On the D-Stability of Linear and Nonlinear Positive Switched Systems

V. S. Bokharaie, O. Mason and F. Wirth

Abstract— **We present a number of results on D-stability of positive switched systems. Different classes of linear and nonlinear positive switched systems are considered and simple conditions for D-stability of each class are presented.**

I. INTRODUCTION

Positive systems have found wide application in different areas of science and engineering, for example, wireless communications, economics, populations dynamics, systems biology [1]. While the stability properties of Positive linear time-invariant systems are thoroughly investigated and well understood, the theory for nonlinear and time-varying systems needs more attention.

Recently, switched systems have attracted a lot of attention [2] [3]. This has been primarily motivated by the fact that many man-made systems and some physical systems can be modelled within this framework. While major advances have been made, many important questions that relate to their behaviour still remain unanswered, even for linear switched systems. Perhaps the most important of these relate to the stability of such systems.

In this paper, we are concerned with the stability of positive switched systems. A popular approach to this issue is to exploit *copositive Lyapunov functions* to investigate the stability of such systems. In particular, linear copositive Lyapunov functions have been considered in [4], while results on quadratic copositive Lyapunov functions were presented in [5]. Linear copositive Lyapunov functions and functionals have also been used to establish strong delay-independent conditions for classes of linear and nonlinear positive systems [6][7][8]. Within the class of switched systems, most of the research in this field to date has focused on linear switched systems. Considering the importance of nonlinear switched systems, there is a need to extend these results to positive nonlinear switched systems.

One of the key properties of positive LTI systems is D-stability, which we define formally below. In the recent paper [9], this concept was extended to positive switched linear systems and separate necessary and sufficient conditions for D-stability for this system class were presented. In this paper we develop the work of [9] in two main directions. First, we show that for irreducible linear systems, the separate necessary and sufficient conditions of [9] can be combined to give a single necessary and sufficient condition for D-stability. Further, we show that positive switched systems with commuting system matrices are D-stable. A major contribution of the paper is to extend the sufficient condition for D-stability for linear switched systems to a class of nonlinear positive switched systems. Specifically, we derive a condition for D-stability for switched systems whose constituent vector fields are homogeneous and cooperative. We also show, as a corollary, that the result on commuting linear systems extends to irreducible, cooperative homogeneous systems.

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II. MATHEMATICAL BACKGROUND

A. Mathematical Notations

Throughout the paper, $\mathbb{R}$ and $\mathbb{R}^n$ denote the field of real numbers and the vector space of all n-tuples of real numbers, respectively. $\mathbb{R}^{n\times n}$ denotes the space of $n\times n$ matrices with real entries. For $x \in \mathbb{R}^n$ and $i = 1,\dots,n$, x_i denotes the i^{th} coordinate of x. Similarly, for $A \in \mathbb{R}^{n\times n}$, a_{ij} denotes the $(i,j)^{th}$ entry of A. Also, for $x \in \mathbb{R}^n$, $diag(x)$ is the $n\times n$ diagonal matrix in which $d_{ii} = x_i$. For a diagonal matrix $D \in \mathbb{R}^{n\times n}$, the notation $D > 0$ indicates that $d_{ii} > 0$ for $i = 1,\dots,n$.

In the interest of brevity, we shall slightly abuse notation and refer to a system as being Globally Asymptotically Stable, GAS for short, when the origin is a GAS equilibrium of the system. Also, as we are dealing with positive systems throughout, when we refer to a system as GAS, it is with respect to initial conditions in $\mathbb{R}^n_+$, where $\mathbb{R}^n_+$ is the set of all vectors in $\mathbb{R}^n$ with non-negative entries:

$$\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_i \geq 0, 1 \leq i \leq n\}.$$

The interior of $\mathbb{R}^n_+$ is given by

$$int(\mathbb{R}^n_+) := \{x \in \mathbb{R}^n : x_i > 0, 1 \leq i \leq n\}.$$

The boundary $bd(\mathbb{R}^n_+) := \mathbb{R}^n_+ \backslash int(\mathbb{R}^n_+)$.

For vectors $x, y \in \mathbb{R}^n$, we write: $x \geq y$ if $x_i \geq y_i$ for $1 \leq i \leq n$; $x > y$ if $x \geq y$ and $x \neq y$; $x \gg y$ if $x_i > y_i, 1 \leq i \leq n$.

