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# A Geometrical Interpretation of the Time Evolution of the Schrödinger Equation for Discrete Quantum Systems

by

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**ABSTRACT.** — The time evolution of a quantum state in the Schrödinger picture of quantum mechanics is presented as a curve in  $\mathbb{C}\mathbb{P}^n$ . An  $(n + 1)$ -state quantum system is considered semi-classically as a  $\mathbb{C}\mathbb{P}^n$  bundle over three dimensional space, with structure group  $SU(n + 1)/\mathbb{Z}_{n+1}$ .

**RÉSUMÉ.** — On identifie l'évolution temporelle d'un état quantique dans la représentation de Schrödinger de la Mécanique Quantique avec une courbe dans  $\mathbb{C}\mathbb{P}^n$ . Un système quantique à  $(n + 1)$  états est considéré du point de vue semi-classique comme un fibré  $\mathbb{C}\mathbb{P}^n$  au-dessus de l'espace à trois dimensions, avec  $SU(n + 1)/\mathbb{Z}_{n+1}$  comme groupe de structure.

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## INTRODUCTION

In this paper, a geometrical interpretation of the time development of the Schrödinger equation for discrete quantum systems is developed. For a Hamiltonian with  $n + 1$  eigenstates, the space of all possible quantum states is interpreted as the complex projective space  $\mathbb{C}\mathbb{P}^n$ . A given set of initial conditions corresponds to a single point on  $\mathbb{C}\mathbb{P}^n$  and if the system is left undisturbed it will follow a continuous one dimensional curve in  $\mathbb{C}\mathbb{P}^n$ , the particular curve and the rate at which it evolves being determined by the initial conditions and the Hamiltonian. The Hamil-

tonian enters as a vector field on  $\mathbb{C}\mathbb{P}^n$ , the vector at each point being the tangent vector to the unique curve (at a given instant of time) through that point describing the evolution of the state that corresponds to that point.

In section one the general correspondence between state vectors and points in  $\mathbb{C}\mathbb{P}^n$  is described. In Section two, the two state quantum system is analysed and described in terms of curves on  $S^2$ , first for a time independent Hamiltonian and then using time dependent perturbation theory; thirdly, non-commuting Hermitian operators, and their eigenstates are considered. Section three treats the three state quantum system and section four the general  $(n + 1)$  state system, both for time independent Hamiltonians. Finally in section five position dependent Hamiltonians are considered semi-classically, viewed as  $\mathbb{C}\mathbb{P}^n$  bundles over three space and the particular case of an electron orbiting a monopole is treated as an example of a non-trivial bundle.

### GENERAL FORMALISM

For a quantum system, a particular state  $|\psi\rangle$  is an element of a complex  $(n + 1)$  dimensional Hilbert space. Choose a basis  $\{e^i, i = 0, \dots, n\}$  of this Hilbert space where each  $e^i$  is an eigenstate of the Hamiltonian (assumed

given). Denote  $|\psi\rangle = \sum_{i=0}^n z_i e^i$  by  $(z_0, \dots, z_n)$  where  $z_i$  are complex numbers.

$|\psi\rangle$  can then be identified with a point in  $\mathbb{C}^{n+1}$ . Any two vectors which are a complex multiple of one another are identified as the same quantum state, i. e.  $(z'_0, \dots, z'_n) = \lambda(z_0, \dots, z_n)$  is considered to be the same state as  $(z_0, \dots, z_n)$ ,  $\lambda \in \mathbb{C} \setminus \{0\}$ . This is the definition of the complex projective space  $\mathbb{C}\mathbb{P}^n$ , with  $n$  complex dimensions (or  $2n$  real dimensions). Thus the space of all possible quantum states of a system with  $n + 1$  eigenstates is topologically  $\mathbb{C}\mathbb{P}^n$ . Each point of  $\mathbb{C}\mathbb{P}^n$  corresponds to a different state of the system. As the system evolves in time, according to the Schrödinger picture, it traces out a one dimensional curve in  $\mathbb{C}\mathbb{P}^n$ . Provided the system is left undisturbed, this curve will be continuous, but if a measurement is made the system will make a discontinuous jump from one point of  $\mathbb{C}\mathbb{P}^n$  to another, the latter being a point corresponding to an eigenstate of the quantity measured.

If the  $e^i$  are normalised to unity, using a metric on the Hilbert space, then this gives a natural metric on  $\mathbb{C}\mathbb{P}^n$ , the Fubini-Study metric [1]. This is obtained by embedding  $\mathbb{C}\mathbb{P}^n$  in  $\mathbb{C}^{n+1}$  and restricting the  $z_i$ , so that

$$\langle \psi | \psi \rangle = 1, \text{ i. e., } \sum_{i=0}^n \bar{z}_i z_i = 1, \text{ thus restricting the } z_i \text{ to lie on the unit}$$

sphere,  $S^{2n+1}$ , in  $\mathbb{C}^{n+1}$ . Using the standard metric on  $S^{2n+1}$ , the Fubini-Study metric on  $\mathbb{C}P^n$  is obtained by projecting out the remaining  $S^1 \approx U(1)$  from  $(z_0, \dots, z_n)$ . This corresponds to the Hopf fibration of  $S^{2n+1}$ .

