DIAS-STP-94-05
February 1994

Co-variant Derivatives And The Renormalisation Group Equation *

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ABSTRACT

The renormalisation group equation for N-point correlation functions can be interpreted in a geometrical manner as an equation for Lie transport of amplitudes in the space of couplings. The vector field generating the diffeomorphism has components given by the β-functions of the theory. It is argued that this simple picture requires modification whenever any one of the points at which the amplitude is evaluated becomes close to any other. This modification requires the introduction of a connection on the space of couplings and new terms appear in the renormalisation group equation involving co-variant derivatives of the β-function and the curvature associated with the connection. It is shown how the connection is related to the operator expansion co-efficients, but there remains an arbitrariness in its definition.

* Work partly supported by an Alexander von Humboldt research stipendium.

§1 Introduction

Geometry has always played a central role in the development of theoretical physics. Recently a new possibility for the application of geometry to physics has emerged, that is the use of geometrical concepts to understand the "space of theories", [1] [2] [3] [4] [5] . In this approach to relativistic quantum field theory or stastical mechanics the couplings which parameterise a theory, e.g. masses, gauge couplings, Yukawa couplings etc., are viewed as parameters on some space $\mathcal{G}$ and it is the geometry of this space which is studied. It is in principle an infinite dimensional space as there are an infinite number of operators that one can introduce into any given theory, but there are circumstances where one might hope that studying a finite dimensional subspace may prove to be sufficient. For example in a renormalisable field theory there are generically only a finite number of operators that can be included in the bare action and the properties of all other operators should be determined by these "basic" operators alone (for an interacting field theory these must include composite operators). In such circumstances $\mathcal{G}$ is finite dimensional and can be parameterised by the couplings g^a associated with each basic operator.

One is then tempted to think of $\mathcal{G}$ as being a differentiable manifold, in which case g^a would be thought of as co-ordinates with $a = 1, \ldots, n$, where n is the dimension of $\mathcal{G}$ - for example it has been proposed that the space of couplings for the dynamics responsible for the quantum Hall effect can usefully be identified with the Lobachevski plane, [6] . What would be the geometrical properties of $\mathcal{G}$ in this approach in general? For example one might seek a consistent definition of a metric on $\mathcal{G}$, this would give a notion of the physical "distance" between two theories. A reasonable criterion for a metric is that it should be related to the two point functions of the theory, [5] [7] . A connection on $\mathcal{G}$ would also be important to give a rule for transporting tensors around. We shall see that physical amplitudes of the theory can be thought of as tensors on $\mathcal{G}$. Co-variant differentiation therefore would give a rule for comparing physical amplitudes for different theories (by this is meant theories with the same field content but different values of the couplings). Unfortunately there is as yet no clear physical definition of a connection, though some suggestions have been made, [1] [2] [3] [4]. The connection is related to the operator expansion co-efficients, and this relationship will be investigated in detail in section five. A precise determination of the connection will not be attempted here, however, rather the existence of a connection will simply be be assumed and some inferences will be drawn.

A more primitive form of differentiation also exists in differential geometry apart from co-variant differentiation, that of Lie differentiation, but the definition of a Lie derivative requires choosing a vector field - it is not intrinsic to the basic geometry of the underlying manifold. It is more primitive in the sense that it does not require either a metric or a connection for its definition and so does not rely on the geometry to the same extent as the co-variant derivative. It has been shown, [8] [9] , that the renormalisation group equation for N-point amplitudes can be thought of as an equation for Lie differentiation of the amplitudes along the vector field defined by the β-functions of the theory. From this perspective the anomalous dimensions are seen as arising from Lie differentiation of the basis vectors for the tangent space. From a geometrical point of view the vanishing of the anomalous dimension of the free energy is due to the fact that it is a scalar function

on $\mathcal{G}$, being the logarithm of the partition function, and is not a tensor.

However this interpretation of the renormalisation group equation as a Lie derivative is only valid when the points in Euclidean space at which the N-point function is evaluated, $x_1, \ldots, x_N$, are well separated with respect to the renormalisation length κ^{-1}.* It will be shown in this paper that there are corrections to this interpretation when some of these points get close together and that these corrections can be expressed by the introduction of a connection. One of the main results presented here is a co-variant generalisation of some formulae appearing in reference [10] for the way in which regularised N-point Green functions mix with lower N-point functions under changes in the renormalisation scale κ.

We shall first state the results. Denote the basic operators, which may be composite, by $[\Phi_a(p)]$ (these will be defined more precisely later) and the regularised Green functions in momentum space by

$$\begin{aligned} G^R(p,q)_{ab} &=< [\Phi_a(p)\Phi_b(q)] > \\ G^R(p,q,r)_{abc} &=< [\Phi_a(p)\Phi_b(q)\Phi_c(r)] > \\ G^R(p,q,r,s)_{abcd} &=< [\Phi_a(p)\Phi_b(q)\Phi_c(r)\Phi_d(s)] > \qquad \text{etc.}, \end{aligned} \tag{1}$$

(square brackets around an operator denote that it is regularised). Then, assuming $< [\Phi_a(p)] >= 0$ for simplicity, it will be shown that the renormalisation group equations for three and four point functions are

$$\begin{aligned} \left[\Big(\kappa\frac{\partial}{\kappa}\Big|_g + \mathcal{L}_\beta\Big)G^R(p,q,r)\right]_{abc} &= \tau_{ab}{}^d G^R_{dc}(p+q,r) + \tau_{bc}{}^d G^R_{da}(q+r,p) + \tau_{ca}{}^d G^R_{db}(r+p,q) \\ &\qquad + \cdots \\ \left[\Big(\kappa\frac{\partial}{\kappa}\Big|_g + \mathcal{L}_\beta\Big)G^R(p,q,r,s)\right]_{abcd} &= \big[\tau_{ab}{}^f G^R_{fcd}(p+q,r,s) + \ 5 \text{ terms}\big] \\ &\qquad - \big[\big(\bigtriangledown_a \tau_{bc}{}^f\big) G^R_{fd}(p+q+r,s) + \ 3 \text{ terms}\big] + \cdots \end{aligned} \tag{2}$$

The notation $\mathcal{L}_\beta$ denotes the Lie derivative e.g.,

$$\mathcal{L}_\beta G^R_{ab}{}^c = \beta^d \partial_d G^R_{ab}{}^c + \big(\partial_a \beta^d\big) G^R_{db}{}^c + \big(\partial_b \beta^d\big) G^R_{ad}{}^c - \big(\partial_d \beta^a\big) G^R_{ab}{}^d. \tag{3}$$

The matrix $\partial_a \beta^b$ is the matrix of anomalous dimensions which mixes the operators under renormalisation. In equation (2) dots denote terms which are monomials of the momenta to the power D, such terms are only significant when all N points of the N-point function are so close to one another that they are unresolvable on the length scale κ^{-1}.

The tensor $\tau_{ab}{}^c$ appearing in the RG equations above is symmetric in a and b and involves the second co-variant derivative of the β-functions,

$$\tau^a{}_{bc} = \bigtriangledown_b \bigtriangledown_c \beta^a - R^a{}_{cbd} \beta^d, \tag{4}$$

* For simplicity we shall take the underlying physical space in which the theory is formulated to be D-dimensional Euclidean space, $\mathbf{R}^D$.

with $R^a{}_{cbd}$ the curvature associated with the connection. This tensor is central to the treatment presented here as it governs the way in which N-point functions (which are rank N co-variant tensors in the co-tangent space $T^*(\mathcal{G})$ at the point g) mix with tensors of lower rank, it is a co-variant generalisation of $\partial_a \partial_b \beta^c$ which appears in [10].

The connection can be related to the operator product expansion co-efficients. The idea that a connection should be related to the OPE co-efficients has been expressed before, [2] [3]. In particular the analysis of the OPE presented in section five is similar in spirit to that of Sonoda [3], but it is implemented in a different way in that reference. The OPE co-efficients have also been related to the second derivatives of the β-functions, at least near a critical point, by Zamolodchikov, [11] . He defines quantities $\tilde{C}^a_{bc}$ which are related to the OPE co-efficients $C^a{}_{bc}(x)$ and shows that $\tilde{C}^a_{bc} = \partial_b \partial_c \beta^a$.

The second main result of this paper is that certain OPE co-efficients satisfy the following renormalisation group equation, in momentum space,

$$\big(\kappa \frac{\partial}{\partial \kappa}\Big|_g + \mathcal{L}_\beta\big) C^{R\,c}_{ab}(p) = \tau_{ab}{}^c + \cdots, \tag{5}$$

where $C^{R\,c}_{ab}(p)$ are regularised OPE co-efficients, in the sense that their integral over all p (or equivalently over all space) is finite. This equation is a co-variant generalisation of a result presented in [12] . It relates the OPE co-efficients to the connection via the tensor $\tau_{ab}{}^c$.

The layout of the paper is as follows. In section two the renormalisation group equation, including the possibility of composite operators, will be discussed from a geometrical point of view. It will be argued that, at least when all of the points are well separated, the equation reduces to nothing more than the definition of the Lie derivative of tensors on $\mathcal{G}$ with respect to the vector field given by the β-functions of the theory. In section three the necessary changes required to regulate Green functions when two of the points get close to one another are discussed and the technique of using position dependent couplings is outlined. The resulting expressions are not co-variant under general co-ordinate transformations on $\mathcal{G}$. Section four is devoted to the development of co-variant expressions and a co-variant renormalisation group equation. It is shown how the renormalisation group flow mixes up tensors of different rank. In section five the operator product expansion co-efficients are discussed and a co-variant renormalisation group equation for them is derived. It is argued that the connection is related to the OPE co-efficients. In section six the results are summarised and some comments are made on possible future directions of development. A derivation of the non-covariant expression for regularised N-point functions with arbitrary N is given in an appendix, for massless theories in four dimensions. A second appendix gives the co-variant renormalisation group equation for four point functions with arbitrary momenta.