B. Cooperative Homogeneous Systems

Given an n-tuple $r = (r_1,\dots,r_n)$ of positive real numbers and $\lambda > 0$, the *dilation map* $\delta^r_\lambda(x) : \mathbb{R}^n \to \mathbb{R}^n$ is given by $\delta^r_\lambda(x) = (\lambda^{r_1}x_1, ..., \lambda^{r_n}x_n)$. If $r = (1,\dots,1)$, then we will have a *standard dilation map*. The vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be *homogeneous* of degree $\tau \geq 0$ with respect to $\delta^r_\lambda(x)$ if

$$\forall x \in \mathbb{R}^n, \lambda \geq 0, \quad f(\delta^r_\lambda(x)) = \lambda^\tau \delta^r_\lambda(f(x)).$$

The following fact, which is known as Euler's formula, will be useful in Section IV.

Proposition 2.1 (Euler's Formula): Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a homogeneous vector field of degree τ with respect to the dilation map δ^r_λ. Then for any $a \in \mathbb{R}^n$:

$$\frac{\partial f}{\partial x}(a)diag(r)a = diag(r+\tau^*)f(a) \tag{1}$$

where $\tau^* := (\tau, \cdots, \tau)$

We say that $f : \mathbb{R}^n \to \mathbb{R}^n$ is *cooperative* if the Jacobian matrix $\frac{\partial f}{\partial x}(a)$ is Metzler for all $a \in \mathbb{R}^n_+$ [12].

Following the definition of [10], $f : \mathbb{R}^n \to \mathbb{R}^n$, is said to be *irreducible* if:

(i) for $a \in int(\mathbb{R}^n_+)$, $\frac{\partial f}{\partial x}(a)$ is irreducible [11];

(ii) for $a \in bd(\mathbb{R}^n_+) \setminus \{0\}$, either $\frac{\partial f}{\partial x}(a)$ is irreducible or $f_i(a) > 0 \ \forall i : a_i = 0$.

The key fact for our later analysis is that cooperative systems are monotone [12]. Formally, if $f : \mathbb{R}^n \to \mathbb{R}^n$ is cooperative and we denote by $x(t, x_0)$ the solution of $\dot{x}(t) = f(x(t))$ satisfying $x(0) = x_0$, then $x_0 \leq y_0$ implies $x(t, x_0) \leq x(t, y_0)$ for all $t \geq 0$. Moreover, as the origin is automatically an equilibrium of a homogeneous system, it follows that homogeneous cooperative systems are positive, meaning that they are $\mathbb{R}^n_+$ invariant.

C. Commuting vector fields

We briefly recall the definition of commuting nonlinear vector fields.

Definition 2.1 (Commuting Nonlinear Vector fields): We say that the vector fields f_1 and f_2 commute if

$$\frac{\partial f_1}{\partial x}(a)f_2(a) = \frac{\partial f_2}{\partial x}(a)f_1(a) \tag{2}$$

for all $a \in \mathbb{R}^n$.

III. D-STABILITY CONDITIONS FOR POSITIVE LINEAR SWITCHED SYSTEMS

A key property of positive LTI systems is that a GAS system is automatically D-stable [1]. The LTI system $\dot{x}(t) = Ax(t)$ is said to be D-stable if $\dot{x}(t) = DAx(t)$ is GAS for all diagonal matrices $D > 0$.

In this section, we consider an extension of the notion of D-stability to positive switched linear systems. First, we recall the definition introduced in [9].

Definition 3.1: Consider the switched system:

$$\dot{x}(t) = A(t)x(t) \quad A(t) \in \{A_1, A_2\} \tag{3}$$

in which A_1 and A_2 are Hurwitz Metzler matrices. We say that (3) is D-stable, if the corresponding switched system

$$\dot{x}(t) = A(t)x(t) \quad A(t) \in \{D_1A_1, D_2A_2\} \tag{4}$$

is GAS for all diagonal $D_1 > 0$ and $D_2 > 0$.

In [9], the following result, giving separate sufficient and necessary conditions for D-stability of positive linear switched systems was presented.

Theorem 3.1: Let $A_1, A_2 \in \mathbb{R}^{n\times n}$ be Metzler and Hurwitz. Then:

(i) If there is some $v \gg 0$ with $A_1v \ll 0$, $A_2v \ll 0$ then the system (3) is D-stable;

(ii) If (3) is D-stable then there exists some non-zero $v > 0$ with $A_1v \leq 0$, $A_2v \leq 0$.