Let  $|\psi(t)\rangle$  be a ket for an  $n + 1$  dimensional quantum system. If the system is left undisturbed, the time evolution of the ket is given by the Schrödinger equation (the notation is that of Ref. [2]).

$$\frac{d}{dt} |\psi(t)\rangle = -\frac{i}{\hbar} H(t) |\psi(t)\rangle$$

where  $H(t)$  is the Hamiltonian — a Hermitian operator on the Hilbert space. Interpreting  $|\psi(t)\rangle$  as a curve  $\gamma(t)$  in  $\mathbb{C}P^n$ , the Schrödinger equation is

$$\frac{d}{dt} \gamma(t) = -\frac{i}{\hbar} H_V(t) [\gamma(t)]$$

Thus we have a geometrical interpretation of the Hamiltonian as a time dependent vector along  $\gamma(t)$  on  $\mathbb{C}P^n$ . (The symbol  $H(t)$  is used here to represent a Hermitian operator in a Hilbert space and  $H_V(t)$  the corresponding vector on  $\mathbb{C}P^n$ . The method of obtaining the correspondence will be developed in the following sections).

By varying  $|\psi\rangle$  we can vary  $\gamma$  so as to cover the whole of  $\mathbb{C}P^n$  and thus extend  $H_V(t)$  to a time dependent vector field on  $\mathbb{C}P^n$ . Note that for  $\gamma(t)$  corresponding to a state satisfying the Schrödinger equation the tangent vector to the curve  $\gamma(t)$  at any point does not depend on the direction of  $\gamma(t)$  at earlier times, since it depends only on  $H_V(t)$ . This corresponds to the fact that the time evolution of a quantum system depends only on the Hamiltonian and the state at a given time — not on the past history of the state.

In particular, if the Hamiltonian is time independent, given any point in  $\mathbb{C}P^n$ , the tangent vector for any  $\gamma(t)$  obeying the Schrödinger equation always points in the same direction, thus  $\gamma(t)$  can never intersect itself. If  $\gamma(t)$  ever passes through the same point of  $\mathbb{C}P^n$  for different values of  $t$ , then it forms a closed loop in  $\mathbb{C}P^n$ , for a time independent Hamiltonian.

## THE TWO STATE QUANTUM SYSTEM

### a) Evolution with time independent Hamiltonian.

Consider a quantum system with only two eigenstates (e. g. the spin of an electron in a magnetic field). The space of all quantum states is thus  $\mathbb{C}P^1$  which is topologically  $S^2$ , the two dimensional sphere.

Choose co-ordinate on  $\mathbb{C}P^1$  as follows. Let  $(z_0, z_1) \in \mathbb{C}^2$ . If  $z_0 \neq 0$ , let  $z_0(1, z_1/z_0) = z_0(1, \xi_1)$ . Then  $\xi_1$  is a single complex co-ordinate and covers

the whole of the southern hemisphere for  $|\xi_1| \leq 1$ . (Indeed  $\xi_1$  can be used to cover the whole sphere except for the north pole where  $|\xi_1| \rightarrow \infty$ ). Similarly, if  $z_1 \neq 0$ , let  $z_1(z_0/z_1, 1) = z_1(\xi_0, 1)$ . Then  $\xi_0$  covers the northern hemisphere for  $|\xi_0| \leq 1$ . In the overlap of the two co-ordinate regions  $\xi_1 = \xi_0^{-1}$ . Writing  $\xi_0 = re^{i\phi}$ ,  $0 \leq r < \infty$ ,  $0 \leq \phi < 2\pi$ , then in standard polar co-ordinates on  $S^2$  (radius unity)  $r = \tan \theta/2$ ,  $0 \leq \theta < \pi$ . (See fig. 1). The magnitude of  $\xi_0$  gives the polar angle on  $S^2$  and the phase the azimuthal angle. This procedure is nothing more than the stereographic projection of  $S^2$  onto  $\mathbb{R}^2 \approx \mathbb{C}$ .

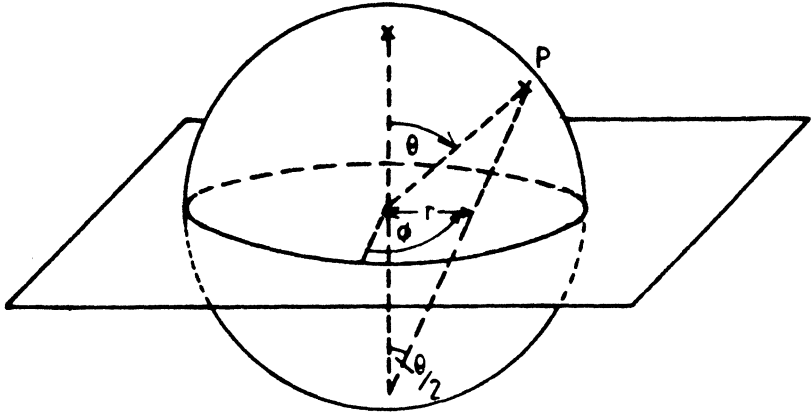


FIG. 1. — P has co-ordinates  $(\theta, \varphi)$  and  $(r, \varphi)$ .

