily small by taking $\mathbf{r}$ to be a rational integer vector with the appropriate number of leading terms taken from the p-adic expansion of $\mathbf{q}X$. Hence the p-adic distance to the origin is considered instead; $W(m, n; \tau)$ is a p-adic analogue of the set $W_0(m, n; \tau)$ considered in [6].

The Hausdorff dimension of $W(2, 1; \tau)$ was obtained by Melnichuk in [9] for the case $\mathbf{x} = (x, 1)$ and this result was extended in two ways by Abercrombie [1]. The first extension was to a more general version of Melnichuk's result and in the second he showed that when $\tau \geq m/n$, the Hausdorff dimension of $W(m,n;\tau)$ was

$$\dim W(m,n;\tau) = (m-1)n + \frac{m}{\tau} \tag{1}$$

for $m > n$. Using ideas from [6] we will complete Abercrombie's result by obtaining the Hausdorff dimension of $W(m,n;\tau)$ for the case $m \leq n$.

Theorem 1.

$$\dim W(m,n;\tau) = \begin{cases} (m-1)n + \frac{m}{\tau} & \text{for } \tau \geq m/n, \\ mn & \text{otherwise.} \end{cases}$$

2 HAUSDORFF DIMENSION AND HAAR MEASURE

A ball $B(a;p^{-h})$ in $\mathbb{Z}_p$ with diameter $\operatorname{diam} B(a;p^{-h}) = p^{-h}$ is defined as

$$B(a;p^{-h}) = \{x \in \mathbb{Z}_p : |x-a|_p \leq p^{-h}\}$$

and a ball in $\mathbb{Z}_p^n$ is similarly defined as

$$B(\mathbf{a};p^{-h}) = \{\mathbf{x} \in \mathbb{Z}_p^n : |\mathbf{x}-\mathbf{a}|_p \leq p^{-h}\} = B(a_1;p^{-h}) \times \cdots \times B(a_n;p^{-h})$$

where $\mathbf{a} = (a_1, \ldots, a_n)$.

Let $\delta > 0$. A δ-cover of a set F in $\mathbb{Z}_p^n$ is a family of balls $B(\mathbf{a}_i;p^{-h_i})$ such that $F \subset \cup_i B(\mathbf{a}_i;p^{-h_i})$ and $p^{-h_i} \leq \delta$. Write $\mathcal{H}_\delta^s(F) = \inf \sum_i p^{-sh_i}$ where the infimum is taken over all δ-covers of F. The Hausdorff outer s-measure of F is defined to be

$$\mathcal{H}^s(F) = \sup_{\delta>0} \mathcal{H}_\delta^s(F)$$

and the Hausdorff dimension of F to be

$$\dim F = \sup\{s : \mathcal{H}^s(F) = \infty\} = \inf\{s : \mathcal{H}^s(F) = 0\}.$$

Let μ be the unique Haar measure on $\mathbb{Q}_p^n$ such that $\mu(\mathbb{Z}_p^n) = 1$ and let χ_B be the characteristic function of a ball B. Then

$$\int_{\mathbb{Z}_p^n} \chi_B(\mathbf{x})d\mu(\mathbf{x}) = \mu(B) = (\operatorname{diam}(B))^n.$$

A more general definition of Hausdorff dimension with respect to Haar measure is given in [1] but for p-adic space it is equivalent to the one above.

3 PROOF OF THEOREM 1

In order to obtain the Hausdorff dimension of $W(m,n;\tau)$ we will consider upper and lower bounds for the dimension separately.

Lemma 1.

$$\dim W(m,n;\tau) \leq \begin{cases} (m-1)n + \frac{m}{\tau} & \text{if } \tau \geq m/n, \\ mn & \text{otherwise.} \end{cases}$$

The first inequality is proved using a standard covering and counting argument and is Lemma 4.5 in [1]. The Hausdorff dimension of $\mathbb{Z}_p^{mn}$ is mn, whence $\dim W(m,n;\tau) \leq mn$.

For the case $m > n$, the lower bound has already been established in [1] and thus from now on we assume that $m \leq n$. To avoid excessive technical detail we first prove the result for $m = n$ and then sketch the extension to $n > m$. The main idea is to map $W(m,m;\tau)$ to the Cartesian product of two spaces, namely $\mathbb{Z}_p^{m-1}$ and the set $W(m,m-1;\tau)$ for which we already know the dimension by (1). Then it will be shown that the map between $W(m,m;\tau)$ and the Cartesian product is bi-Lipschitz. Using the lemmas below this gives the Hausdorff dimension.