§2 The Renormalisation Group Equation

The regularisation of quantum amplitudes involving composite operators is more subtle than for those involving just elementary fields, at short distances new divergences appear over and above those present in the standard treatments. In reference [10] a technique was described for dimensional regularisation of amplitudes involving only those composite operators associated with couplings appearing in the original Lagrangian of the theory and this was extended in [12] to include more general composite operators. For example if we consider the renormalised composite operator $[\phi^4]$ in $\lambda\phi^4$ theory in four dimensions then the operator $[\phi^4(x)][\phi^4(y)]$ is singular as $x \to y$ and requires a new subtraction (here and subsequently square brackets around an operator or product of operators means that it is regularised, thus $[\phi^4(x)][\phi^4(y)] \neq [\phi^4(x)\phi^4(y)]$). The technique adopted in [10] involves defining 'basic' composite operators, one associated with every coupling g^{a_0} appearing in the bare Langrangian density $L_0(x)$. These basic operators are given by $\Phi_{a_0}(x) = \partial_{a_0} L_0(x)$. For example in $\lambda\phi^4$ in four dimensions for $g^{a_0} = \lambda_0$ one has $\Phi_{\lambda_0} = \frac{1}{4!}\phi_0^4$. In a renormalisable theory there are a finite number n of these operators, where n is the number of couplings $a_0 = 1, \ldots, n$. Φ_{a_0} are of course bare operators. Renormalised operators can be defined by $[\Phi_a(x)] = Z_a{}^{b_0}\Phi_{b_0}(x)$ where $Z_a{}^{b_0}$ is a matrix of renormalisation co-efficients which mixes operators. This matrix can be interpreted as a co-ordinate transformation matrix

$$Z_a{}^{b_0} = \frac{\partial g^{b_0}}{\partial g^a} \tag{6}$$

in which g^a are renormalised couplings and $[\Phi_a(x)] = \partial_a L_0(x)$. Thus the space of couplings $\mathcal{G}$ is viewed as a n-dimensional differentiable manifold with g^a and g^{a_0} being different co-ordinate systems on $\mathcal{G}$. The bare couplings, $g^{a_0}(g^a, \epsilon)$, are analytic functions of g^a and of the regularisation parameter ϵ, provided $\epsilon \neq 0$ ($\epsilon = D - 4$ in dimensional regularisation). The matrix $Z_a{}^{b_0}$ is a co-ordinate transformation matrix. Viewed from this geometric perspective the quantities

$$\Phi(x) = [\Phi_a(x)]dg^a = \Phi_{a_0}(x)dg^{a_0} \tag{7}$$

are operator valued one-forms on the co-tangent space $T^*(\mathcal{G})$. This picture has also proven useful in conformal field theories in two dimensions where the operators $[\Phi_a]$ are primary fields, [11].

N-point Green functions are now rank N tensors on $\mathcal{G}$. Provided all the points x_i are well separated,

$$G^{(N)}_{a_1\cdots a_N}(x_1, \ldots, x_N) = < [\Phi_{a_1}(x_1)] \cdots [\Phi_{a_N}(x_N)] > . \tag{8}$$

Note that in general the tensor $G^{(N)}_{a_1\cdots a_N}(x_1, \ldots, x_N)$ has no particular symmetry properties. When all the x_i are well separated the renormalisation group equation has a very simple geometrical interpretation, it is simply the Lie derivative of $G^{(N)}_{a_1\cdots a_N}$ with respect to the vector field on $T(\mathcal{G})$ given by the β-functions of the theory $\vec{\beta} = \beta^a \frac{\partial}{\partial g^a}$, [8] [9]. To see

this we simply write the N-point functions with the basis dg^a for real valued one-forms included

$$\begin{aligned} G^{(N)} &=< [\Phi_{a_1}(x_1)]\cdots[\Phi_{a_N}(x_N)] > dg^{a_1}\cdots dg^{a_N} \\ &=< \Phi_{a_{01}}(x_1)\cdots\Phi_{a_{0N}}(x_N) > dg^{a_{01}}\cdots dg^{a_{0N}}. \end{aligned} \tag{9}$$

The usual renormalisation group argument is now applied to $G^{(N)}$, it should be independent of the renormalisation point κ. Thus

$$\kappa\frac{d}{d\kappa}G^{(N)} = \big(\kappa\frac{\partial}{\partial\kappa}\Big|_g + \mathcal{L}_\beta\big)G^{(N)} = 0. \tag{10}$$

This immediately leads to

$$\begin{aligned} &\kappa\frac{\partial}{\partial\kappa}\Big|_g G^{(N)}_{a_1\cdots a_N}(x_1,\ldots,x_N) = \\ &\qquad -\beta^b\partial_b G^{(N)}_{a_1\cdots a_N}(x_1,\ldots,x_N) - \sum_{i=1}^{N}\big(\partial_{a_i}\beta^b\big)G^{(N)}_{a_1\cdots a_{i-1}ba_{i+1}\cdots a_N}(x_1,\ldots,x_N), \end{aligned} \tag{11}$$

where we have used

$$\kappa\frac{d}{d\kappa}dg^a = d\Big(\kappa\frac{dg^a}{d\kappa}\Big) = d\beta^a = \partial_b\beta^a dg^b. \tag{12}$$

The matrix of anomalous dimensions, $\partial_b\beta^a$, is thus seen to come from Lie dragging of the basis one-forms dg^a. Note that equation (11) is co-variant under general co-ordinate transformations, even though the derivatives on the right hand side are not co-variant, because the Lie derivative is co-variant by construction, [13] . There is no need to introduce a connection to define Lie derivatives. However the interpretation of the matrix $\partial_a\beta^b$ as having physical significance is tied in to a special choice of co-ordinates. More generally one would expect a co-variant generalisation of this matrix, $\bigtriangledown_a\beta^b$, to have the physical interpretation of a matrix of anomalous dimensions, [8].

This treatment of the RG equation, though conceptually simple, is not the whole story. We must be careful to regularise $[\Phi_{a_i}(x_i)][\Phi_{a_j}(x_j)]$ whenever any two of the points x_i and x_j start getting close to one another. The operator product expansion co-efficients clearly play an important role here and this combination becomes a single renormalised composite operator as $x_i \to x_j$. Thus the regularised Green functions,

$$G^{R(N)}_{a_1\cdots a_N}(x_1,\cdots x_N) =< [\Phi_{a_1}(x_1)\cdots\Phi_{a_N}(x_N)] >, \tag{13}$$

are linear combinations of all the lower, unregularised ones, $G^{(M)}$ for $M \leq N$ including $M = 0$. This phenomenon manifests itself at the level of the renormalisaion group by the fact that $G^{R(N)}$ gets mixed up with tensors of lower rank under RG flow. This mixing was exhibited in [10], but the tensor expressions in that reference were not co-variant. For example the mixing co-efficients involved the second derivative of the β-functions, $\partial_a\partial_b\beta^c$ which is clearly not a tensor and this can only be consistent if it is legitimate to put a flat connection on $T(\mathcal{G})$ and a co-ordinate system can be found in which the connection

co-efficients vanish (e.g. if $T(\mathcal{G})$ admits a flat metric, with g^a Cartesian co-ordinates, and the connection is the Levi-Civita connection). The remedy for this problem is pointed out in [10], a connection on $T(\mathcal{G})$ must be introduced. It is even indicated how this should be done, but the authors do not do it because they do not know how to calculate the connection, in general. However a connection must be introduced, whether it be flat or not, in order to make the renormalisation group equation co-variant and the approach adopted here will be to introduce one, without any prescription as to how it might be calculated, and co-variant expressions will be derived.

To appreciate the necessity of the aforementioned regularisation consider the two point functions, $G_{ab}(x_1, x_2)$, for a theory in flat Euclidean space, $\mathbf{R}^D$, where $D = 4 - \epsilon$ and a and b are indices associated with dimensionless couplings. In dimensional regularisation $[\Phi_a]$ are dimension $4-\epsilon$ operators. The renormalised two point Green functions are of the form,* [14]

$$G^R_{ab}(x, y) = G_{ab}(x, y) + \kappa^{-\epsilon} A_{ab} \Box\Box\, \delta(x - y), \tag{14}$$

where $A_{ab}(g, \epsilon)$ is a tensor on $\mathcal{G}$, independent of x and y, but depending on g^a and containing poles in ϵ in general. $\Box$ is the four dimensional Laplacian in Euclidean space, $\Box = \partial^\mu \partial_\mu$. The tensor $A_{ab}(g, \epsilon)$ is chosen to cancel the singularities at $x \approx y$ in $G_{ab}(x, y)$, so as to render $\int d^D x G^R_{ab}(x, y)$ finite. These counterterms introduce corrections into the RG equation which will be developed in section four. First we develope a technique for determining regularised N-point functions for general N.

§3 Regularised N-point Functions

The technique developed in [10] for handling the counterterms described in the previous section will now be summarised, leaving out the connection on $T^*(\mathcal{G})$ until the next section. Expressions for the regularised N-point function, in the absence of a connection, can be derived by induction. The renormalised Green functions are obtained by introducing position dependent renormalised couplings, $g^a(x)$ so that $[\Phi_a(x)] = \frac{\delta S_0}{\delta g_a(x)}$, where S_0 is the action $S_0 = \int d^D x L_0(x)$ with x^μ Cartesian co-ordinates on $\mathbf{R}^D$. Next a counterterm proportional to the identity, which involves derivatives of $g^a(x)$, is subtracted from the bare action,

$$\tilde{S}_0(g, \epsilon) = S_0(g, \epsilon) - \frac{\kappa^{-\epsilon}}{2} \int_{\mathbf{R}^D} d^D x A_{ab} \Box\, g^a \Box\, g^b. \tag{15}$$

Defining the generating functional in the usual way,

$$W = -\ln \mathcal{Z} \qquad \text{where} \qquad \mathcal{Z} = \int \mathcal{D}\varphi e^{-\tilde{S}_0}, \tag{16}$$

allows the regularised N-point functions to obtained by functional differentiation. Thus

$$G^R_{a_1 \cdots a_N}(x_1, \ldots, x_N) = (-1)^{N+1} \frac{\delta^N W}{\delta g^{a_1}(x_1) \cdots \delta g^{a_N}(x_N)}. \tag{17}$$

* Henceforth the superscript (N) on N-point functions will be omitted since it is clear from the index structure on G which value of N is under consideration.

Now the two point functions (14) can be obtained from

$$G^R_{ab}(x,y) = \frac{\delta^2 \ln Z}{\delta g^a(x)\delta g^b(y)}\Big|_{\partial_\mu g = 0} = \Big\langle \frac{\delta \tilde{S}_0}{\delta g^a(x)} \frac{\delta \tilde{S}_0}{\delta g^b(y)} \Big\rangle\Big|_{\partial_\mu g = 0} - \Big\langle \frac{\delta^2 \tilde{S}_0}{\delta g^a(x)\delta g^b(y)} \Big\rangle\Big|_{\partial_\mu g = 0}, \tag{18}$$

where it is assumed for simplicity that

$$< [\phi_a(x)] > |_{\partial_\mu g^a = 0} = 0. \tag{19}$$

This assumption simplifies some of the equations. Non-zero expectation values are easily accounted for by using (11) with $N = 1$ giving

$$\kappa \frac{\partial}{\partial \kappa}\Big|_g G_a = -\beta^b \partial_b G_a - \partial_a \beta^b G_b \tag{20}$$

(one point Green functions, G_a, are of course independent of position because of translational invariance).

