A conjecture on the existence of common quadratic Lyapunov functions for positive linear systems

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Abstract

We present a conjecture concerning necessary and sufficient conditions for the existence of a common quadratic Lyapunov function (CQLF) for a switched linear system obtained by switching between two positive linear time-invariant (LTI) systems. We conjecture that these conditions are also necessary and sufficient for the exponential stability of such switched linear systems; namely, the existence of a CQLF is a non-conservative stability condition in this case. A number of new results supporting this conjecture are described.

1 Introduction

The problem of determining necessary and sufficient conditions for the existence of a common quadratic Lyapunov function (CQLF) for a set of stable linear time-invariant (LTI) systems

$$\Sigma_{A_i}: \dot{x}(t) = A_i x(t),\ A_i \in \mathbb{R}^{n\times n},\ 1 \le i \le k$$

plays an important role in the study of switched linear systems of the form:

$$\dot{x}(t) = A(t)x(t),\ A(t) \in \{A_1, ..., A_k\}. \qquad (1)$$

Formally, if there is a symmetric positive definite matrix P that simultaneously satisfies the Lyapunov inequalities

$$A_i^T P + PA_i \ = \ -Q_i < 0,\ i \in \{1, 2, ...k\} \qquad (2)$$

then $V(x) = x^T Px$ is a CQLF for the system (1) and the associated LTI systems Σ_{A_i}. The existence of a CQLF is sufficient to guarantee global uniform exponential stability of (1) for arbitrary switching sequences. It is well known that requiring the existence of a CQLF for a switched linear system is, in general, a conservative stability condition [1]. However, it has recently been established that entire system classes exist for which the existence of a CQLF is not necessarily a conservative stability condition [2, 3]. In view of this observation, a problem of considerable interest and importance is to identify precisely those system classes for which the existence of a CQLF is a non-conservative stability condition. The work of this paper is primarily motivated by such considerations.

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2 Notation and Preliminaries

For a matrix A in $\mathbb{R}^{n\times n}$, a_{ij} denotes the element in the (i,j) position of A, and we shall write $A \succeq 0$ if $a_{ij} \ge 0$ for $1 \le i,j \le n$. The matrix $A \in \mathbb{R}^{n\times n}$ is said to be *Hurwitz* if all the eigenvalues of A have negative real parts, and for P in $\mathbb{R}^{n\times n}$ the notation $P > 0$ means that the matrix P is positive definite.

A matrix A in $\mathbb{R}^{n\times n}$ is a *Metzler* matrix if all of the off-diagonal elements of A are non-negative; that is $a_{ij} \ge 0$ for $i \ne j$. The LTI system Σ_A is *positive* [1] [4] if and only if A is a Metzler matrix. The associated class of *M-matrices* [5, 6] is defined to consist of matrices A with non-positive off-diagonal elements, all of whose eigenvalues lie in the open right half-plane.

A conjecture:

Let Σ_{A_i}, $i = 1, 2$ be a pair of stable positive LTI systems. Recent work carried out by the authors suggests that the matrix product $A_1 A_2^{-1}$ having no negative eigenvalues is a necessary and sufficient condition for:

(i) the existence of a CQLF for the LTI systems Σ_{A_1} Σ_{A_2};

(ii) global exponential stability of the switched linear system (1).

3 Sufficient conditions for CQLF existence

In this section we state without proof a number of sufficient conditions for a pairs of stable positive LTI systems to possess a CQLF. Details of the proofs can be found in [7]. The result stated in the next lemma is not new [8] but is included here for the sake of comparison with the main result of this note (Theorem 3.1 below).

[1]An LTI system is positive if, for any initial conditions where the state variables are all non-negative, the state variables remain non-negative for all time

Lemma 3.1 Let $\Sigma_{A_1}, \Sigma_{A_2}$ be stable positive LTI systems, with $A_1 - A_2 \succeq 0$. Then Σ_{A_1} and Σ_{A_2} have a CQLF $V(x) = x^T P x$, with P diagonal.

Theorem 3.1 Let $\Sigma_{A_1}, \Sigma_{A_2}$ be stable positive LTI systems. If both $A_1 A_2^{-1}$ and $A_2^{-1} A_1$ are M-matrices, then Σ_{A_1} and Σ_{A_2} have a CQLF, $V(x) = x^T P x$, and moreover, P may be taken to be a diagonal matrix.

Note that within the class of matrices with non-positive off-diagonal elements, a non-singular matrix having no eigenvalues on the negative real axis is equivalent to it being an M-matrix ([5]).

Theorem 3.2 Let Σ_{A_1}, Σ_{A_2} be stable positive LTI systems. Suppose that $A_1 A_2^{-1} \succeq 0$ and $A_2^{-1} A_1 \succeq 0$. Then Σ_{A_1} and Σ_{A_2} have a CQLF.

It was noted in [5] that if A_1, A_2 are both Hurwitz Metzler matrices with $A_1 \succeq A_2$, then $A_1 A_2^{-1}$ and $A_2^{-1} A_1$ are both M-matrices. Thus the class of matrices covered by Lemma 3.1 is a subclass of the class covered by Theorem 3.1. (In fact, Theorem 3.2 covers a still larger class of systems than Theorem 3.1.) The next example shows that it is a strict subclass.

Example: Consider the two Metzler matrices in $\mathbb{R}^{3\times 3}$ given by

$$A_1 = \begin{pmatrix} -1.1686 & 0.5618 & 0.3837 \\ 0.9512 & -1.7425 & 0.7293 \\ 0.9460 & 0.4830 & -1.8474 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} -1.7697 & 0.3163 & 0.1496 \\ 0.1599 & -0.9759 & 0.2794 \\ 0.2167 & 0.1769 & -1.0543 \end{pmatrix}.$$

It is evident that neither $A_1 - A_2 \succeq 0$ nor $A_2 - A_1 \succeq 0$ is true, so Lemma 3.1 does not apply. However, it is a simple matter to check that both $A_1 A_2^{-1}$ and $A_2^{-1} A_1$ are M-matrices. Thus by Theorem 3.1 we can conclude that A_1 and A_2 have a CQLF $x^T P x$ with P diagonal.

In [9] it is shown that LTI systems whose system matrices commute have a CQLF $x^T P x$. The next result shows that P may be chosen to be diagonal if the LTI systems are positive.

Theorem 3.2 Let Σ_{A_1}, Σ_{A_2} be two positive LTI systems with $A_1 A_2 = A_2 A_1$. Then there is a CQLF $V(x) = x^T P x$ for $\Sigma_{A_1}, \Sigma_{A_2}$ with P diagonal.

4 Conclusions

In this paper, we have proposed a conjecture concerning CQLF existence for a pair of stable positive LTI systems. It was also conjectured that for switched linear systems obtained by switching between stable positive LTI systems, the existence of a CQLF is a non-conservative stability criterion. A number of new results in this direction were presented. The authors have also gathered considerable empirical evidence supporting the conjecture.

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