Definition 1. A function $f : E \to F$ is (p-adically) bi-Lipschitz on E if there exist positive constants c and C such that

$$c|\mathbf{x}-\mathbf{y}|_p \leq |f(\mathbf{x})-f(\mathbf{y})|_p \leq C|\mathbf{x}-\mathbf{y}|_p.$$

We need two more lemmas:

Lemma 2. *Let $f : E \to F$ be a bi-Lipschitz function which is one-one and onto. Then*

$$\dim E = \dim F.$$

The proof of this lemma follows that for the real case which can be found in [7, page 30].

The next result is a p-adic analogue of a lemma which appears in [3] and [8]; the proof uses ideas from [3].

Lemma 3. *Let D be an r-dimensional ball, and E a set in $\mathbb{Z}_p^k$ with Hausdorff dimension d, then the Hausdorff dimension of the Cartesian product $E \times D$ is*

$$\dim(E \times D) = d + r.$$

Proof. The proof will be done for $r = 1$ and a simple induction argument extends it for general r.

Let the Hausdorff dimension of $E \times D$ be s and assume that $s < d + 1$. Then for each $\varepsilon > 0$ there exists a positive δ and a cover $\mathcal{C}$ of $E \times D$ by $k+1$-dimensional balls B_i, such that

$$(2) \qquad \sum_{i=1}^{\infty} (\operatorname{diam} B_i)^{d+1-\delta} < \varepsilon.$$

For each $\theta \in D$ there is a cover $\mathcal{C}(\theta) = \{\hat{B}_i(\theta) = B_i \cap (\mathbb{Z}_p^k \times \{\theta\}) : B_i \in \mathcal{C}\}$ of $E \times \{\theta\}$ by k-dimensional balls, obtained by taking the cross-section of $\mathcal{C}$ through θ. Define $\hat{B}_i$ by $\hat{B}_i(\theta) = \hat{B}_i \times \{\theta\}$. In what follows B_i will be a $k+1$-dimensional ball in the cover $\mathcal{C}$ and B_i^1 will be a ball in $\mathbb{Z}_p$ such that $B_i = \hat{B}_i \times B_i^1$. Let $\chi_i : \mathbb{Z}_p \to \mathbb{R}$ be defined by

$$\chi_i(\theta) = \begin{cases} 1 & \text{if } \theta \in B_i^1 \text{ i.e., if } B_i \cap (\mathbb{Z}_p^k \times \{\theta\}) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

so that

$$\int_{\mathbb{Z}_p} \chi_i(\theta) d\mu(\theta) = \text{diam } B_i^1 = \text{diam } B_i$$

and

$$\sum_{i=1}^{\infty} (\text{diam } \hat{B}_i(\theta))^{d-\delta} = \sum_{i=1}^{\infty} \chi_i(\theta)(\text{diam } B_i)^{d-\delta}.$$

Then, by the Monotone Convergence Theorem,

$$\int_{\mathbb{Z}_p} \sum_{i=1}^{\infty} (\text{diam } \hat{B}_i(\theta))^{d-\delta} d\mu(\theta) = \sum_{i=1}^{\infty} \int_{\mathbb{Z}_p} \chi_i(\theta) d\mu(\theta)(\text{diam } B_i)^{d-\delta}$$
$$= \sum_{i=1}^{\infty} (\text{diam } B_i)^{d+1-\delta} < \varepsilon.$$

Hence for some θ in D, $E \times \{\theta\}$ has a cover $\mathcal{C}(\theta)$ such that

$$\sum_{i=1}^{\infty} (\text{diam } \hat{B}_i(\theta))^{d-\delta} \leq \varepsilon.$$

Translating $\mathcal{C}(\theta)$ appropriately gives a cover of E with

$$\sum_{i=1}^{\infty} (\text{diam } \hat{B}_i)^{d-\delta} \leq \varepsilon$$

which, as ε is arbitrarily small, implies by definition that the Hausdorff dimension of E is strictly less than d. It follows from this contradiction that $s \geq d+1$.