The new counter term in (15) gives rise to

$$\begin{aligned} \frac{\delta^2 \tilde{S}_0}{\delta g^a(x)\delta g^b(y)}\Big|_{\partial_\mu g = 0} &= \left(\frac{\delta}{\delta g^a(x)} [\phi_b(y)] \right)\Big|_{\partial_\mu g = 0} \\ &= \delta(x-y) K^c{}_{ab}[\phi_c(x)]\Big|_{\partial_\mu g = 0} - \kappa^{-\epsilon} A_{ab} \Box\Box \delta(x-y), \end{aligned} \tag{21}$$

where it is assumed that the bare basic operators are independent of the couplings, $\frac{\delta \phi_{b_0}(y)}{\delta g_{a_0}(x)} = 0$. The quantities $K^c{}_{ab}$ here are defined by

$$K^c{}_{ab}(g,\epsilon) = \partial_a \partial_b g^{d_0} \left(\frac{\partial g^c}{\partial g^{d_0}} \right) = \partial_a Z_b{}^{d_0} (Z^{-1})_{d_0}{}^c, \tag{22}$$

and contain poles in ϵ. Combining equations (18) and (21) now gives the regularised two point functions (14).

In four dimensions there are other counterterms proportional to the identity that can be added to S_0 which are necessary for regularisation when $N > 2$. For simplicity we shall assume that only couplings that are dimensionless in four dimensions appear in the Lagrangian (no masses). Including masses introduces more terms but is straightforward in principle.

By simple dimensional analysis only terms involving the appropriate number of derivatives of the dimensionless couplings can contribute. The most general counterterm, invariant under parity transformations, consists of the following combination (modulo integration by parts),

$$\tilde{S}_0(g,\epsilon) = S_0(g,\epsilon) - I_0(g,\epsilon),$$

where

$$\begin{aligned} I_0(g,\epsilon) &= \int_{\mathbf{R}^D} d^D x \mathcal{I}_0 \\ &= \frac{1}{2}\int_{\mathbf{R}^D} d^D x A_{ab}\Box g^a \Box g^b + \frac{1}{2}\int_{\mathbf{R}^D} d^D x B_{abc}\partial_\mu g^a \partial^\mu g^b \Box g^c \\ &\qquad + \frac{1}{4}\int_{\mathbf{R}^D} d^D x C_{abcd}\partial_\mu g^a \partial^\mu g^b \partial_\nu g^c \partial^\nu g^d. \end{aligned} \tag{23}$$

$B_{abc}(g,\epsilon)$ and $C_{abcd}(g,\epsilon)$ are new quantities with no explicit x and y dependence, but depending on $g^a(x)$ and containing poles in ϵ. As emphasised in [10] they are not tensors because they do not transform co-variantly but this will be remedied later when a connection on $T^*(\mathcal{G})$ is included. Note the symmetries $B_{abc} = B_{bac}$ and $C_{abcd} = C_{cdab} = C_{bacd} = C_{abdc}$.

The structure of this counterterm would be more complicated if there were masses around, but these can be treated by similar techniques and (23) will be sufficient for the purposes of illustration. I_0 also depends on the number of dimensions D, for example in two dimensions B_{abc} and C_{abcd} do not appear and

$$\mathcal{I}_0 = \frac{1}{2}A_{ab}\partial_\mu g^a \partial^\mu g^b. \tag{24}$$

The arguments here will be illustrated using (23). Thus, for example, the operator which gives a finite 3-point function is,

$$\begin{aligned} &\frac{\delta\tilde{S}_0}{\delta g^a(x)}\frac{\delta\tilde{S}_0}{\delta g^b(y)}\frac{\delta\tilde{S}_0}{\delta g^c(z)} - \frac{\delta^2\tilde{S}_0}{\delta g^a(x)\delta g^b(y)}\frac{\delta\tilde{S}_0}{\delta g^c(z)} - \frac{\delta^2\tilde{S}_0}{\delta g^a(x)\delta g^c(z)}\frac{\delta\tilde{S}_0}{\delta g^b(y)} \\ &\qquad - \frac{\delta^2\tilde{S}_0}{\delta g^b(y)\delta g^c(z)}\frac{\delta\tilde{S}_0}{\delta g^a(x)} + \frac{\delta^3\tilde{S}_0}{\delta g^a(x)\delta g^b(y)\delta g^c(z)}, \end{aligned}$$

and it involves B_{abc} as well as derivatives of A_{ab}.

In momentum space the regularised two, three and four point functions can thus be determined in terms of their unregularised counterparts and B_{abc} and C_{abcd} by setting $\partial_\mu g^a = 0$ in the appropriate finite operators. The two and three point functions are derived in reference [10] and are, after Fourier transforming to momentum space,

$$G^R_{ab}(p,q) = G_{ab}(p,q) + \kappa^{-\epsilon}A_{ab}p^2q^2, \qquad \text{with} \quad p+q=0 \tag{25}$$

$$\begin{aligned} G^R_{abc}(p,q,r) &= G_{abc}(p,q,r) - K^d{}_{ab}G_{dc}(p+q,r) - K^d{}_{bc}G_{da}(q+r,p) - K^d{}_{ca}G_{db}(r+p,q) \\ &\quad - \kappa^{-\epsilon}\big(p^2q^2A_{ab,c} + q^2r^2A_{bc,a} + r^2p^2A_{ca,b} + r^2p.qB_{abc} + p^2q.rB_{bca} + q^2r.pB_{cab}\big) \end{aligned}$$

with $p+q+r=0$. Using the same techniques the four point function can also be determined to be

$$\begin{aligned} G^R_{abcd}(p,q,r,s) =& G_{abcd}(p,q,r,s) - \big(K^e{}_{ab}G_{ecd}(p+q,r,s) + 5 \text{ permutations}\big) \\ &+ \big(K^e{}_{ab}K^f{}_{cd}G_{ef}(p+q,r+s) + 2 \text{ permutations}\big) \\ &+ \big(K^e{}_{abc}G_{ed}(-s,s) + 3 \text{ permutations}\big) \\ &+ \kappa^{-\epsilon}\big(r^2s^2A_{cd,ab} + 5 \text{ permutations}\big) \\ &+ \kappa^{-\epsilon}\big(p^2(r.s)B_{cda,b} + 11 \text{ permutations}\big) \\ &+ 2\kappa^{-\epsilon}\big((p.q)(r.s)C_{abcd} + 2 \text{ permutations}\big), \end{aligned} \tag{26}$$

with $p+q+r+s=0$. In these expressions a comma denotes partial differentiation with respect to the couplings g^a and $K^e{}_{abc}$ is defined as

$$K^a{}_{bcd} = \big(\frac{\partial^3 g^{f_0}}{\partial g^b \partial g^c \partial g^d}\big)\frac{\partial g^a}{\partial g^{f_0}} = \partial_d K^a{}_{bc} + K^e{}_{bc}K^a{}_{ed}. \tag{27}$$

In equation (26) some terms involving exceptional momenta (e.g. $p+q=r+s=0$) have been omitted for clarity. The reason such momenta cause extra terms to arise is that N-point amplitudes (17) are not simple expectation values of the basic operators, for $N \geq 4$. For example the four point function in position space, $G_{abcd}(x,y,z,t)$, involves products of two point functions $G_{ab}(x,y)G_{cd}(z,t)$. However such terms are discarded when momenta are non-exceptional because both the two point fucntions are separately translationally invariant. Thus in momentumm space this reads $G_{ab}(p,q)G_{cd}(r,s)\delta(p+q)\delta(r+s)$ and excluding exceptional momenta excludes such terms. This is also true of the expectation values themselves, $<[\Phi_a(p)]>$. Translational invariance demands that these vanish except at $p=0$, thus if they were to appear as factors in a N-point amplitude momentum conservation would require exceptional momenta among the other $N-1$ momenta. Hence excluding exceptional momenta automatically excludes expectation values for the basic operators. This simplifies some of the following formulae. In all of the ensuing expressions it will be assumed that none of the momenta is exceptional.

The structure of the terms involving the unregularised Green functions on the right hand side of equations (25) and (26) is independent of the dimension D in which we are working and is not affected by the introduction of masses. In other dimensions, or in theories with masses, only the A, B and C terms differ. e.g. in two dimensions only the A term is there (with $p.q$ instead of p^2q^2) since there are no B or C terms in two dimensions. Also extra terms independent of the $G^{(M)}$'s appear when there are masses.

The various terms in equation (26) can be given the following physical interpretation. The integral of the left hand side with respect to any of its arguments is finite and the terms on the right hand side involving three point functions are necessary in order to cancel singularities in the integral of the unregularised G_{abcd} that occur whenever one of the three independent momenta gets large. This happens if two of the four points get close in space so that the unregularised Green function becomes effectively a three point function multiplied by divergent operator product expansion co-efficients. The divergent part is extracted as a δ-function in position space and the three point functions appearing on the right hand side of (26) cancel this divergence in momentum space. These three point functions only depend on two of the momenta because of overall momentum conservation, e.g. $G_{ecd}(p+q,r,s) = G_{ecd}(-r-s,r,s)$ and independence of the third momentum corresponds to δ-function singularities in position space. The $K^a{}_{bc}$ are thus related to the operator product expansion co-efficients in a manner which will be analysed more fully later. Similarly the terms on the right hand side involving two point functions cancel singularities in double integrals of G_{abcd} that arise when two of the momenta get large. Lastly the terms involving momenta to the fourth power cancel the singularities that occur when all the four points in position space collapse to a single point in a triple integral - the momentum structure of these terms indicates that they correspond to fourth derivatives of δ-functions and are thus more singular than the other terms.

The regularised N-point functions can be obtained from this technique by induction and are given in an appendix. The expressions given here are however not co-variant under co-ordinate transformations on $\mathcal{G}$.