It was highlighted in [9] that there is a clear gap between the two conditions given above. In the next result, we show that under the additional assumption of irreducibility, it is possible to give a single necessary and sufficient condition for D-stability.

Theorem 3.2: Let $A_1, A_2 \in \mathbb{R}^{n\times n}$ be Metzler, irreducible and Hurwitz. Then the switched system (3) is D-stable if and only if there exists a vector $v \gg 0$ such that $A_1v \leq 0$ and $A_2v \leq 0$.

Proof:

Necessity:

Based on Theorem 3.1, we already know that if (3) is D-stable, then there exists a $v > 0$ such that $A_1v \leq 0$ and $A_2v \leq 0$. We shall show that if A_1 and A_2 are irreducible, then any such v must be strictly positive.

To this end, assume that $v > 0$, $A_iv \leq 0$ for $i = 1, 2$ and v is not strictly positive. Without loss of generality, we assume that precisely the first k elements of v are non-zero, so $v_i > 0$ for $i = 1, \cdots, k$ and $v_i = 0$ for $i = k+1, \cdots, n$. Now we partition A_1 and A_2 as follows:

$$A_1 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} A'_{11} & A'_{12} \\ A'_{21} & A'_{22} \end{bmatrix}$$

In which A_{11} and A'_{11} are $k \times k$, A_{22} and A'_{22} are $(n-k) \times (n-k)$ and A_{21} and A'_{21} are $(n-k) \times k$ submatrices. Please note that A_{11}, A'_{11}, A_{22} and A'_{22} are Metzler and A_{12}, A'_{12}, A_{21} and A'_{21} are element-wise non-negative. We know that $A_1v \leq 0$ and since the last $n-k$ elements of v are zero, then we should have $A_{21}v' \leq 0$, in which $v' = [v_1, \cdots, v_k]^T$. Since we know A_{21} is a non-negative matrix and $v' \gg 0$, then the only way that this inequality can hold is that $A_{21} = 0$ in which 0 refers to a matrix with zero entries and appropriate dimensions. Using the same method, we can easily conclude that $A'_{21} = 0$. This implies that both A_1 and A_2 are reducible, which is a contradiction. Therefore, v cannot have zero entries and we must have $v \gg 0$ as claimed.

Sufficiency:

Let σ be a given switching signal with switching instances $t_0, t_1, t_2, ...$ and, as is standard in the literature on switched systems, we assume that there is some $\tau > 0$ such that $t_{j+1} - t_j \geq \tau$ for all j. We shall also write $i_j = \sigma(t_j)$ for $j = 0, 1, ...$

For $x_0 \in \mathbb{R}^n_+$, we denote by $x(\sigma, t, x_0)$ the solution of (3) corresponding to the switched signal σ and initial condition x_0.

Now note that for an irreducible Metzler matrix A, $e^{At} \gg 0$ for all $t > 0$. Consider for any such A, the system $\dot{x}(t) = Ax(t)$. Then for any solution $x(t)$ of this system, $y(t) = Ax(t)$ also satisfies $\dot{y}(t) = Ay(t)$. As $e^{At} \gg 0$ for all $t > 0$, it immediately follows that if $y(0) < 0$ then we must have $y(t) \ll 0$ for all $t > 0$. In terms of the original system, this means that $Ax(0) < 0$ implies that $Ax(t) \ll 0$ for all $t > 0$.

This argument guarantees that there is some $\alpha < 1$ such that for $i = 1, 2$:

$$e^{A_i \tau} v \leq \alpha v. \quad (5)$$

Further, as $t_{j+1} - t_j \geq \tau$ for all j, we can also conclude that for $i = 1, 2$ and $j = 0, 1, 2, 3, ...$

$$e^{A_i(t_{j+1}-t_j)} v \leq e^{A_i(\tau)} v. \quad (6)$$

Now consider any time $t > 0$ and assume that t_K is the final switching instant before t. Then

$$x(\sigma, t, v) = e^{A_{i_K}(t-t_K)} e^{A_{i_{K-1}}(t_K - t_{K-1})} \cdots e^{A_{i_0}(t_1 - t_0)} v. \quad (7)$$

It follows from (6) and (7) that $x(\sigma, t, v) \leq \alpha^K v$. A little thought (we can "lose" at most one power of α per switch) shows that if we define N_t to be the largest integer less than or equal to $\frac{t}{2\tau}$, then for any switching signal (whether there are finitely many switches or infinitely many switches) we must have

$$x(\sigma, t, v) \leq \alpha^{N_t} v$$

implying that $x(\sigma, t, v) \to 0$ as $t \to \infty$.