Next, we prove that $s \leq d+1$. By definition, given positive δ and ε there exists a cover $\mathcal{C}$ of E of k-dimensional balls B_j with

$$\mathcal{H}_\delta^s(\mathcal{C}) = \sum_{j=1}^{\infty} (\text{diam } B_j)^{d+\delta} < \varepsilon/\mu(D)$$

with diam $B_j = p^{-t_j} < \delta$. For each B_j in $\mathcal{C}$, construct a cover $\mathcal{C}_j$ of D by $p^{t_j}\mu(D)$ one-dimensional balls B'_{j_i} of diameter p^{-t_j} and let

$$B_{ij} = B_j \times B'_{j_i}, \quad 1 \leq i \leq p^{t_j}\mu(D).$$

The collection

$$\mathcal{C}^* = \{B_{ij} : B_j \in \mathcal{C}, B'_{j_i} \in \mathcal{C}_j\}$$

covers $E \times D$ and hence

$$\sum_{B_{ij} \in \mathcal{C}^*} (\text{diam } B_{ij})^{d+1+\delta} \leq \mu(D) \sum_{B_j \in \mathcal{C}} (\text{diam } B_j)^{-1}(\text{diam } B_j)^{d+1+\delta}$$
$$= \mu(D) \sum_{B_j \in \mathcal{C}} (\text{diam } B_j)^{d+\delta} < \varepsilon,$$

whence $s \leq d+1$ and the lemma is proved. $\square$

3.1. Auxilliary results. It is only possible determine a lower bound for the case $m > n$ on a small ball in $\mathbb{Z}_p^{mn}$, but clearly this will also be a lower bound for $\mathbb{Z}_p^{mn}$. The method uses Abercrombie's result [1] for $\mathbb{Z}_p^{m(m-1)}$; he actually proves the result for any ball in $\mathbb{Z}_p^{m(m-1)}$. Without loss of generality we restrict ourselves to a ball for which all the points in the ball (defined in $\mathbb{Z}_p^{m(m-1)}$) have linearly independent columns. To show that such a ball exists, consider the following two lemmas.

Lemma 4. *Let A be a matrix in $\mathbb{Z}_p^{n^2}$ such that $|\det A|_p = p^{-t}$ for some fixed integer t. Then*

$$|\det\,(A+\varepsilon)|_p = |\det A|_p = p^{-t}$$

for any matrix $\varepsilon = (\varepsilon_{ij})$ with $|\varepsilon_{ij}|_p < p^{-t}$.

Proof. By expanding the determinant in the usual way it is readily verified that

$$\det(A+\varepsilon) = \det\begin{pmatrix} a_{11}+\varepsilon_{11} & a_{12}+\varepsilon_{12} & \dots & a_{1n}+\varepsilon_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n1}+\varepsilon_{n1} & a_{n2}+\varepsilon_{n2} & \dots & a_{nn}+\varepsilon_{nn} \end{pmatrix}$$
$$= \det A + R(\varepsilon),$$

where $R(\varepsilon)$ is the sum of the remaining terms of the determinant, in which each term contains an ε_{ij} for some i and j. Thus $|R(\varepsilon)|_p < p^{-t}$ whence $|\det(A+\varepsilon)|_p = |\det A|_p$. $\square$

For the rest of this paper, $\tilde{M}$ will denote a $k \times (k+r)$ matrix M with some r rows deleted. The next result states in essence that linear independence is an open property in $\mathbb{Z}_p^m$ and is a familiar fact for Euclidean space.

Lemma 5. *Let $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \dots, \mathbf{a}^{(m-1)}$ be a set of linearly independent vectors in $\mathbb{Z}_p^m$. Then the matrix*

$$A = (\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \dots, \mathbf{a}^{(m-1)}) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1,m-1} \\ a_{21} & a_{22} & \dots & a_{2,m-1} \\ \vdots & \vdots & & \vdots \\ a_{m-1,1} & a_{m-1,2} & \dots & a_{m-1,m-1} \\ a_{m1} & a_{m2} & \dots & a_{m,m-1} \end{pmatrix}$$

is of maximal rank $m-1$ with $|\det \tilde{A}|_p = p^{-t}$ for some integer t and square matrix $\tilde{A}$. Consider the ball

$$B(A;p^{-r}) = B(\mathbf{a}^{(1)};p^{-r}) \times B(\mathbf{a}^{(2)};p^{-r}) \times \cdots \times B(\mathbf{a}^{(m-1)};p^{-r})$$

for any $r > t$. Then every set of vectors $\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, \dots, \mathbf{y}^{(m-1)}$ in $B(A;p^{-r})$ is linearly independent.