§4 A Co-variant Renormalisation Group Equation

The regularised N-point functions in the previous section have been derived assuming the connection vanishes. Everything can now be made co-variant by introducing a connection as described in [10]. The basic idea is the following. For position dependent couplings the matrix $M_\mu{}^a = \partial_\mu g^a$ gives a map between $T^*(\mathcal{G})$ and $T^*(\mathbf{R}^D)$,

$$\begin{array}{rccc} M_\mu{}^a: & T^*(\mathcal{G}) & \to & T^*(\mathbf{R}^D) \\ & \omega_a & \mapsto & \omega_\mu := M_\mu{}^a \omega_a, \end{array} \tag{28}$$

where ω_a is a one-form on $T^*(\mathcal{G})$ and ω_μ a one-form on $T^*(\mathbf{R}^D)$. Introduce a connection $\Gamma^a{}_{bc}$ on $T(\mathcal{G})$. Then for any vector $V^a \in T(\mathcal{G})$ a co-variant derivative $\bigtriangledown_\mu$ mapping $T(\mathcal{G}) \to T(\mathcal{G}) \otimes T^*(\mathbf{R}^D)$ can be defined by

$$\bigtriangledown_\mu V^a := \partial_\mu V^a + \mathcal{A}_\mu{}^a{}_b V^b \qquad \text{where} \qquad \mathcal{A}_\mu{}^a{}_b = M_\mu{}^c \Gamma^a{}_{cb}. \tag{29}$$

In this expression $\partial_\mu V^a$ is to be interpreted as $\partial_\mu g^b \partial_b V^a$. Thus $\bigtriangledown_\mu V^a = M_\mu{}^b \bigtriangledown_b V^a$. Now the Laplacian in flat Euclidean space acting on couplings is modified to read

$$\Box g^a \to \bigtriangledown^2 g^a = \bigtriangledown_\mu \partial^\mu g^a = \partial_\mu \partial^\mu g^a + \Gamma^a{}_{cb} \partial_\mu g^b \partial^\mu g^c. \tag{30}$$

The idea here is that the co-ordinates g^a are not vectors, they are n functions on $\mathcal{G}$, and $\partial^\mu g^a$ are n vectors on $T(\mathbf{R}^D)$. The co-varaint derivative is now used in the definition of I_0 in (23) to replace $\partial_\mu g^a$. Thus (23) now reads

$$\mathcal{I}_0 = \frac{1}{2} A_{ab} \bigtriangledown^2 g^a \bigtriangledown^2 g^b + \frac{1}{2} B_{abc} \partial_\mu g^a \partial^\mu g^b \bigtriangledown^2 g^c + \frac{1}{4} C_{abcd} \partial_\mu g^a \partial^\mu g^b \partial_\nu g^c \partial^\nu g^d \,, \tag{31}$$

where all three of A_{ab}, B_{abc} and C_{abcd} are now tensors. The A, B and C in (31) are not the same as those in (23) - they differ by terms involving the connection. From now on these symbols will refer exclusively to the co-variant forms in equation (31). Note that only the symmetric part of the connection, $\Gamma^a{}_{cb} = \Gamma^a{}_{bc}$, is relevant and so it is sufficient for our needs to take the connection to be symmetric.

One defines a curvature tensor in the usual way,

$$R^a{}_{bcd} = \partial_c \Gamma^a{}_{db} - \partial_d \Gamma^a{}_{cb} + \Gamma^a{}_{ec} \Gamma^e{}_{db} - \Gamma^a{}_{ed} \Gamma^e{}_{cb}. \tag{32}$$

It would be wrong to call $R^a{}_{bcd}$ a Riemann tensor as there is no definition of a metric on $T(\mathcal{G})$ here and hence no Riemannian structure, only a connection.

There is probably a natural definition of a connection for any given theory, e.g. the Knizhnik-Zamolodchikov connection for certain conformal theories (for which $R^a{}_{bcd} = 0$

despite global holonomy), [1], but the question of a general definition will not be addressed here. Rather it will just be assumed that one exists and no prescription will be given for calculating it, though it will be argued in the next section that it must be related to the operator product expansion co-efficients. Connections on the space of couplings in other theories are also discussed in, [1] [2] [3] [4].

The calculation of N-point functions proceeds in principle as before, though the introduction of the connection causes some extra complications. Consider the covariant analogue of (21)

$$\begin{aligned}\Big(\frac{\delta}{\delta g^a(x)}[\phi_b(y)] - \Gamma^c{}_{ab}[\phi_c(x)]\delta(x-y)\Big)\Big|_{\partial_\mu g=0} \\ = \delta(x-y)\big(K^c{}_{ab} - \Gamma^c{}_{ab}\big)[\phi_a(x)]\Big|_{\partial_\mu g=0} - \kappa^{-\epsilon}A_{ab}\Box\Box\,\delta(x-y).\end{aligned} \tag{33}$$

The difference $K^c{}_{ab} - \Gamma^c{}_{ab}$ is a tensor symmetric in a and b which will be denoted by $T^c{}_{ab}$. Of course $T^c{}_{ab}$ may contain poles in ϵ since $K^c{}_{ab}$ does, although $\Gamma^c{}_{ab}$ is assumed independent of ϵ and finite.

The regularised two point functions are the same as in equation (25) but the three point function is now

$$\begin{aligned}G^R_{abc}(p,q,r) = G_{abc}(p,q,r) - T^d{}_{ab}G_{dc}(p+q,r) - T^d{}_{bc}G_{da}(q+r,p) - T^d{}_{ca}G_{db}(r+p,q) \\ - \kappa^{-\epsilon}\big(p^2q^2A_{ab;c} + q^2r^2A_{bc;a} + r^2p^2A_{ca;b} + r^2p.qB_{abc} + p^2q.rB_{bca} + q^2r.pB_{cab}\big),\end{aligned} \tag{34}$$

where a semi-colon denotes co-variant differentiation, $A_{ca;b} = \bigtriangledown_b A_{ca}$.

For the four point function (26), however, the situation is much more complicated. The order in which the second derivatives on A_{ab} is taken is important. For simplicity, we shall restrict ourselves to the symmetric point in momentum space,

$$p_i.p_j = \frac{\mu^2}{3}\left(4\delta_{ij} - 1\right). \tag{35}$$

The co-variant result is

$$\begin{aligned}G^R_{abcd}(p,q,r,s) =& G_{abcd}(p,q,r,s) - \big[T^e{}_{ab}G_{ecd}(p+q,r,s) + 5 \text{ permutations}\big] \\ &+ \big[T^e{}_{ab}T^f{}_{cd}G_{ef}(p+q,r+s) + 2 \text{ permutations}\big] \\ &+ \left[\big(T^e{}_{bc;a} + T^f{}_{bc}T^e{}_{af}\big)G_{ed}(-s,s) + \left\{\begin{array}{c} b\leftrightarrow d \\ q\leftrightarrow s\end{array}\right\} + \left\{\begin{array}{c} c\leftrightarrow d \\ r\leftrightarrow s\end{array}\right\}\right] \\ &+ \big(T^e{}_{cd;b} + T^f{}_{cd}T^e{}_{bf}\big)G_{ea}(-p,p) \\ &+ \kappa^{-\epsilon}\mu^4\Big[\frac{1}{3}\big((R^f{}_{cba} + R^f{}_{bca})A_{fd} + (b\leftrightarrow d) + (c\leftrightarrow d)\big) \\ &+ \frac{1}{3}(R^f{}_{cdb} + R^f{}_{dcb})A_{fa} + \frac{1}{2}\big(A_{cd;a;b} + \text{ 11 permutations}\big) \\ &- \frac{1}{3}\big(B_{cdb;a} + 11 \text{ permutations}\big) + \frac{2}{9}\big(C_{abcd} + 2 \text{ permutations}\big)\Big].\end{aligned} \tag{36}$$

Note that the indices a and b now occur on a different footing to c and d. The expression is not so symmetric as (26) because co-variant derivatives do not commute.

The renormalisation group equation for regularised N-point functions will now be derived, for $N = 2, 3, 4$. The simple result (11), for the case when the points $x_1, \ldots, x_N$ are well separated, is modified by the mixing with the lower M-point functions ($M \leq N$).

The derivation involves β-functions for the counterterms (23),

$$\begin{aligned}\kappa\frac{d}{d\kappa}\mathcal{I}_0 &= \left(\kappa\frac{\partial}{\partial\kappa}\Big|_g + \beta^a\partial_a\right)\mathcal{I}_0 \\ &= -\frac{1}{2}\chi_{ab}\nabla^2 g^a \nabla^2 g^b - \frac{1}{2}\chi_{abc}\partial_\mu g^a \partial^\mu g^b \nabla^2 g^c - \frac{1}{4}\chi_{abcd}\partial_\mu g^a \partial^\mu g^b \partial_\nu g^c \partial^\nu g^d,\end{aligned} \tag{37}$$

where χ_{ab}, χ_{abc} and χ_{abcd} are finite functions of the renormalised couplings g, independent of ϵ. Using the expression (31) for $\mathcal{I}_0$ leads to the following equations for the χ's in terms of A_{ab}, B_{abc} and C_{abcd},

$$\begin{aligned}\chi_{ab} =& \epsilon A_{ab} - \big(\mathcal{L}_\beta A\big)_{ab} \\ \chi_{abc} =& \epsilon B_{abc} - \big(\mathcal{L}_\beta B\big)_{abc} - 2\big(\nabla_a\nabla_b\beta^d - R^d{}_{bae}\beta^e\big)A_{dc} \\ \chi_{abcd} =& \epsilon D_{abcd} - \big(\mathcal{L}_\beta D\big)_{abcd} - B_{abe}\left(\nabla_c\nabla_d\beta^e - R^f{}_{dce}\beta^e\right) - B_{cde}\left(\nabla_a\nabla_b\beta^e - R^e{}_{baf}\beta^f\right).\end{aligned} \tag{38}$$

Again the symbol $\mathcal{L}_\beta$ here denotes Lie differentiation with respect to the vector field β. The combination $\nabla_a\nabla_b\beta^c - R^c{}_{baf}\beta^f$ will occur so frequently in the sequel that it will be convenient to define $\tau_{ab}{}^c = \nabla_a\nabla_b\beta^c - R^c{}_{baf}\beta^f$. $\tau_{ab}{}^c$ is a tensor symmetric in the indices a and b.

