A result on the existence of common quadratic Lyapunov functions for pairs of stable discrete-time LTI systems

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Abstract

This paper deals with the existence of weak and strong common quadratic Lyapunov functions (CQLFs) for pairs of stable discrete-time linear time-invariant (LTI) systems. The main result of the paper provides a simple characterisation of pairs of such systems for which a weak CQLF of a given form exists but for which no strong CQLF exists. An application of this result to second order discrete-time LTI systems is presented.

1 Introduction

In recent years, the area of systems theory and control has witnessed a considerable growth of interest in systems characterised by a combination of continuous dynamics and logic-based switching. A major issue for such systems has been the determination of easily verifiable and interpretable conditions that guarantee stability. For an overview of some recent approaches to this issue see [1], [2], [3], [4], [5], [6], [7]. In this context the problem of the existence or non-existence of a common quadratic Lyapunov function (CQLF) for a family of linear time-invariant (LTI) systems is of considerable interest. There is already a substantial body of literature dedicated to this question for both discrete-time and continuous-time systems [8], [9], [10], [11], [12], [13], [7]. The main result presented here is concerned with the CQLF existence problem for a family of two discrete-time LTI systems.

2 Notation and Preliminaries

Throughout $\mathbb{R}$ and $\mathbb{C}$ will denote the fields of real and complex numbers respectively and $\mathbb{R}^{n\times n}$ ($\mathbb{C}^{n\times n}$) denotes the space of $n\times n$ matrices with real (complex) entries. For a matrix A in $\mathbb{R}^{n\times n}$, A^T denotes its transpose, $det(A)$ its determinant and a_{ij} the entry in the (i,j) position of A. Similarly for a vector x in $\mathbb{R}^n$, x_i denotes the i^{th} component of x. A matrix $A\in\mathbb{R}^{n\times n}$ is said to be symmetric if $A=A^T$. The notation $P>0$ ($P\geq 0$) is used to denote that the matrix P is positive (semi-)definite with $P<0$, ($P\leq 0$) meaning that $-P>0$ ($-P\geq 0$). Recall that $A\in\mathbb{R}^{n\times n}$ is said to be *Schur-stable* (*Hurwitz*) if the eigenvalues of A all lie in the open unit disk (open left half plane) in $\mathbb{C}$.

We note that a matrix $A\in\mathbb{R}^{n\times n}$ is Schur-stable if and only if the associated discrete-time LTI system

$$\Sigma_A : x(k+1) = Ax(k)$$

is asymptotically stable [14].

In the spirit of [15], we now define strong and weak CQLFs for a set of stable discrete-time LTI systems.

Strong and weak CQLFs:

Consider the set of stable discrete-time LTI systems

$$\Sigma_{A_i} : x(k+1) = A_i x(k),\ 1\leq i\leq M. \quad (1)$$

where the A_i are Schur-stable matrices in $\mathbb{R}^{n\times n}$. If there is a simultaneous solution $P=P^T>0$ to the discrete-time Lyapunov inequalities [1]

$$A_i^T P A_i - P = -Q_i < 0 \quad 1\leq i\leq M \quad (2)$$

then the scalar function $V(x)=x^TPx$ is said to be a *strong CQLF* for the systems Σ_{A_i}. If $M=1$, then $V(x)$ is said to be a strong quadratic Lyapunov function for the system Σ_{A_1}.

Similarly, if $P=P^T>0$ simultaneously satisfies the non-strict inequalities

$$A_i^T P A_i - P = -Q_i \leq 0 \quad 1\leq i\leq M \quad (3)$$

we say that $V(x)=x^TPx$ is a *weak CQLF* for the systems Σ_{A_i}. A weak quadratic Lyapunov function for a single system is defined in the obvious manner.

The notion of a matrix pencil, defined below, will be convenient for expressing our later results.