Proof. As $\mathbf{y}^{(j)} \in B(\mathbf{a}^{(j)}; p^{-t})$ for each $j = 1, \ldots, m-1$, we have

$$\mathbf{y}^{(j)} = \begin{pmatrix} a_{1j} + \varepsilon_{1j} \\ a_{2j} + \varepsilon_{2j} \\ \vdots \\ a_{m-1,j} + \varepsilon_{m-1,j} \\ a_{mj} + \varepsilon_{mj} \end{pmatrix}$$

where $|\varepsilon_{ij}|_p \leq p^{-r} < p^{-t}$. By relabelling if necessary we can take $\tilde{A}$ to be A with the bottom row deleted. Let $\tilde{Y}$ represent the matrix $(y_i^{(j)})$ for $i = 1, \ldots, m-1$ and $j = 1, \ldots, m-1$ and $\tilde{\varepsilon}$ the matrix ε without the bottom row. Then

$$|\det \tilde{Y}|_p = |\det(\tilde{A} + \tilde{\varepsilon})|_p = |\det \tilde{A}|_p = p^{-t} > 0$$

from Lemma 4 implying that the vectors $\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, \ldots, \mathbf{y}^{(m-1)}$ are linearly independent. □

Now we give some definitions regarding *continuously differentiable* functions. More details can be found in [10]. From now on F is a non-empty set without isolated points.

Definition 2. Let $F \subset \mathbb{Q}_p$. The function $f : F \to \mathbb{Q}_p$ is continuously differentiable at a point $a \in F$ if

$$\lim_{(x,y) \to (a,a)} \frac{f(x) - f(y)}{x - y}$$

exists. The function f is continuously differentiable if f is continuously differentiable at a for all $a \in F$.

In other words f is continuously differentiable at a if f is differentiable at a and if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $x, y \in B(a; \delta)$ with $x \neq y$ then

$$\left| \frac{f(x) - f(y)}{x - y} - f'(a) \right|_p \leq \varepsilon.$$

Definition 3. Let $F \subset \mathbb{Q}_p^u$, $f : F \to \mathbb{Q}_p$ and $k \in \{1, \ldots, u\}$. Given a point $\mathbf{x} = (x_1, \ldots, x_u)$ in F, let $\mathbf{x}' = (x_1', \ldots, x_u')$ be another point in F such that $x_i = x_i'$ for $i \neq k$ and $x_k \neq x_k'$ and then consider the limit

$$(3) \qquad \lim_{(x_k', x_k) \to (a_k, a_k)} \frac{f(\mathbf{x}') - f(\mathbf{x})}{x_k' - x_k},$$

where $\mathbf{a} = (a_1, \ldots, a_u) \in F$. If this limit exists it is called the partial derivative of f with respect to the k'th coordinate. If the partial derivatives exist for each coordinate then f is continuously differentiable at $\mathbf{a}$.

Definition 4. Let $F \subset \mathbb{Q}_p^u$ and $f_i : F \to \mathbb{Q}_p$ be p-adic functions for $i = 1, \ldots, v$. If f_i is continuously differentiable at $\mathbf{a} \in F$ for $i = 1, \ldots, v$ then $f = (f_1, \ldots, f_v)$ is said to be continuously differentiable at $\mathbf{a} \in F$.

Let $J(f)(\mathbf{a})$ denote the Jacobian matrix $(f_{i,j}(\mathbf{a}))$ of f at $\mathbf{a}$, where $f_{i,j}$ denotes the j'th partial derivative of the function f_i, that is

$$f_{i,j}(\mathbf{a}) = \lim_{\substack{x_j \to a_j \\ x_j' \to a_j}} \frac{f_i(\mathbf{x}') - f_i(\mathbf{x})}{x_j' - x_j},$$

where $x_l' = x_l$ for all $l \neq j$ and $x_j \neq x_j'$.