The RG equation for N-point functions now follows by application of the operator

$$\kappa\frac{d}{d\kappa} = \kappa\frac{\partial}{\partial\kappa}\Big|_g + \mathcal{L}_\beta \tag{39}$$

to the regularised N-point functions. A considerable simplification is introduced by noting that $\kappa\frac{d}{d\kappa}$ acting on unregularised Green functions gives zero, $\kappa\frac{d}{d\kappa}G^{(N)} = 0$ (equation (11)). Another useful identity in the derivation is

$$\big(\mathcal{L}_\beta T\big)^a{}_{bc} = -\tau_{bc}{}^a. \tag{40}$$

Note that this expression is finite although $T^a{}_{bc}$ itself is not. Equation (40) is the co-varaint generalisation of equation (2.22) in reference [10].

For two and three point functions the renormalisation group equation is obtained by applying $\kappa\frac{d}{d\kappa}$ to equation (34) and using (38). The result is

$$\begin{aligned}\left[\Big(\kappa\frac{\partial}{\kappa}\Big|_g + \mathcal{L}_\beta\Big)G^R(p,q)\right]_{ab} &= -\kappa^{-\epsilon}(p.q)\chi_{ab} \\ \left[\Big(\kappa\frac{\partial}{\kappa}\Big|_g + \mathcal{L}_\beta\Big)G^R(p,q,r)\right]_{abc} &= \big[\tau_{ab}{}^d G^R_{dc}(p+q,r) + \text{ 2 permutations}\big] \\ &\quad + \kappa^{-\epsilon}\big(p^2q^2\nabla_c\chi_{ab} + r^2 p.q\chi_{abc} + \text{ 2 permutations}\big).\end{aligned} \tag{41}$$

The RG equation for four point functions be obtained from (36) in a similar fashion. The result for general momenta is however rather long and for simplicity it is again given only at the symmetric point,

$$\begin{aligned}\Big[\Big(\kappa\frac{\partial}{\kappa}\Big|_g + \mathcal{L}_\beta\Big)G^R(p,q,r,s)\Big]_{abcd} &= \big[\tau_{ab}{}^f G^R_{fcd}(p+q,r,s) + \text{ 5 terms}\big] \\ &- \Big[(\triangledown_a\tau_{bc}{}^f)G^R_{fd}(-s,s) + \left\{\begin{array}{c} b\leftrightarrow d \\ q\leftrightarrow s\end{array}\right\} + \left\{\begin{array}{c} c\leftrightarrow d \\ r\leftrightarrow s\end{array}\right\}\Big] - (\triangledown_b\tau_{cd}{}^f)G^R_{fa}(-p,p) \\ &- \kappa^{-\epsilon}\mu^4\Big[\frac{1}{2}\Big(\triangledown_a\triangledown_b\chi_{cd} + \text{ 11 terms}\Big) \\ &\quad + \frac{1}{3}\Big(\big\{\big(R^e{}_{bca} + R^e{}_{cba}\big)\chi_{ed} + (b\leftrightarrow d) + (c\leftrightarrow d)\big\} + \big(R^e{}_{cdb} + R^e{}_{dcb}\big)\chi_{ea}\Big) \\ &\quad - \frac{1}{3}\big(\triangledown_a\chi_{bcd} + 11 \text{ permutations}\big) + \frac{2}{9}\big(\chi_{abcd} + 2 \text{ permutations}\big)\Big]\end{aligned} \tag{42}$$

Again the general form of the terms involving the G^R's will be the same for any theory, only the structure of the μ^4 terms will be different in different theories. The general expression, away from the symmetric point is given in appendix two.

§5 Operator Product Expansion Co-efficients

The fact that, for a given N, the renormalisation group flow induces mixing with Green functions of lower order is intimately related to the operator product expansion (OPE). The connection between the OPE co-efficients and $K^a{}_{bc}$ was mentioned previously and this will now be made more explicit. In particular the RG equation obeyed by the OPE co-efficients will be shown to involve the tensor $\tau_{ab}{}^c$.

In general the OPE involves an infinite number of operators and the basis $[\Phi_a(x)]$ should be extended to include higher dimension operators,

$$[\Phi_a(x)][\Phi_b(y)] = C_{ab}{}^A(x-y)\big[O_A\big(\frac{x+y}{2}\big)\big], \tag{43}$$

where $[O_A(x)]$ are a complete set of operators, in general an infinite set, but certainly containing all of the $[\Phi_a(x)]$ as a subset. $C_{ab}{}^A(x-y)$ are the OPE expansion co-efficients, which of course are singular as $x\to y$. If the (mass) dimensions of the operators $[\Phi_a], [\Phi_b]$ and $[O_A(x)]$ are d_a, d_b and d_A respectively (including anomalous dimensions) then dimensional counting gives the short distance behaviour for the OPE co-efficients as

$$C_{ab}{}^A(x-y) \approx |x-y|^{d_A-d_a-d_b}. \tag{44}$$

Thus the most singular behaviour, for given a and b, is for operators on the right hand side with the smallest values of d_A. Using naive dimensions the operators of lowest dimension are precisely those that appear in the original bare Lagrangian, and the same conclusion will hold for the full dimensions provided none of the anomalous dimensions is too large. A large anomalous dimension would probably be indicative of having chosen unphysical

degrees of freedom in the original Lagrangian. Thus, for example, in scalar $\lambda\varphi^4$ theory in four dimensions the operators φ_0^p for $p = 1, 2, 3, 4$ are allowed to appear in the Lagrangian but higher powers of p would give a non-renormalisable theory and are excluded. The most singular terms in the OPE expansion are therefore given by

$$[\Phi_a(x)][\Phi_b(y)] = C_{ab}{}^c(x-y)\big[\Phi_c\big(\frac{x+y}{2}\big)\big] \quad + \quad \text{less singular terms.} \tag{45}$$

The less singular terms can be investigated by using the non-linear source source renormalisation techniques of reference [12]. It is clear that $C_{ab}{}^c(x-y)$ are tensors on $\mathcal{G}$.

Consider the unregularised N-point functions in position space $G_{a_1\cdots a_N}(x_1, \ldots, x_N)$ when only two of the points get close but all of the others remain well separated, e.g. $x_1 \approx x_2$. Close here means that $|x_1 - x_2| \approx \kappa^{-1}$ where κ is the renormalisation point. From the above discussion we have

$$G_{a_1\cdots a_N}(x_1, \ldots, x_N) = C_{a_1a_2}{}^d(x_1-x_2)G_{da_3\cdots a_N}\big(\frac{x_1+x_2}{2}, x_3, \ldots, x_N\big) \; + \; \text{less singular. terms} \tag{46}$$

The purpose of the counterterms is to tame the singularity as $x_1 \approx x_2$. Referring to the regularised three point functions, (34), in position space,

$$\begin{aligned} &G^R_{abc}(x,y,z) = \\ &G_{abc}(x,y,z) - \delta(x-y)T^d{}_{ab}G_{dc}(x,z) - \delta(y-z)T^d{}_{bc}G_{da}(y,x) - \delta(z-x)T^d{}_{ca}G_{db}(z,y) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \cdots \end{aligned} \tag{47}$$

it is clear that we want the combination $\int d^Dx_1 C_{a_1a_2}{}^d(x_1-x_2) - T^d{}_{a_1a_2}$ to be finite. To this end we shall define a new tensor

$$C^R_{a_1a_2}{}^c(x) = C_{a_1a_2}{}^d(x) - \delta(x)T^d{}_{a_1a_2}, \tag{48}$$

whose integral over all space is finite. Thus $\int d^Dx C^R_{a_1a_2}{}^d(x)$ is finite, whereas $\int d^Dx C_{a_1a_2}{}^d(x)$ is not.

Now recall the definition of the tensors $T^a{}_{bc}$,

$$T^a{}_{bc} = K^a{}_{bc} - \Gamma^a{}_{bc}, \tag{49}$$

where $K^a{}_{bc}$ is given in terms of the renormalisation matrix $Z^{a_0}{}_b$ in equation (22). Equation (49) can be inverted to give an expression for $\Gamma^a{}_{bc}$ in terms of computable quantities and C^{Ra}_{bc},

$$\Gamma^a{}_{bc} = K^a{}_{bc} + \int d^Dx\big(C^{Ra}_{bc}(x) - C_{bc}{}^a(x)\big). \tag{50}$$

This equation is similar to the definition of a connection used by Sonoda, [3], except that the K-terms are not present in that work since it assumed there that the basic operators are independent of the couplings. This has the consequence that the regularised OPE co-efficients defined in [3] are not tensors, instead they transform inhomogeneously under

general co-ordinate transformations. The K-terms in (50) are present because the renormalised basis operators do depend on the couplings, in general. Note that equation (50) does not determine the connection but merely expresses it in terms of the undetermined finite tensor $\int d^D x C^{R\,a}_{bc}(x)$.

More generally one could define a position dependent connection by smearing out the δ-functions $\delta(x)K^a{}_{bc} \to K^a{}_{bc}(x)$ and defining

$$\Gamma^a{}_{bc}(x) = K^a{}_{bc}(x) + C^{R\,a}_{bc}(x) - C_{bc}{}^a(x). \tag{51}$$

Such a position dependent connection appears in the version of the RG equation presented in [1].

Returning to equation (51), combining (46), (47) and (48) now leads to a regularised version of (46),

$$G^R_{a_1\cdots a_N}(x_1,\ldots,x_N) = C^R_{a_1a_2}{}^d(x_1-x_2)G_{da_3\cdots a_N}\big(\frac{x_1+x_2}{2},x_3,\ldots,x_N\big) \quad +\cdots \tag{52}$$

where the dots denote terms that are negligible provided that none of the x_i is close to x_1 or x_2 for $i \geq 3$. We now follow the standard argument that the OPE co-efficients also satisfy a RG equation. To this end consider the action of $\kappa\frac{d}{d\kappa} = \kappa\frac{\partial}{\partial\kappa} + \mathcal{L}_\beta$ on (52) when $N = 4$,

$$\big(\kappa\frac{\partial}{\partial\kappa}\Big|_g + \mathcal{L}_\beta\big)G^R_{abcd}(x,y,z,t) = \Big[\big(\kappa\frac{\partial}{\partial\kappa}\Big|_g + \mathcal{L}_\beta\big)C^{R\,e}_{ab}(x-y)\Big]G_{ecd}\Big(\frac{x+y}{2},z,t\Big) + \cdots \tag{53}$$

where equation (11) with $N = 3$ has been used. In momentum space this reads

$$\big(\kappa\frac{\partial}{\partial\kappa}\Big|_g + \mathcal{L}_\beta\big)G^R_{abcd}(p,q,r,s) = \Big[\big(\kappa\frac{\partial}{\partial\kappa}\Big|_g + \mathcal{L}_\beta\big)C^{R\,e}_{ab}\Big(\frac{p-q}{2}\Big)\Big]G^R_{ecd}(p+q,r,s) + \cdots \tag{54}$$

where we have replaced the three point function on the right hand side with its regularised counterpart - the difference only affects the omitted terms. Now it is clear from (42) that the renormalisation group equation for $x \approx y$ and all other points well seperated takes the form

$$\big(\kappa\frac{\partial}{\partial\kappa}\Big|_g + \mathcal{L}_\beta\big)G^R_{abcd}(p,q,r,s) = \tau_{ab}{}^e G^R_{ecd}(p+q,r,s) + \cdots. \tag{55}$$

Thus we deduce that, for large momenta,

$$\big(\kappa\frac{\partial}{\partial\kappa}\Big|_g + \mathcal{L}_\beta\big)C^{R\,c}_{ab}(p) = \tau_{ab}{}^c + \cdots, \tag{56}$$

where the dots refer to terms that fall off with momentum, the term exhibited on the right hand side is the most significant at small distances. A similar equation for the singular OPE co-efficients $C_{ab}{}^c$ is presented in reference [12], but with vanishing connection so $\tau_{ab}{}^c$ reduces to $\partial_a\partial_b\beta^c$.