Now let $x_0 \in \mathbb{R}^n_+$ be given. Choose $\lambda > 0$ with $x_0 \leq \lambda v$. It follows from $e^{A_i t} \gg 0$ for all $t > 0$ and $i = 1, 2$ that

$$x(\sigma, t, x_0) \leq x(\sigma, t, \lambda v) = \lambda x(\sigma, t, v)$$

and hence $x(\sigma, t, x_0) \to 0$ also. The result now follows as it is immediate that $D_i A_i v < 0$ for $i = 1, 2$, for any diagonal matrices $D_1 > 0$, $D_2 > 0$.

It has been previously shown [13] that switched linear systems with commuting system matrices are GAS. In the following result, we show that for positive switched linear systems, commutativity implies the stronger property of D-stability.

Theorem 3.3: Let $A_1 \in \mathbb{R}^{n \times n}$ and $A_2 \in \mathbb{R}^{n \times n}$ be Metzler and Hurwitz. Further, assume that $A_1 A_2 = A_2 A_1$. Then the switched system (3) is D-stable under arbitrary switching.

Proof: Recall that for Metzler, Hurwitz matrices A_1 and A_2, $A_1^{-1} < 0$ and $A_2^{-1} < 0$ [14]. Now let $w \gg 0$ in $\mathbb{R}^n$ be given. Then $v = A_1^{-1} A_2^{-1} w \gg 0$, and therefore:

$$A_1 v = A_1 A_1^{-1} A_2^{-1} w = A_2^{-1} w \ll 0$$

and

$$A_2 v = A_2 A_1^{-1} A_2^{-1} w = A_2 A_2^{-1} A_1^{-1} w = A_1^{-1} w \ll 0$$

Thus, we have $v \gg 0$ such that $A_1 v \ll 0$ and $A_2 v \ll 0$, and it follows from Theorem (3.1) that the switched system is D-stable.

IV. D-STABILITY CONDITIONS FOR POSITIVE NONLINEAR SWITCHED SYSTEMS

In this section, we consider extensions of the previous results on D-stability to classes of nonlinear positive switched systems; specifically, we consider switched systems defined by cooperative homogeneous vector fields.

Throughout this section, all vector fields are assumed to be cooperative and homogeneous *of degree* 0 with respect to a fixed dilation map δ^r_λ. Note that for the standard dilation map, this amounts to assuming that $f(\lambda x) = \lambda f(x)$ for all $x \in \mathbb{R}^n$ and all $\lambda > 0$. Further, we shall assume that all vector fields are continuous on $\mathbb{R}^n$ and C^1 on $\mathbb{R}^n \backslash \{0\}$. As noted in [10], this ensures existence and uniqueness of solutions for the associated autonomous system. As we are interested in D-stability and stability under arbitrary switching, we also assume that all constituent systems $\dot{x} = f_i(x)$ are GAS throughout the section.

To begin, we state the following definition, which is a natural extension of the Definition (3.1) for linear switched systems.

Definition 4.1: Consider the switched system

$$\dot{x}(t) = f(x(t), t); \quad f(\cdot, t) \in \{f_1(\cdot), \ldots, f_m(\cdot)\} \quad (8)$$

We call this system D-stable if the associated switched system

$$\dot{x}(t) = f(x(t), t); \quad f(\cdot, t) \in \{D_1 f_1(\cdot), \ldots, D_m f_m(\cdot)\} \quad (9)$$

is GAS for any diagonal matrices D_j with $D_j > 0$ for $j = 1, \cdots, m$.

For the proof of the next result, we will need the following proposition, which is Proposition 3.2.1 in [12].

Proposition 4.1: Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be cooperative. Then if $f(x_0) \leq 0$ ($f(x_0) \geq 0$) the trajectory $x(t, x_0)$ of $\dot{x} = f(x)$ with initial condition x_0 is nonincreasing (nondecreasing) for $t \geq 0$.