Lemma 6. *Let $f = (f_1, \ldots, f_{k+r})$ be continuously differentiable at $\mathbf{a} \in F$ (where F is a bounded non-empty subset of $\mathbb{Q}_p^k$) such that $0 < \max |f_{i,j}(\mathbf{a})|_p \leq 1$ and $J(f)(\mathbf{a})$ is of maximal rank k. Then there exists $\delta > 0$ such that f is bi-Lipschitz on $B(\mathbf{a}; \delta)$. More precisely there exists an integer t such that for any $\mathbf{x}, \mathbf{y} \in B(\mathbf{a}; \delta)$*

$$p^{-t}|\mathbf{x} - \mathbf{y}|_p \leq |f(\mathbf{x}) - f(\mathbf{y})|_p \leq |\mathbf{x} - \mathbf{y}|_p.$$

Proof. As $J(f)(\mathbf{a})$ is of maximal rank k, there exists a $k \times k$ matrix with non-zero determinant. Let this be

$$\widetilde{J(f)(\mathbf{a})} = \begin{pmatrix} f_{\beta_1,1}(\mathbf{a}) & \cdots & f_{\beta_1,k}(\mathbf{a}) \\ \vdots & & \vdots \\ f_{\beta_k,1}(\mathbf{a}) & \cdots & f_{\beta_k,k}(\mathbf{a}) \end{pmatrix},$$

where $\beta_i \in \{1, \ldots, k + r\}$ and let $|\det \widetilde{J(f)(\mathbf{a})}|_p = p^{-t}$. Suppose that $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{a}$ are points in $\mathbb{Q}_p^k$. In what follows $\mathbf{x}'$ will represent the point $(x_1, \ldots, x_{j-1}, y_j, x_{j+1}, \ldots, x_k)$. From the definition of continuous differentiability there exists $\delta_{ij} > 0$ such that if $x_j, y_j \in B(a_j; \delta_{ij})$ then

$$\left| \frac{f_i(\mathbf{x}') - f_i(\mathbf{x})}{x_j - y_j} - f_{i,j}(\mathbf{a}) \right|_p < p^{-t}.$$

i.e.,

$$\frac{f_i(\mathbf{x}') - f_i(\mathbf{x})}{x_j - y_j} = f_{i,j}(\mathbf{a}) + \sum_{l=t+1}^{\infty} c_l p^l = f_{i,j}(\mathbf{a}) + \varepsilon_{ij}, \tag{4}$$

where $c_l \in \{0, 1, \ldots, p - 1\}$ and $|\varepsilon_{ij}|_p < p^{-t}$. Let $\delta = \min(\delta_{ij})$. Then for any $\mathbf{x}$, $\mathbf{x}' \in B(\mathbf{a}; \delta)$, the equation (4) holds for all $i = 1, \ldots, k + r$ and $j = 1, \ldots, k$. Now assume $\mathbf{x}$, $\mathbf{y} \in B(\mathbf{a}; \delta)$. Then

$$\begin{aligned} f_i(\mathbf{y}) - f_i(\mathbf{x}) &= f_i(y_1, y_2, \dots, y_k) - f_i(x_1, x_2, \dots, x_k) \\ &= f_i(y_1, \dots, y_k) - f_i(x_1, y_2, \dots, y_k) + f_i(x_1, y_2, \dots, y_k) \\ &\quad - f_i(x_1, x_2, y_3, \dots, y_k) + \cdots - f_i(x_1, \dots, x_k) \\ &= \frac{f_i(y_1, \dots, y_k) - f_i(x_1, y_2, \dots, y_k)}{y_1 - x_1}(y_1 - x_1) \\ &\quad + \frac{f_i(x_1, y_2, \dots, y_k) - f_i(x_1, x_2, \dots, y_k)}{y_2 - x_2}(y_2 - x_2) + \dots \\ &\quad \dots + \frac{f_i(x_1, \dots, x_{k-1}, y_k) - f_i(x_1, \dots, x_k)}{y_k - x_k}(y_k - x_k) \\ &= (f_{i,1}(\mathbf{a}) + \varepsilon_{i1})(y_1 - x_1) + (f_{i,2}(\mathbf{a}) + \varepsilon_{i2})(y_2 - x_2) + \dots \\ &\quad \dots + (f_{i,k}(\mathbf{a}) + \varepsilon_{ik})(y_k - x_k) \end{aligned}$$

with $|\varepsilon_{ij}|_p < p^{-t}$. Evidently

$$\begin{aligned} f(\mathbf{y}) - f(\mathbf{x}) &= (f_1(\mathbf{y}) - f_1(\mathbf{x}), f_2(\mathbf{y}) - f_2(\mathbf{x}), \dots, f_{k+r}(\mathbf{y}) - f_{k+r}(\mathbf{x})) \\ &= (J(f)(\mathbf{a}) + \varepsilon)(\mathbf{x} - \mathbf{y}) \end{aligned}$$

where ε is the matrix (ε_{ij}), implying that $|f(\mathbf{x}) - f(\mathbf{y})|_p \leq K|\mathbf{x} - \mathbf{y}|_p$.