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[1]referred to as the Stein inequalities by some authors

0-7803-7896-2/03/$17.00

Proceedings of the American Control Conference Denver, Colorado June 4-6, 2003

The matrix pencil $\sigma_{\gamma[0,\infty)}[A_1, A_2]$:

Given $A_1, A_2 \in \mathbb{R}^{n\times n}$, the matrix pencil $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ is the parameterised family of matrices

$$\sigma_{\gamma[0,\infty)}[A_1, A_2] = \{A_1 + \gamma A_2 : \gamma \in [0, \infty)\}. \qquad (4)$$

We say that the pencil is *non-singular* if $A_1 + \gamma A_2$ is non-singular for all $\gamma \geq 0$. Otherwise the pencil is said to be *singular*.

3 Some Preliminary Lemmas

In this section, we state without proof a number of lemmas that are needed to establish the results of the following sections. For details of the proofs of these lemmas, consult the report [16]. The following well-known lemma provides a convenient test for singularity of a matrix pencil.

Lemma 2.1 : Let A_1, A_2 be non-singular matrices in $\mathbb{R}^{n\times n}$. The pencil $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ is singular if and only if the matrix product $A_1 A_2^{-1}$ has a negative (real) eigenvalue.

We next record the simple observation that the quadratic Lyapunov functions for the stable discrete-time LTI systems Σ_A and Σ_{-A} coincide. This relates to the result in [17] identifying the quadratic Lyapunov functions for Σ_A and $\Sigma_{A^{-1}}$ for continuous-time systems. A related observation was also made in [18].

Lemma 2.2 : Consider the stable discrete-time LTI systems

$$\begin{aligned} \Sigma_A : x(k+1) &= Ax(k) \\ \Sigma_{-A} : x(k+1) &= (-A)x(k). \end{aligned}$$

Then, any quadratic Lyapunov function for Σ_A is also a quadratic Lyapunov function for Σ_{-A}.

Lemma 2.3 : Let $u, v, x, y \in \mathbb{R}^n$ be any four non-zero vectors. There exists a non-singular $T \in \mathbb{R}^{n\times n}$ such that each component of the vectors Tu, Tv, Tx, Ty is non-zero.

The next result (Lemma 2.4) establishes a convenient relationship between two parameterizations of the same hyperplane in the space of symmetric matrices in $\mathbb{R}^{n\times n}$.

Lemma 2.4 : Let x, y, u, v be 4 non-zero vectors in $\mathbb{R}^n$. Suppose that there is some $k > 0$ such that for all $n \times n$ real symmetric matrices H

$$x^T H y = -k u^T H v.$$

Then either

$$x = \alpha u \text{ for some real scalar } \alpha, \text{ and } y = -(\frac{k}{\alpha})v$$

or

$$x = \beta v \text{ for some real scalar } \beta \text{ and } y = -(\frac{k}{\beta})u.$$

4 Main results

The principal result of this paper concerns two stable discrete-time LTI systems for which no strong CQLF exists but for which a weak CQLF exists with each of the $Q_i, i \in \{1, 2\}$ in (3) of rank $n-1$. In Theorem 3.1 we provide a simple algebraic characterisation of this situation. The result is of interest for any class of systems where the transition from the existence of a CQLF to the non-existence of a CQLF passes through the situation described in the theorem.

Remark: It is possible to show that for any Schur-stable matrix $A \in \mathbb{R}^{n\times n}$, the set of matrices $P = P^T$ satisfying

$$A^T PA - P = -Q \quad Q \geq 0, \, rank(Q) = n-1$$

is dense in the set of matrices satisfying

$$A^T PA - A = -Q \quad Q \geq 0, \, det(Q) = 0.$$

This indicates that the situation described in the theorem is potentially of great importance in providing insight into the existence question for strong and weak CQLFs.

Before stating Theorem 3.1, we introduce the notation $C(A) = (A - I)(A + I)^{-1}$ for A in $\mathbb{R}^{n\times n}$. Note that $C(A)$ is well-defined for any Schur-stable A.