We can now present and prove the main Theorem of this section, which extends the sufficient condition of Theorem 3.1 to cooperative homogeneous systems.

Theorem 4.1 (Main Theorem): Consider the switched system (8). If there exists a $v \gg 0$ such that $f_i(v) \ll 0$ for all $i \in \{1, 2, ..., m\}$, then (8) is D-stable under arbitrary switching.

Proof: We will prove the Theorem in a number of steps. The first step is to show Lyapunov stability of the system.

(i) Proof of Stability

Let an arbitrary switching signal $\sigma : [0, \infty) \to \{1, \ldots, m\}$ be given with switching instances $0 = t_0, t_1, t_2, \cdots$. For $x_0 \in \mathbb{R}^n_+$, let $x(\sigma, t, x_0)$ denote the solution of (8) corresponding to the initial condition x_0 and the switching signal σ.

To begin with, from the homogeneity of the vector fields f_i it follows that for any $\lambda > 0$, $f_i(\delta^r_\lambda(v)) \ll 0$ for $i = 1, \ldots, m$. Thus as each f_i is cooperative, Proposition 4.1 implies that the trajectory starting from the initial condition $x_0 = \delta^r_\lambda(v)$ is non-increasing for $0 \leq t < t_1$. In particular, for $0 \leq t \leq t_1$, $x(\sigma, t, \delta^r_\lambda(v)) < x(\sigma, 0, \delta^r_\lambda(v)) = \delta^r_\lambda(v)$. At $t = t_1$, we switch to a new vector field, which starts from initial condition equal to $x(\sigma, t_1, \delta^r_\lambda(v))$. Since this new system is also cooperative it follows from $f_i(\delta^r_\lambda(v)) \ll 0$ for $i = 1, \ldots, m$ and $x(\sigma, t_1, \delta^r_\lambda(v)) < \delta^r_\lambda(v)$ that

$$x(\sigma, t, \delta^r_\lambda(v)) < \delta^r_\lambda(v)$$

for $t_1 \leq t \leq t_2$. Continuing in this way it is clear that the trajectory $x(\sigma, t, \delta^r_\lambda(v)) < \delta^r_\lambda(v)$ for all $t \geq 0$.

Let $\epsilon > 0$ be given. Then we can choose $\lambda > 0$ so that $\|\delta^r_\lambda(v)\|_\infty < \epsilon$. Now putting

$$\delta = \min_i (\delta^r_\lambda(v))_i$$

we see that if $x_0 \geq 0$ and $\|x_0\|_\infty < \delta$, then $x_0 \leq \delta^r_\lambda(v)$ and the above argument guarantees that

$$\|x(\sigma, t, x_0)\|_\infty < \epsilon$$

for all $t \geq 0$. Note that our choice of δ does not depend on the switching signal σ.

(ii) Proof of Global Asymptotic Stability

We next show that the origin is uniformly attractive for (8). To this end choose $\alpha > 0$ such that

$$f_i(v) + \alpha v \ll 0$$

for $1 \leq i \leq m$ and for $i = 1, \ldots, m$, define $g_i : \mathbb{R}^n \to \mathbb{R}^n$ by

$$g_i(y) = f_i(y) + \alpha diag(r) y$$

and consider the switched system

$$\dot{y}(t) = g(y(t), t) \;\; g(y, .) \in \{g_1, \ldots, g_m\}. \tag{10}$$

Let $x(\sigma, t, x_0)$ be a solution of (8) with initial condition x_0. Then it can be verified by direct calculation that

$$y(t) = \delta^r_\alpha(x(t)) = (e^{r_1 \alpha t} x_1, \cdots, e^{r_n \alpha t} x_n) \tag{11}$$

is a solution of (10) with $y(0) = x_0$.

It follows from the argument given in part (i) that for any x_0 there exists some $\lambda > 0$ with $y(\sigma, t, x_0) \leq \delta^r_\lambda(v)$ for all $t \geq 0$. This immediately implies that

$$x(\sigma, t, x_0) \to 0$$

as $t \to \infty$.

(iii) Proof of D-stability

Let diagonal matrices $D_1, \ldots, D_m$ be given with $D_i > 0$ for $i = 1, \ldots m$. Then it is simple to check that each vector field $D_i f_i(\cdot)$ is cooperative and homogeneous of degree 0 with respect to δ^r_λ. Further,

$$D_i f_i(v) \ll 0$$

for $1 \leq i \leq m$. It now follows from the previous arguments, that (9) is GAS and hence (8) is D-stable as claimed.