For the other direction the inverse function must be determined. The function

$$g : B(\mathbf{a}; \delta) \to B(\mathbf{a}; \delta) \times \{0\}^r$$

given by $g(\mathbf{x}) = (\mathbf{x}, 0, \dots, 0)$ is clearly one-one, onto and bi-Lipschitz. Define a second function $\tilde{f} : B(\mathbf{a}; \delta) \times \{0\}^r \to \mathbb{Q}_p^{k+r}$ by $\tilde{f}(\mathbf{x}, 0, \dots, 0) = f(\mathbf{x})$. Let I' denote the following matrix: $I'_{\beta_i, j} = 0$ for each $i = 1, \dots, k$ and $j = 1, \dots, k + r$; for each $\gamma_i \in \{1, \dots, k + r\} \backslash \{\beta_1, \dots, \beta_k\}$, $I'_{\gamma_i, i} = 1$ and $I'_{\gamma_i, j} = 0$ for $i \neq j$ (this matrix has been chosen for convenience). Let $M_{\tilde{f}} = M_{\tilde{f}}(\mathbf{x}, \mathbf{y})$ be the matrix $J(f)(\mathbf{a}) + \varepsilon$ augmented by I' so that $M_{\tilde{f}}$ is square. It can be readily verified that

$$(5) \qquad f(\mathbf{y}) - f(\mathbf{x}) = \tilde{f}(\mathbf{y}) - \tilde{f}(\mathbf{x}) = M_{\tilde{f}} \begin{pmatrix} \mathbf{x} - \mathbf{y} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and that the determinant of $M_{\tilde{f}}$ satisfies

$$|\det M_{\tilde{f}}|_p = |\det(\widetilde{J(f)(\mathbf{a})} + \tilde{\varepsilon})|_p = |\det \widetilde{J(f)(\mathbf{a})}|_p = p^{-t} > 0$$

(from Lemma 4). Therefore the inverse matrix $M_{\tilde{f}}^{-1}$ exists and

$$\begin{pmatrix} \mathbf{x} - \mathbf{y} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = M_{\tilde{f}}^{-1}(f(\mathbf{x}) - f(\mathbf{y})).$$

It follows that

$$M_{\tilde{f}}^{-1} = \frac{\begin{pmatrix} b_{11} & \cdots & b_{1,k+r} \\ \vdots & & \vdots \\ b_{k+r,1} & \cdots & b_{k+r,k+r} \end{pmatrix}}{\det M_{\tilde{f}}}$$

where (b_{ij}) is the adjoint matrix of $M_{\tilde{f}}$. Each b_{ij} is the ij'th cofactor of $M_{\tilde{f}}$ implying, by virtue of the ultrametric $|b_{ij}|_p \leq 1$. Hence

$$|\mathbf{x} - \mathbf{y}|_p \leq \frac{1}{|\det M_{\tilde{f}}|_p} |f(\mathbf{x}) - f(\mathbf{y})|_p = p^t |f(\mathbf{x}) - f(\mathbf{y})|_p$$

completing the proof. □

It would also be possible to prove this lemma using a p-adic maximal rank theorem as in the real case [6]. Such a theorem exists over local fields when the number of variables is at least the number of functions, see [2, Chapter 2, 10.12] but it is not immediately obvious how to adapt the proof for the complementary case. Inverse function theorems over local fields differ from those of the real case, see [10, page 75, Example 26.6] and [2, Chapter 2, 10.10]. Recently however de Smedt has obtained a C^r p-adic higher dimensional inverse function theorem [5] and it seems likely that this could be extended to a maximal rank result.