Theorem 3.1 : Let $\Sigma_{A_1}, \Sigma_{A_2}$ be two stable discrete-time LTI systems such that a solution $P = P^T > 0$ exists to the non-strict Lyapunov equations

$$A_1^T PA_1 - P = -Q_1 \leq 0, \qquad (5)$$

$$A_2^T PA_2 - P = -Q_2 \leq 0, \qquad (6)$$

for some positive semi-definite matrices Q_1, Q_2 both of rank $n-1$ ($n \geq 2$). Furthermore suppose that the systems $\Sigma_{A_1}, \Sigma_{A_2}$ do not have a strong CQLF. Under these conditions, at least one of the pencils $\sigma_{\gamma[0,\infty)}[C(A_1), C(A_2)]$, $\sigma_{\gamma[0,\infty)}[C(A_1), C(A_2)^{-1}]$ is singular, and equivalently, at least one of the matrix products $C(A_1)C(A_2)$ and $C(A_1)C(A_2)^{-1}$ has a real negative eigenvalue.

Proof: As Q_1 and Q_2 are of rank $n-1$, there are non-zero vectors x_1, x_2 such that

$$x_i^T Q_i x_i = 0, \; i = 1, 2 \qquad (7)$$

The proof of Theorem 3.1 is split into two main stages.

Stage 1 : The first stage in the proof is to show that if there exists a real symmetric matrix H satisfying

$$x_1^T(A_1^T H A_1 - H)x_1 < 0 \quad (8)$$

$$x_2^T(A_2^T H A_2 - H)x_2 < 0 \quad (9)$$

then $\Sigma_{A_1}, \Sigma_{A_2}$ would have a strong CQLF.

So assume that there is some H satisfying (8), (9), and consider the set

$$\Omega_1 = \{x \in \mathbb{R}^n : \|x\| = 1, x^T(A_1^T H A_1 - H)x \geq 0\}.$$

(Here $\|x\|$ is the usual Euclidean norm on $\mathbb{R}^n$.)

We shall show that there is a positive constant $C_1 > 0$ such that $A_1^T(P + \delta_1 H)A_1 - (P + \delta_1 H)$ is negative definite provided that $0 < \delta_1 < C_1$.

Firstly suppose that Ω_1 was empty. Then $A_1^T(P + \delta_1 H)A_1 - (P + \delta_1 H)$ is negative definite for any $\delta_1 > 0$. So any positive constant C_1 will work in this case.

Now, assume that the set Ω_1 is non-empty. The function that takes x to $x^T(A_1^T H A_1 - H)x$ is continuous. Thus Ω_1 is closed and bounded, hence compact. Furthermore x_1 (or any non-zero multiple of x_1) is not in Ω_1 and thus $x^T(A_1^T P A_1 - P)x$ is strictly negative on Ω_1.

Let M_1 be the maximum value of $x^T(A_1^T H A_1 - H)x$ on Ω_1, and let M_2 be the maximum value of $x^T(A_1^T P A_1 - P)x$ on Ω_1. Then by the final remark in the previous paragraph, $M_2 < 0$. Choose any constant $\delta_1 > 0$ such that

$$\delta_1 < \frac{|M_2|}{M_1 + 1}$$

and consider the matrix

$$P + \delta_1 H.$$

By separately considering the cases $x \in \Omega_1$ and $x \notin \Omega_1$, $\|x\| = 1$, it is easy to see that for all non-zero vectors x of norm 1

$$x^T(A_1^T(P + \delta_1 H)A_1 - (P + \delta_1 H))x < 0$$

provided $0 < \delta_1 < \frac{|M_2|}{M_1+1}$. Let C_1 denote the value $\frac{|M_2|}{M_1+1}$. Thus we have shown that there is some positive constant C_1 such that $A_1^T(P + \delta_1 H)A_1 - (P + \delta_1 H)$ is negative definite provided that $0 < \delta_1 < C_1$.