Remark: While we have assumed that all vector fields are homogeneous with respect to the same dilation map, this does not appear to be necessary and extending the above result in this and other directions is the work of ongoing research.

Now, we present the result on commuting vector fields, but before that, we need the following Theorem which is Theorem 5.2 in [10].

Theorem 4.2: Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be cooperative and irreducible. Further assume that f is homogeneous of degree 0 with respect to δ^r_λ. Then there exists a $\gamma \in \mathbb{R}$ and a vector $v \gg 0$, such that $f(v) = \gamma diag(r) v$ and $\dot{x} = f(x)$ is GAS if and only if $\gamma < 0$.

It is known that a switched system with GAS commuting vector fields is GAS itself. Our next result, provides a condition for D-stability of a class of commuting vector fields.

Corollary 4.1: Consider the switched system:

$$\dot{x}(t) = f(x(t), t) \quad f(\cdot, t) \in \{f_1(\cdot), f_2(\cdot)\}. \tag{12}$$

Assume that f_1 and f_2 commute and are irreducible. Then the switched system (12) is D-stable under arbitrary switching.

Proof: Since f_2 is GAS, homogeneous, cooperative and irreducible, then based on Theorem (4.2), we know that there exists a $v \gg 0$ such that

$$f_2(v) = \gamma diag(r) v \tag{13}$$

in which $\gamma < 0$ is a scalar. Now, applying Euler's formula to f_2 and evaluating it at v, we have:

$$\frac{\partial f_2}{\partial x}(v) diag(r) v = diag(r) f_2(v) \tag{14}$$

Substituting $f_2(v)$ in the right-hand side of (14) from (13), we have:

$$\frac{\partial f_2}{\partial x}(v) diag(r) v = diag(r) \gamma diag(r) v$$

$$\Rightarrow diag(r)^{-1} \frac{\partial f_2}{\partial x}(v) diag(r) v = \gamma diag(r) v$$

therefore $diag(r) v$ is an eigenvector of $diag(r)^{-1} \frac{\partial f_2}{\partial x}(v)$. Since $\frac{\partial f_2}{\partial x}(v)$ and therefore $diag(r)^{-1} \frac{\partial f_2}{\partial x}(v)$ is irreducible and Metzler, and since $diag(r) v \gg 0$, the Perron-Frobenius Theorem for irreducible matrices [1] implies that γ is the right-most eigenvalue of $diag(r)^{-1} \frac{\partial f_2}{\partial x}(v)$ and $diag(r) v$ is its unique eigenvector (up to scalar multiple).

On the other hand, by evaluating the commutativity equality at $x = v$, we have:

$$\frac{\partial f_1}{\partial x}(v) f_2(v) = \frac{\partial f_2}{\partial x}(v) f_1(v)$$

By applying (13) and Euler's formula to the left-hand side of the above equation, we have:

$$\gamma diag(r) f_1(v) = \frac{\partial f_2}{\partial x}(v) f_1(v)$$

$$\Rightarrow \gamma f_1(v) = diag(r)^{-1} \frac{\partial f_2}{\partial x}(v) f_1(v)$$

Therefore, $f_1(v)$ is also an eigenvector corresponding to γ. Since the eigenvector corresponding to this eigenvalue is unique up to scalar multiple, then we should have:

$$f_1(v) = \kappa diag(r) v$$

where κ is a scalar. Since f_1 is GAS, homogeneous, cooperative and irreducible, then based on Theorem (4.2) $\kappa < 0$. Thus $f_1(v) \ll 0$ and from (13) we know $f_2(v) \ll 0$. It now follows from Theorem 4.1 that the switched system (12) is D-stable under arbitrary switching.

V. CONCLUSIONS

We have shown that the separate necessary sufficient conditions for D-stability for switched positive linear systems previously presented in [9] can be combined into a single necessary and sufficient condition in the case of irreducible systems. Further, we have shown that switched positive linear systems with commuting system matrices are D-stable. A simple extension of the concept of D-stability for switched nonlinear positive systems has been considered and a sufficient condition for cooperative homogeneous (of degree 0) switched systems to be D-stable has been derived. The result on commuting system matrices has also been extended to irreducible vector fields in this case.

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