3.2. Completing the proof of the theorem. Let $V(m,n;\tau)$ denote the set of points $X = (\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)})$ in $\mathbb{Z}_p^{mn}$ such that $(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m-1)}) \in W(m, m-1; \tau)$ and $\mathbf{x}^{(j)} = \sum_{i=1}^{m-1} w_i^{(j)} \mathbf{x}^{(i)}$ with $w_i^{(j)} \in \mathbb{Z}_p$ for $j = m, \ldots, n$. It is readily verified, by checking that $|\mathbf{q} \cdot \mathbf{x}^{(j)}|_p < |\mathbf{q}|^{-\tau}$, that $V(m,n;\tau) \subseteq W(m,n;\tau)$. The rest of the paper involves constructing a particular function f, proving that it is bi-Lipschitz, and showing that

$$f(W(m-1, m; \tau) \times \mathbb{Z}_p^{(m-1)(n-m+1)}) \subseteq V(m,n;\tau).$$

Then Lemmas 2 and 3 can be used to obtain the Hausdorff dimension of $V(m,n;\tau)$ and hence a lower bound for the Hausdorff dimension of $W(m,n;\tau)$. For simplicity we start with the case $m = n$.

Let $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(m-1)}$ be linearly independent column vectors in $\mathbb{Z}_p^m$. Then the $m \times m - 1$ matrix

$$A = (\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(m-1)})$$

is of maximal rank with $|\det \tilde{A}|_p = p^{-t}$ for some $t \in \mathbb{Z}$, where by relabelling if necessary $\tilde{A}$ is the matrix A without its bottom row. Choose δ such that every matrix in the ball $B(A;\delta)$ has $m-1$ linearly independent column vectors (this is possible by Lemma 5). Define the function

$$f : B(A;\delta) \times \mathbb{Z}_p^{m-1} \to B(A;\delta) \times \mathbb{Z}_p^m$$

by

$$(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(m-1)}, w_1, w_2, \dots, w_{m-1}) \mapsto \left(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(m-1)}, \sum_{r=1}^{m-1} w_r \mathbf{x}^{(r)}\right).$$

This can be written more concisely as $f(X, \mathbf{w}) = (X, X\mathbf{w}^T)$, where $\mathbf{w}^T$ denotes the transpose of $\mathbf{w}$. Now

$$f\big((W(m, m-1; \tau) \cap B(A; \delta)) \times \mathbb{Z}_p^{m-1}\big) = V(m, m; \tau) \cap (B(A; \delta) \times \mathbb{Z}_p^m)$$

and f is one-one as points in $B(A; \delta)$ have linearly independent column vectors. Thus to obtain the Hausdorff dimension it suffices to prove that f is bi-Lipschitz.

Lemma 7. *There exists $\delta > 0$ such that for any $(X, \mathbf{v}), (Y, \mathbf{w}) \in B(A; \delta) \times \mathbb{Z}_p^{m-1}$*

$$p^{-t}|(Y, \mathbf{w}) - (X, \mathbf{v})|_p \le |f(Y, \mathbf{w}) - f(X, \mathbf{v})|_p \le |(Y, \mathbf{w}) - (X, \mathbf{v})|_p,$$

where $A = (\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \dots, \mathbf{a}^{(m-1)})$ is as above.

Proof. It is easy to verify that f is continuously differentiable at any point in $B(A; \delta) \times \mathbb{Z}_p^{m-1}$. It is also readily verified that $J(f)(A, \mathbf{w})$ is of maximal rank $m^2 - 1$ for any $\mathbf{w} \in \mathbb{Z}_p^{m-1}$ with $|f_{i,j}(A, \mathbf{w})|_p \le 1$. In fact

$$J(f)(A, \mathbf{w}) = \begin{pmatrix} I & 0 \\ E & A \end{pmatrix},$$

where I is the identity matrix and E represents a matrix depending on $\mathbf{w}$ so that

$$|\det \widetilde{J(f)(\mathbf{a})}|_p = |\det \tilde{A}|_p = p^{-t}.$$

Hence from Lemma 6 there exists $\delta' > 0$ such that

$$p^{-t}|(X, \mathbf{v}) - (Y, \mathbf{w})|_p \le |f(Y, \mathbf{w}) - f(X, \mathbf{v})|_p \le |(X, \mathbf{v}) - (Y, \mathbf{w})|_p$$

for any $(X, \mathbf{v}), (Y, \mathbf{w}) \in B(A; \delta') \times \mathbb{Z}_p^{m-1}$. □

Thus from Lemmas 2 and 3, we get that

$$\begin{aligned} \dim W(m, m; \tau) \ge \dim V(m, m; \tau) &= \dim(W(m, m-1; \tau) \times \mathbb{Z}_p^{m-1}) \\ &= \dim W(m, m-1; \tau) + m - 1 = (m-1)m + \frac{m}{\tau} \end{aligned}$$

from (1) with $n = m - 1$. This, together with Lemma 1, proves the theorem for $m = n$.