Now the same argument can be used to guarantee the existence of a positive constant C_2 such that

$$x^T(A_2^T(P + \delta_1 H)A_2 - (P + \delta_1 H))x < 0.$$

for all non-zero x provided we choose $0 < \delta_1 < C_2$. So, if we choose $\delta > 0$ less than the minimum of C_1, C_2, we would have a real symmetric matrix

$$P_1 = P + \delta H$$

satisfying (2) with $Q_1, Q_2 > 0$. This implies that $P_1 > 0$ [14] and thus $V(x) = x^T P_1 x$ would be a strong CQLF for $\Sigma_{A_1}, \Sigma_{A_2}$.

Stage 2 : So, under our assumptions there is no real symmetric matrix H such that

$$x_1^T(A_1^T H A_1 - H)x_1 < 0 \quad (10)$$

$$x_2^T(A_2^T H A_2 - H)x_2 < 0. \quad (11)$$

Thus, the two linear functionals defined on the space of real symmetric matrices in $\mathbb{R}^{n \times n}$ by

$$H \rightarrow x_i^T(A_i^T H A_i - H)x_i \quad i \in \{1, 2\}$$

must have the same kernel. This together with the fact that there is no H satisfying (10), (11) implies that there is some positive constant k such that

$$x_1^T(A_1^T H A_1 - H)x_1 = -k x_2^T(A_2^T H A_2 - H)x_2 \quad (12)$$

for all real symmetric matrices H.

Expanding the expression

$$(A_i x_i - x_i)^T H (A_i x_i + x_i)$$

and noting that, for symmetric H,

$$x_i^T A_i^T H x_i - x_i^T H A_i x_i = 0,$$

we see that the two expressions

$$x_i^T(A_i^T H A_i - H)x_i \quad (13)$$
$$(A_i x_i - x_i)^T H (A_i x_i + x_i) \quad (14)$$

are identical for all symmetric $H \in \mathbb{R}^{n \times n}$ for $i = 1, 2$.

Combining this fact with (12) and applying Lemma 2.4 now shows that either

$$\begin{aligned}(A_1 x_1 + x_1) &= \alpha(A_2 x_2 + x_2), \\ (A_1 x_1 - x_1) &= -\frac{k}{\alpha}(A_2 x_2 - x_2)\end{aligned} \quad (15)$$

or

$$\begin{aligned}(A_1 x_1 + x_1) &= \alpha(A_2 x_2 - x_2), \\ (A_1 x_1 - x_1) &= -\frac{k}{\alpha}(A_2 x_2 + x_2).\end{aligned} \quad (16)$$

In the first case (15), we have

$$x_1 = \alpha(A_1 + I)^{-1}(A_2 + I)x_2$$

and substituting this into the second identity in (15) yields

$$(A_1 - I)(A_1 + I)^{-1}(A_2 + I)x_2 = -\frac{k}{\alpha^2}(A_2 - I)x_2$$

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Letting $y = (A_2 + I)x_2$ we see that

$$(C(A_1) + \frac{k}{\alpha^2}C(A_2))y = 0$$

and hence the pencil $\sigma_{\gamma[0,\infty)}[C(A_1), C(A_2)]$ is singular and the product $C(A_1)C(A_2)^{-1}$ has a negative eigenvalue. A similar argument shows that in the case (16), the pencil $\sigma_{\gamma[0,\infty)}[C(A_1), C(A_2)^{-1}]$ is singular and the product $C(A_1)C(A_2)$ has a negative eigenvalue. This completes the proof of Theorem 3.1.

Remarks:

(i) It is worth noting that the so-called *bilinear transform*, $C(A) = (A - I)(A + I)^{-1}$ [19, 20], appears naturally in the course of the proof of Theorem 3.1.

(ii) The positive definite P assumed in the statement of Theorem 3.1 need only be semi-definite. The conclusions of the theorem are still valid in this case.

(iii) A crucial point in relation to Theorem 3.1 is that there is a hyperplane separating the two convex sets $\{P : A_i^T PA_i - P < 0\}$, $i = 1, 2$. Essentially, the effect of the rank $n - 1$ assumption is that this hyperplane is unique.

5 Second order systems

In this section we present an example to illustrate the use of Theorem 3.1.