The second derivative of the β function also appears in the treatment of the OPE by Zamolodchikov [11], where a Taylor expansion of the β-functions near a conformal theory is performed and it is shown that the OPE co-efficients are essentially the quadratic terms in this expansion. For a conformal field theory Zamolodchikov shows that

$$C_{ab}{}^{c}(x-y) = \tilde{C}^{c}_{ab}\frac{1}{|x-y|^{d_a+d_b-d_c}}. \tag{57}$$

d_a, d_b and d_c here are the dimensions (including anomalous dimensions) of the operators concerned and $\tilde{C}^{c}_{ab}$ are independent of $|x-y|$. Zamolodchikov argues that the basis operators can be chosen so that

$$\tilde{C}^{c}_{ab} = \partial_a\partial_b\beta^c. \tag{58}$$

His argument assumes that a metric exists and that the connection is Levi-Civita. Riemann normal co-rdinates, compatible with (57), can then be chosen so that the connection vanishes and (58) ensues. Clearly this arguement cannot always be applied. Even if co-ordinates can be chosen so that the connection vanishes, it is not true that derivatives of the connection vanish, unless the space is flat, and these are important when more than one derivative is taken. It would seem that the correct tensor to use is $\tau_{ab}{}^{c}$ rather than $\partial_a\partial_b\beta^c$ unless one has reasons to believe that the curvature vanishes. It may be that a flat connection is reasonable for fixed points (i.e. the curvature vanishes at fixed points) but this is not yet clear and, even if this subsequently proves to be the case, it seems unlikely to be true away from fixed points.

An important point of physics in the analysis presented in this section is that the definition of the regularised OPE co-efficients (48) requires integrating over all of space and for *large* separations the less singular terms in equation (45) may become important. In other words the assumption that the operators $[\Phi_a]$ give the most important contributions in the general OPE (43) might not hold for large separations and other operators might become significant for describing the physical degrees of freedom of the theory at larger scales. Such a phenomenon occurs in QCD, for example, where quarks and gluons are believed to be the physical degrees of freedom at short distances whereas mesons and hadrons are more appropriate for larger scales. If one tries to integrate gluonic degrees of freedom over all space one is hit by the infra-red problem. In perturbation theory, at least, this would present insurmountable problems. One must therefore include an infra-red cut-off, for example integrating over only a finite volume, and hope that the volume can be made large enough that finite volume effects are not important, but that $[\Phi_a]$ still give the most important contribution to the OPE within the whole volunme. Such a procedure, if valid, allows the determination of at least the short distance behaviour of the theory using the techniques here, but it must be borne in mind that it may not always give sensible answers.

§6 Conclusions

In conclusion it has been argued that N-point amplitudes, $G^{(N)}$, should be thought of as tensors on the space of couplings, $\mathcal{G}$, and the renormalisation group equation mixes up

tensors of different rank, $G^{(N)}$ being related to linear combinations of $G^{(M)}$ with $M \leq N$. The crucial quantity that determines this mixing is a tensor given by the second co-variant derivative of the β-functions of the theory,

$$\tau^a{}_{bc} = \nabla_b \nabla_c \beta^a - R^a{}_{cbd} \beta^d. \tag{4}$$

The RG equations for two, three and four point functions for a massless theory in four dimensions are given in equations (41) and (42). When all the points are well separated in space it reduces to the definition of a Lie derivative with respect to the vector field given by the β-functions of the theory, equation (11). However when any of the points start to get close to one another, relative to the renormalisation length κ^{-1}, there are extra contributions. The form of the terms involving the tensors χ (defined in equation (38)) is specific to massless theories in four dimensions and result from a subtraction which is necessary in the circumstance when all the spatial points in the Green function are degenerate. The other terms on the right hand side of (41) and (42) are present in any theory and the M-point functions with $M < N$ reflect singularities that arise when some of the points start getting close to one another. It should be observed that the mixing between tensors of different rank is *linear*. This is only true for the Green functions of the theory (excluding exceptional momenta). Were one to consider the composite operator analogues of the proper vertices, $\Gamma^{(N)}$, then the resulting mixing is *non-linear* even when exceptional momenta are excluded, see reference [12].

No prescription as to how the connection might be calculated in general has been given, it is merely assumed that one must exist, but it has been argued that it should be related to the operator product expansion co-efficients, through equation (51),

$$\Gamma^a{}_{bc} = K^a{}_{bc} + \int d^D x \big(C^{R\,a}_{bc}(x) - C_{bc}{}^a(x) \big), \tag{51}$$

where $K^a{}_{bc} = \partial_a \partial_b g^{d_0} \left(\frac{\partial g^c}{\partial g^{d_0}} \right)$, equation (22). In this expression $C^{R\,a}_{bc}(x)$ is a regularised OPE co-efficient whose integral over all space is finite. If one could calculate $\Gamma^a{}_{bc}$ then one would immediately know $\int d^D x C^{R\,a}_{bc}(x)$ and vice versa. This will not be attempted here but is clearly an interesting programme with much scope for development.

Further questions concerning the nature of the connection present themselves. Would it be metric compatible, if one were to give a physically reasonable definition of a Riemannian metric on $\mathcal{G}$? For example the Zamolodchikov metric constructed from the two point functions of the theory

$$g_{ab} = G_{ab}(x, y,) \big|_{|x-y|=\kappa^{-1}} \tag{59}$$

might be a candidate. It is expected to be positive definite for unitary theories. It is not clear if the Levi-Civita connection associated with this definition of a metric would provide useful physical information for a theory, or perhaps it would have to be supplemented by more structure. As mentioned earlier, the connection is symmetric so if it is not Levi-Civita then it cannot be metric compatible - the extra structure is not simply a torsion tensor, it would be given by the regularised OPE co-efficients, $\int d^D x C^{R\,a}_{bc}(x)$.

Lastly it should be emphasised that everything that has been presented here is in terms of the local geometry, the global structure of $\mathcal{G}$ has not beem addressed at all, but clearly it would be very interesting to be able to ascertain something about it.

It is a pleasure to thank Denjoe O'Connor for stimulating discussions on the renormalisation group, and also Prof. N. Dragon for his hospitality at the Institut für Theoretische Physik, Hanover where this investigation was begun.

Appendix 1

For completeness we include the non-covariant expression for the regularised N-point Green functions in terms of their unregularised counterparts, for a massless theory in four dimensions. Unregularised in this context does not mean bare - it is always assumed that renormalised operators $[\Phi_a(x)]$ are used in all Green functions - rather it means regularisation in the sense of regularisation of the infinities that occur when two or more points get close together in the Green function.

The regularised Green functions are obtained by functionally differentiating the generating functional,

$$\mathrm{e}^{-W[g^a]} = \int \mathcal{D}\varphi \mathrm{e}^{-\tilde{S}_0(\varphi, g^a)}. \tag{16}$$

Thus

$$G^R_{a_1\cdots a_N}(x_1,\ldots,x_N) = (-1)^{N+1}\frac{\delta^N W}{\delta g^{a_1}(x_1)\cdots\delta g^{a_N}(x_N)}. \tag{17}$$

The regularised N-point function can be obtained by induction. We first write down the formula for the N-point function in momentum space. It reads

$$\begin{aligned}
G^R_{a_1\cdots a_N}(p_1,\ldots,p_N)|_{\partial_\mu g=0} = &\sum_{s=0}^{[N/2]}\sum_{\text{partitions}}\frac{(-1)}{s!}^{N+s+r_0} K^{m_1}_{\pi_1}\cdots K^{m_s}_{\pi_s} \\
&\times G_{\pi_0 m_1\cdots m_s}\Big(p_{\pi_0(1)},\ldots,p_{\pi_0(r_0)},\Sigma_{k=1}^{r_1}p_{\pi_1(k)},\ldots,\Sigma_{k=1}^{r_s}p_{\pi_s(k)}\Big) \\
&+(-1)^N\frac{\kappa^{-\epsilon}}{2(N-2)!}\sum_{\text{permutations}}\partial^{(N-2)}_{\{a_3\cdots a_N}A_{a_1a_2}p^2_{a_1}p^2_{a_2\}} \\
&+(-1)^N\frac{\kappa^{-\epsilon}}{2(N-3)!}\sum_{\text{permutations}}\partial^{(N-3)}_{\{a_4\cdots a_N}B_{a_1a_2a_3}\big(p_{a_1}.p_{a_2}\big)p^2_{a_3\}} \\
&+(-1)^N\frac{\kappa^{-\epsilon}}{(2!)^2(N-4)!}\sum_{\text{permutations}}\partial^{(N-4)}_{\{a_5\cdots a_N}C_{a_1a_2a_3a_4}\big(p_{a_1}.p_{a_2}\big)\big(p_{a_3}.p_{a_4\}}\big),
\end{aligned} \tag{60}$$

where the sum over partitions involves splitting $a_1,\ldots,a_N$ up into $s+1$ sets, $\pi_0,\ldots,\pi_s$ each with r_j elements $a_{\pi_j(1)},\ldots,a_{\pi_j(r_j)}$ such that $\sum_{j=0}^s r_j = N$, $0\le r_0\le N$ and $2\le r_j\le N$ for $1\le j\le s$. Thus $\pi_j = \{a_{\pi_j(1)},\ldots,a_{\pi_j(r_j)}\}$ is some subset of $a_1,\ldots,a_N$ consisting of r_j elements. The number of sets lies between 0 and $[N/2]$ where $[N/2]$ is the integral part of