There now remains the case $m < n$. This is done exactly as above but using the function

$$h : B(A; \delta) \times \mathbb{Z}_p^{(m-1)(n-m+1)} \to B(A; \delta) \times \mathbb{Z}_p^{m(n-m+1)}$$

defined by

$$h(X, \mathbf{w}^{(1)}, \dots, \mathbf{w}^{(n-m+1)}) = (X, X(\mathbf{w}^{(1)})^T, \dots, X(\mathbf{w}^{(n-m+1)})^T).$$

Plainly

$$h\big((W(m,m-1;\tau)\times B(A;\delta))\times \mathbb{Z}_p^{(m-1)(n-m+1)}\big)$$
$$= V(m,n;\tau)\cap (B(A;\delta)\times \mathbb{Z}_p^{m(n-m+1)}).$$

It is readily verified that the Jacobian $J(f)(A,\mathbf{w}^{(1)},\ldots,\mathbf{w}^{(n-m+1)})$ is of maximal rank for any $(\mathbf{w}^{(1)},\ldots,\mathbf{w}^{(n-m+1)})$ in $\mathbb{Z}_p^{(m-1)(n-m+1)}$. By Lemma 6 this implies that there exists $\delta'>0$ such that h is bi-Lipschitz on $B(A,W;\delta')$. Hence from Lemmas 2 and 3 we obtain that

$$\begin{aligned}\dim W(m,n;\tau) &\geq \dim V(m,n;\tau) = \dim\big(W(m,m-1;\tau)\times \mathbb{Z}_p^{(m-1)(n-m+1)}\big)\\ &= \dim W(m,m-1;\tau) + (n-m+1)(m-1) = (m-1)n + \frac{m}{\tau}\end{aligned}$$

again from (1). If $\tau \leq m/n$ then it is shown in [1] that $W(m,n;\tau)$ has full Haar measure which implies that its Hausdorff dimension is mn and completes the proof of Theorem 1. □

ACKNOWLEDGEMENTS

We are grateful to Dr. A. Abercrombie, Dr. S. de Smedt, Dr. J.B. Fountain, Dr. R. Nair, Professor R. Odoni, and Professor W. Schikhof for their helpful comments and discussions. The Royal Society provided funds for a Study Visit for Jin Yuan to spend three months in York where this work was carried out and H. Dickinson is supported by the Engineering and Physical Sciences Research Council (grant GR/K56407).

REFERENCES

1. Abercrombie, A.G. – The Hausdorff dimension of some exceptional sets of p-adic integer matrices. J. Number Th. **53**, no. 2, 311–341 (1995).
2. Abhyankar, S.S. – Local Analytic Geometry. Academic Press, New York and London (1964).
3. Bovey, J.D. and M.M. Dodson – The fractional dimension of sets whose simultaneous rational approximations have errors with a small product. Bull. London Math. Soc. **10**, 213–218 (1978).
4. Bovey, J.D. and M.M. Dodson – The Hausdorff dimension of systems of linear forms. Acta Arith. **45**, 337–358 (1986).
5. de Smedt, S. – Local invertibility of non-Archimedean vector-valued functions. Annales Mathematiques Blaise Pascal **5**, 13–23 (1998).
6. Dickinson, H – The Hausdorff dimension of systems of simultaneously small linear forms. Mathematika **40**, 367–374 (1993).
7. Falconer, K. – The geometry of fractal sets. Cambridge Tracts in Mathematics, vol. 85. Cambridge University Press (1985).
8. Larman, D.G. – On Hausdorff measures in finite dimensional compact metric spaces. Proc. London Math. Soc. **17**, 193–206 (1967).
9. Melnichuk, Y.V. – Hausdorff dimension in Diophantine approximations of p-adic numbers. Ukrain. Mat. Zh. **32**, no. 1 (in Russian), 118–124 (1980).
10. Schikhof, W.H. – Ultrametric calculus; an introduction to p-adic analysis. Cambridge University Press (1984).

(Received June 1998)