Example: Second order systems

Let Σ_{A_1} and Σ_{A_2} be stable discrete-time LTI systems with $A_1, A_2 \in \mathbb{R}^{2\times2}$. We note the following readily verifiable facts.

(a) If $P = P^T$ satisfies $A^T PA - P = -Q \leq 0$, then $C(A)^T P + PC(A) = -Q' \leq 0$ with $Q' = 2(A + I)^{-T} Q(A + I)^{-1}$ having the same rank as Q.

(b) If a CQLF exists for Σ_{A_1} and Σ_{A_2}, then a CQLF (in the continuous-time sense) exists for the systems $\Sigma_{C(A_1)}$, $\Sigma_{C(A_2)}$, and the pencils $\sigma_{\gamma[0,\infty)}[C(A_1), C(A_2)]$ and $\sigma_{\gamma[0,\infty)}[C(A_1), C(A_2^{-1})]$ must consist entirely of Hurwitz matrices.

(c) If a CQLF does not exist for Σ_{A_1} and Σ_{A_2} then a CQLF (in the continuous-time sense) does not exist for the continuous-time systems $\Sigma_{C(A_1)}$, $\Sigma_{C(A_2)}$. However by choosing $d > 0$ sufficiently large, we can ensure that a CQLF (in the continuous-time sense) exists for $\Sigma_{C(A_1)-dI}$ and $\Sigma_{C(A_2)}$. A continuity argument can be employed to show that there is some d_1 with $0 < d_1 < d$ such that $C^{-1}(C(A_1) - d_1 I)$ and A_2 satisfy Theorem 3.1, and thus one of the pencils $\sigma_{\gamma[0,\infty)}[C(A_1) - d_1 I, C(A_2)]$ and $\sigma_{\gamma[0,\infty)}[C(A_1) - d_1 I, C(A_2)^{-1}]$ is necessarily singular. Hence, it follows that one of the pencils $\sigma_{\gamma[0,\infty)}[C(A_1), C(A_2)]$, $\sigma_{\gamma[0,\infty)}[C(A_1), C(A_2)^{-1}]$ is not Hurwitz.

Items (a) - (c) establish the following facts. Given two stable discrete-time second order LTI systems Σ_{A_1} and Σ_{A_2}, a necessary condition for the existence of a CQLF is that the pencils $\sigma_{\gamma[0,\infty)}[C(A_1), C(A_2)]$ and $\sigma_{\gamma[0,\infty)}[C(A_1), C(A_2)^{-1}]$ are Hurwitz. Conversely, if a CQLF does not exist for Σ_{A_1}, Σ_{A_2}, then one of the pencils $\sigma_{\gamma[0,\infty)}[C(A_1), C(A_2)]$, $\sigma_{\gamma[0,\infty)}[C(A_1), C(A_2)^{-1}]$ is not Hurwitz. Together these conditions yield the following result which is closely related to that presented in [11] and is the discrete-time counterpart of results presented in [10].

A necessary and sufficient condition for the stable discrete-time LTI systems Σ_{A_1} and Σ_{A_2}, $A_1, A_2 \in \mathbb{R}^{2\times2}$, to have a CQLF is that the pencils $\sigma_{\gamma[0,\infty)}[C(A_1), C(A_2)]$ and $\sigma_{\gamma[0,\infty)}[C(A_1), C(A_2)^{-1}]$ are Hurwitz.

6 Concluding remarks

In this paper we have derived a CQLF non-existence theorem. We have applied this theorem to derive a CQLF existence result for a pair of stable LTI systems that belong to a certain system class. We believe that our result can be applied to derive similar results for pairs of stable LTI systems belonging to other important system classes.

Acknowledgements: This work was partially supported by the European Union funded research training network *Multi-Agent Control*, HPRN-CT-1999-00107[2] and by the Enterprise Ireland grant SC/2000/084/Y. Neither the European Union or Enterprise Ireland is responsible for any use of data appearing in this publication.

The authors gratefully acknowledge several discussions with Professor Kumpati S. Narendra.

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