$N/2$. In any given partition each subset occurs only once, regardless of the ordering of its elements. For example, for $N = 4$, s has three possible values, $0, 1$ or 2, and the partitions are

$$
\begin{aligned}
& s = 0: \qquad \pi_0 = \{a_1, a_2, a_3, a_4\} \\
& s = 1: \quad \begin{cases}
\pi_0 = \{a_1, a_2\}\ \pi_1 = \{a_3, a_4\}, & \pi_0 = \{a_1, a_3\}\ \pi_1 = \{a_2, a_4\}, \\
\pi_0 = \{a_1, a_4\}\ \pi_1 = \{a_2, a_3\}, & \pi_0 = \{a_2, a_3\}\ \pi_1 = \{a_1, a_4\}, \\
\pi_0 = \{a_2, a_4\}\ \pi_1 = \{a_1, a_3\}, & \pi_0 = \{a_3, a_4\}\ \pi_1 = \{a_1, a_2\}, \\
\pi_0 = \{a_1\}\ \pi_1 = \{a_2, a_3, a_4\}, & \pi_0 = \{a_2\}\ \pi_1 = \{a_1, a_3, a_4\}, \\
\pi_0 = \{a_3\}\ \pi_1 = \{a_1, a_2, a_4\}, & \pi_0 = \{a_4\}\ \pi_1 = \{a_1, a_2, a_3\}, \\
\pi_0 = \emptyset\ \pi_1 = \{a_1, a_2, a_3, a_4\}
\end{cases} \\
& s = 2: \quad \begin{cases}
\pi_0 = \emptyset\ \pi_1 = \{a_1, a_2\}\ \pi_1 = \{a_3, a_4\}, & \pi_0 = \emptyset\ \pi_1 = \{a_1, a_3\}\ \pi_2 = \{a_2, a_4\}, \\
\pi_0 = \emptyset\ \pi_1 = \{a_1, a_4\}\ \pi_2 = \{a_2, a_3\}, & \pi_0 = \emptyset\ \pi_1 = \{a_2, a_3\}\ \pi_2 = \{a_1, a_4\}, \\
\pi_0 = \emptyset\ \pi_1 = \{a_2, a_4\}\ \pi_2 = \{a_1, a_3\}, & \pi_0 = \emptyset\ \pi_1 = \{a_3, a_4\}\ \pi_2 = \{a_1, a_2\}.
\end{cases}
\end{aligned}
\tag{61}
$$

The co-efficients $K^{m_j}_{\pi_j} = K^{m_j}_{a_{\pi_j(1)} \cdots a_{\pi_j(r_j)}}$ in equation (60) are defined analogously to (22),

$$
K^{m}_{a_{\pi_j(1)} \cdots a_{\pi_j(r_j)}} = \left(\partial_{a_{\pi_j(1)}} \cdots \partial_{a_{\pi_j(r_j)}} g^{d_0}\right) \left(\frac{\partial g^m}{\partial g^{d_0}}\right). \tag{62}
$$

Terms involving exceptional momenta are omitted from the above expression. It is straightforward to show that equation (60) reproduces the regularised two, three and four point functions in the text (equations (25) and (26)), provided $G_a(p) =< \Phi_a(p) >= 0$.

It will be more useful to work in position space in order to construct an inductive proof. Equation (60) translates as

$$
\begin{aligned}
G^R_{a_1 \cdots a_N}(x_1, \ldots, x_N)|_{\partial_\mu g = 0} = \sum_{s=0}^{[N/2]} \sum_{\text{partitions}} & \frac{(-1)^{N+s+r_0}}{s!} K^{m_1}_{\pi_1} \cdots K^{m_s}_{\pi_s} \\
& \times G^{(r_0+s)}_{\pi_0 m_1 \cdots m_s}(x_{\pi_0(1)}, \ldots, x_{\pi_0(r_0)}, x_{\pi_1(1)}, \ldots, x_{\pi_s(1)}) \prod_{j=1}^{s} \prod_{m=2}^{r_j} \delta_{x_{\pi_j(1)}, x_{\pi_j(m)}} \\
+ \frac{(-1)^N \kappa^{-\epsilon}}{2(N-2)!} & \Big[\partial^{(N-2)}_{a_3 \cdots a_N} A_{a_1 a_2} \delta''_{x_{a_1}, x_{a_N}} \delta''_{x_{a_2}, x_{a_N}} \prod_{j=3}^{N-1} \delta_{x_{a_j}, x_{a_N}} \ + \text{ permutations}\Big] \\
+ \frac{(-1)^N \kappa^{-\epsilon}}{2(N-3)!} & \Big[\partial^{(N-3)}_{a_4 \cdots a_N} B_{a_1 a_2 a_3} \left(\delta'_{x_{a_1}, x_{a_N}} . \delta'_{x_{a_2}, x_{a_N}}\right) \delta''_{x_{a_3}, x_{a_N}} \prod_{j=4}^{N-1} \delta_{x_{a_j}, x_{a_N}} \ + \text{ permutations}\Big] \\
+ \frac{(-1)^N \kappa^{-\epsilon}}{(2!)^2(N-4)!} & \Big[\partial^{(N-4)}_{a_5 \cdots a_N} C_{a_1 a_2 a_3 a_4} \left(\delta'_{x_{a_1}, x_{a_N}} . \delta'_{x_{a_2}, x_{a_N}}\right) \left(\delta'_{x_{a_3}, x_{a_N}} . \delta'_{x_{a_4}, x_{a_N}}\right) \prod_{j=5}^{N-1} \delta_{x_{a_j}, x_{a_N}} \\
& \qquad + \text{ permutations}\Big],
\end{aligned}
\tag{63}
$$

where the δ-function notation used here is a shorthand for $\delta_{x_i, x_j} = \delta(x_i - x_j)$ and a prime denotes differentiation with respect to the first argument of the δ-function. Thus

$\delta''_{x_i,x_j} = \Box_{x_i}\delta(x_i - x_j)$ and $\delta'_{x_i,x_j}.\delta'_{x_i,x_k} = \partial^{\mu}_{x_i}\delta(x_i - x_j)\partial_{x_i\mu}\delta(x_i - x_k)$. The number of arguments in the unregularised Green functions on the right hand side has been shown explicity as a superscript in order to try to make the formulae easier to interpret - thus $G^{(r_0+s)}$ is a (r_0+s)-point function. The fact that exceptional momenta are being excluded is interpreted in position space as meaning that terms which factorise into products of amplitudes which are separately translationally invariant are omitted from (63).

Proceeding inductively, we relax the condition $\partial_\mu g^a = 0$ in the regularised Green functions and functionally differentiate (63) with respect to $g^{a_{N+1}}(x_{N+1})$ and then check that the resulting expression agrees with (63) with N replaced by $N+1$. The regularised $N+1$-point function is thus given by

$$-\frac{\delta G^R_{a_1\cdots a_N}(x_1,\ldots,x_N)}{\delta g^{a_{N+1}}(x_{N+1})}\Big|\partial_\mu g = 0. \tag{64}$$

Consider, therefore, a generic term from the right hand side of (63),

$$F_s[g^a(x)] := (-1)^{N+s+r_0}K^{m_1}_{\pi_1}\cdots K^{m_s}_{\pi_s}G^{(r_0+s)}_{\pi_0 m_1\cdots m_s}\prod_{j=1}^{s}\prod_{m=2}^{r_j}\delta_{x_{\pi_j(1)},x_{\pi_j(m)}}, \tag{65}$$

where the argument of G has been omitted for brevity, thus

$$G^{(r_0+s)}_{\pi_0 m_1\cdots m_s} = G^{(r_0+s)}_{a_{\pi_0(1)}\cdots a_{\pi_0(r_0)}m_1\cdots m_s}(x_{\pi_0(1)},\ldots,x_{\pi_0(r_0)},x_{\pi_1(1)},\ldots,x_{\pi_s(1)}). \tag{66}$$

The index structure on G is sufficient to deduce its arguments.

Functionally differentiating F_s and subsequently setting the couplings to be independent of position gives

$$\begin{aligned}\frac{\delta F_s}{\delta g^{a_{N+1}}(x_{N+1})}\Big|\partial_\mu g^{a_i} = 0 &= (-1)^{N+s+r_0}\sum_{j=1}^{s}K^{m_1}_{\pi_1}\cdots K^{m_{j-1}}_{\pi_{j-1}}\left[\partial_{a_{N+1}}K^{m_j}_{\pi_j}\right]K^{m_{j+1}}_{\pi_{j+1}}\cdots K^{m_s}_{\pi_s}\\ &\qquad\times G^{(r_0+s)}_{\pi_0 m_1\cdots m_s}\delta_{x_{N+1},x_{\pi_j(1)}}\prod_{j=1}^{s}\prod_{m=2}^{r_j}\delta_{x_{\pi_j(1)},x_{\pi_j(m)}}\\ &\quad+(-1)^{N+s+r_0}K^{m_1}_{\pi_1}\cdots K^{m_s}_{\pi_s}\left[\delta_{a_{N+1}}G^{(r_0+s)}_{\pi_0 m_1\ldots m_s}\right]\prod_{j=1}^{s}\prod_{m=2}^{r_j}\delta_{x_{\pi_j(1)},x_{\pi_j(m)}}\end{aligned} \tag{67}$$

where $\delta_{a_{N+1}} := \frac{\delta}{\delta g^{a_{N+1}}(x_{N+1})}$. From the definition (62) we have

$$\partial_b K^m_{a_1\cdots a_j} = K^m_{ba_1\cdots a_j} - K^c_{a_1\cdots a_j}K^m{}_{cb}. \tag{68}$$

We also observe that, from equation (16),

$$\begin{aligned}\delta_{a_{N+1}}G^{(r_0+s)}_{\pi_0 m_1\ldots m_s} &= -G^{(r_0+s+1)}_{\pi_0 a_{N+1}m_1\cdots m_s}\\ &\quad+\sum_{k=1}^{r_0}K^c{}_{a_{\pi_0(k)}a_{N+1}}G^{(r_0+s)}_{a_{\pi_0(1)}\cdots a_{\pi_0(k-1)}ca_{\pi_0(k+1)}\cdots a_{\pi_0(r_0)}m_1\cdots m_s}\delta_{x_{N+1},x_{\pi_0(k)}}\\ &\quad+\sum_{k=1}^{s}K^c{}_{m_k a_{N+1}}G^{(r_0+s)}_{a_{\pi_0(1)}\cdots a_{\pi_0(r_0)}m_1\cdots m_{k-1}cm_{k+1}\cdots m_s}\delta_{x_{N+1},x_{\pi_k(1)}}\end{aligned} \tag{69}$$

(in general there are other terms on the right hand side here involving A_{ab}, from equation (21), but these only contribute to the final result if there are exceptional momenta and they will be omitted from this analysis). The (r_0+s+1)-point function in (69) is a shorthand notation for

$$G^{(r_0+s+1)}_{\pi_0 a_{N+1} m_1 \cdots m_s} = G^{(r_0+s+1)}_{a_{\pi_0(1)} \cdots a_{\pi_0(r_0)} a_{N+1} m_1 \cdots m_s}(x_{\pi_0(1)}, \ldots, x_{\pi_0(r_0)}, x_{N+1}, x_{\pi_1(1)}, \ldots, x_{\pi_s(1)}).$$

Thus both the terms on the right hand side of equation (67) involve

$$K^{m_1}_{\pi_1} \cdots K^{m_s}_{\pi_s} \sum_{k=1}^{s} K^{c}{}_{m_k a_{N+1}} G^{(r_0+s)}_{\pi_0 m_1 \cdots m_{k-1} c m_{k+1} m_s}, \tag{70}$$

but with opposite sign so that they cancel.

Equation (69) can now be re-arranged as

$$\begin{aligned}
\frac{\delta F_s}{\delta g^{a_{N+1}}(x_{N+1})}\Big|_{\partial_\mu g^{a_i}=0} &= (-1)^{N+1+s+r_0} K^{m_1}_{\pi_1} \cdots K^{m_s}_{\pi_s} G^{(r_0+s+1)}_{\pi_0 a_{N+1} m_1 \cdots m_s} \prod_{j=1}^{s} \prod_{m=2}^{r_j} \delta_{x_{\pi_j(1)}, x_{\pi_j(m)}} \\
&+ (-1)^{N+s+r_0} \sum_{j=1}^{s} K^{m_1}_{\pi_1} \cdots K^{m_{j-1}}_{\pi_{j-1}} K^{m_j}_{\pi_j a_{N+1}} K^{m_{j+1}}_{\pi_{j+1}} \cdots K^{m_s}_{\pi_s} G^{(r_0+s)}_{\pi_0 m_1 \cdots m_s} \\
&\qquad \times \delta_{x_{N+1}, x_{\pi_j(1)}} \prod_{j=1}^{s} \prod_{m=2}^{r_j} \delta_{x_{\pi_j(1)}, x_{\pi_j(m)}} \\
&+ (-1)^{N+s+r_0} K^{m_1}_{\pi_1} \cdots K^{m_s}_{\pi_s} \sum_{k=1}^{r_0} K^{m_{s+1}}_{a_{\pi_0(k)} a_{N+1}} G^{(r_0+s)}_{a_{\pi_0(1)} \cdots a_{\pi_0(k-1)} m_{s+1} a_{\pi_0(k+1)} \cdots a_{\pi_0(r_0)} m_1 \cdots m_s} \\
&\qquad \times \delta_{x_{N+1}, x_{\pi_0(k)}} \prod_{j=1}^{s} \prod_{m=2}^{r_j} \delta_{x_{\pi_j(1)}, x_{\pi_j(m)}}
\end{aligned} \tag{71}$$

where the summation variable c has been replaced in a suggestive manner by m_{s+1}.

We now note that the indices on $G^{(r_0+s)}$ can be permuted, provided that one also understands the arguments to be permuted as well, thus

$$\begin{aligned}
&G^{(r_0+s)}_{a_{\pi_0(1)} \cdots a_{\pi_0(k-1)} m_{s+1} a_{\pi_0(k+1)} \cdots a_{\pi_0(r_0)} m_1 \cdots m_s} \\
&= G^{(r_0+s)}_{a_{\pi_0(1)} \cdots a_{\pi_0(k-1)} m_{s+1} a_{\pi_0(k+1)} \cdots a_{\pi_0(r_0)} m_1 \cdots m_s}(x_{\pi_0(1)}, .., x_{\pi_0(k-1)}, x_{N+1}, x_{\pi_0(k+1)}, .., x_{\pi_0(r_0)}, x_{\pi_1(1)}, .., x_{\pi_s(1)}) \\
&= G^{(r_0+s)}_{a_{\pi_0(1)} \cdots a_{\pi_0(k-1)} a_{\pi_0(k+1)} \cdots a_{\pi_0(r_0)} m_1 \cdots m_s m_{s+1}}(x_{\pi_0(1)}, .., x_{\pi_0(k-1)}, x_{\pi_0(k+1)}, .., x_{\pi_0(r_0)}, x_{\pi_1(1)}, .., x_{\pi_s(1)}, x_{N+1}) \\
&= G^{(r_0+s)}_{a_{\pi_0(1)} \cdots a_{\pi_0(k-1)} a_{\pi_0(k+1)} \cdots a_{\pi_0(r_0)} m_1 \cdots m_s m_{s+1}}.
\end{aligned} \tag{72}$$

Thus, introducing an overall minus sign, we arrive at

$$
\begin{aligned}
-\left.\frac{\delta F_s}{\delta g^{a_{N+1}}(x_{N+1})}\right|_{\partial_\mu g^{a_i}=0} &= (-1)^{N+1+s+r_0+1} K^{m_1}_{\pi_1}\cdots K^{m_s}_{\pi_s} G^{(r_0+s+1)}_{\pi_0 a_{N+1} m_1\cdots m_s} \prod_{j=1}^{s}\prod_{m=2}^{r_j}\delta_{x_{\pi_j(1)},x_{\pi_j(m)}} \\
&+ (-1)^{N+1+s+r_0}\sum_{j=1}^{s} K^{m_1}_{\pi_1}\cdots K^{m_{j-1}}_{\pi_{j-1}} K^{m_j}_{\pi_j a_{N+1}} K^{m_{j+1}}_{\pi_{j+1}}\cdots K^{m_s}_{\pi_s} G^{(r_0+s)}_{\pi_0 m_1\cdots m_s} \\
&\qquad\qquad \times \delta_{x_{N+1},x_{\pi_j(1)}} \prod_{j=1}^{s}\prod_{m=2}^{r_j}\delta_{x_{\pi_j(1)},x_{\pi_j(m)}} \\
&+ (-1)^{N+1+s+1+r_0+1} K^{m_1}_{\pi_1}\cdots K^{m_s}_{\pi_s}\sum_{k=1}^{r_0} K^{m_{s+1}}_{a_{\pi_0(k)}a_{N+1}} G^{(r_0+s)}_{a_{\pi_0(1)}\cdots a_{\pi_0(k-1)}a_{\pi_0(k+1)}\cdots a_{\pi_0(r_0)}m_1\cdots m_s m_{s+1}} \\
&\qquad\qquad \times \delta_{x_{N+1},x_{\pi_0(k)}} \prod_{j=1}^{s}\prod_{m=2}^{r_j}\delta_{x_{\pi_j(1)},x_{\pi_j(m)}}.
\end{aligned}
\tag{73}
$$

Using this in equation (63) and (64) and re-arranging the summations one sees the desired structure emerging, but there are more terms to be taken into account. These come from extra contributions to the regularised Green functions (63) when $\partial_\mu g^b \neq 0$. One only need consider the terms linear in $\partial_\mu g^b \neq 0$, as higher order contributions vanish when the condition $\partial_\mu g^b = 0$ is imposed after one functional differentiation. These extra terms have the effect of symmetrising the result between all $N+1$ indices, and the full expression (63) is recovered with N replaced by $N+1$. The A, B and C terms can be verified without diffuclty.

The form of equation (60) is basically the same in dimensions other than four and/or when masses are included - all that changes are the terms involving A, B and C.

Appendix 2

In this appendix we give the full expression for the co-variant renormalisation group equation for regularised four point functions, not just at the symmetric point. The derivation is a straightforward, but tedious, application of the techniques described in the text.

The result, in momentum space, is

$$\begin{aligned}
&\Big[\Big(\kappa\frac{\partial}{\partial\kappa}\Big|_g + \mathcal{L}_\beta\Big)G^R(p,q,r,s)\Big]_{abcd} = \big[\tau_{ab}{}^f G^R_{fcd}(p+q,r,s) + \text{ 5 terms}\big] \\
&- \Big[\big(\nabla_a\tau_{bc}{}^f\big)G^R_{fd}(-s,s) + \left\{\begin{array}{c} b\leftrightarrow d \\ q\leftrightarrow s\end{array}\right\} + \left\{\begin{array}{c} c\leftrightarrow d \\ r\leftrightarrow s\end{array}\right\}\Big] - \big(\nabla_b\tau_{cd}{}^f\big)G^R_{fa}(-p,p) \\
&- \frac{\kappa^{-\epsilon}}{2}\Big(p^2q^2\nabla_a\nabla_b\chi_{cd} + \text{ 11 terms}\Big) \\
&- \frac{\kappa^{-\epsilon}}{2}\Big[\Big(s^2(q^2+p^2+4q.p)R^e{}_{bca} + s^2(r^2+p^2+4r.p)R^e{}_{cba}\Big)\chi_{ed} + \left\{\begin{array}{c} b\leftrightarrow d \\ p\leftrightarrow s\end{array}\right\} + \left\{\begin{array}{c} c\leftrightarrow d \\ r\leftrightarrow s\end{array}\right\}\Big] \\
&- \frac{\kappa^{-\epsilon}}{2}\Big(p^2(q^2+r^2+4q.r)R^e{}_{cdb} + p^2(q^2+s^2+4q.s)R^e{}_{dcb}\Big)\chi_{ea} \\
&- \kappa^{-\epsilon}\big[(q.r)s^2\nabla_a\chi_{bcd} + 11 \text{ permutations}\big] - \kappa^{-\epsilon}\big[2(p.q)(r.s)\chi_{abcd} + 2 \text{ permutations}\big].
\end{aligned} \tag{74}$$

Once again the structure of the χ-terms is peculiar to massless theories in four dimensions, but the τ-terms are the same for all theories.

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