Consider a time independent Hamiltonian, with eigenvalues  $E_0$  and  $E_1$ . Take  $|\psi_0\rangle = (1, 0)$  and  $|\psi_1\rangle = (0, 1)$  as eigenvectors in the Hilbert space,  $\mathbb{C}^2$ , with eigenvalues  $E_0$  and  $E_1$  respectively. With the co-ordinate identification above, the north pole corresponds to the eigenvector  $|\psi_0\rangle$  and the south pole to  $|\psi_1\rangle$ . The values of  $\theta/2$  is the mixing angle for a general state. In general, if the polar co-ordinates had been chosen with a different orientation relative to  $\xi_0$ , the eigenstates would not be at the poles, but they are always diametrically opposite one another.

The Schrödinger equation reads

$$\frac{d}{dt} \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = -\frac{i}{\hbar} \begin{pmatrix} E_0 & 0 \\ 0 & E_1 \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}$$

giving

$$z_0(t) = z_0(0)e^{-iE_0t/\hbar}$$

$$z_1(t) = z_1(0)e^{-iE_1t/\hbar}$$

and

$$\xi_0(t) = z_0/z_1 = \xi_0(0)e^{i(E_1 - E_0)t/\hbar}$$

$$\xi_1(t) = z_1/z_0 = \xi_1(0)e^{i(E_0 - E_1)t/\hbar}$$

Hence once the initial state is given,  $\zeta_0(0)$  if  $z_1(0) \neq 0$ , or  $\xi_1(0)$  if  $z_0(0) \neq 0$ , the subsequent magnitudes of  $\zeta_0(t)$  or  $\xi_1(t)$  do not change, only their phases. Hence the evolution of a quantum state is described by circles of constant latitude, the state orbits at a rate  $\dot{\phi} = (E_1 - E_0)/\hbar$  (see fig. 2) and these are the flow lines of the Hamiltonian vector field on  $S^2$ .

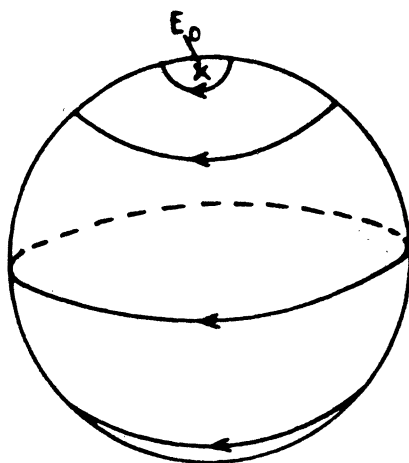


FIG. 2. —  $E_0 > E_1$ .

The state does not move from its initial position on  $CP^1$  if  $E_0 = E_1$  or if it is at one of the zeros of the Hamiltonian vector field at  $t = 0$ . The zeros of the Hamiltonian vector field therefore are at the eigenstates.

The Fubini-Study metric on  $CP^1$  (i. e. the standard metric on  $S^2$ ) has  $SO(3)$  as its group of isometries and the flow lines of the Hamiltonian vector field correspond to the flow lines of the Killing vector  $\frac{\partial}{\partial \phi}$  of this metric.

### b) Time Dependent Perturbation Theory.

Following the usual construction of perturbation theory [2] suppose that the Hamiltonian can be decomposed into the sum of a time independent part,  $H$ , and a time dependent part,  $V$ , proportional to some small parameter. The Schrödinger equation is now

$$i\hbar \frac{d}{dt} \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = (H + V) \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}$$

Define the evolution operator,  $T$ , by

$$\begin{pmatrix} z_0(t) \\ z_1(t) \end{pmatrix} = T(t) \begin{pmatrix} z_0(0) \\ z_1(0) \end{pmatrix}$$

Let

$$T^* = \exp(iHt/\hbar) T = \begin{pmatrix} e^{iE_0 t/\hbar} & 0 \\ 0 & e^{iE_1 t/\hbar} \end{pmatrix} T$$

where  $E_0$  and  $E_1$  are the eigenvalues of the unperturbed Hamiltonian,  $H$ . Also let

$$V^* = \exp(iHt/\hbar) V \exp(-iHt/\hbar)$$

Then the Schrödinger equation reduces to

$$\frac{dT^*}{dt} = -\frac{i}{\hbar} V^* T^*$$

Now expand  $T^*$  in the small parameter

$$T^* = 1_{2 \times 2} + T_1^* + T_2^* + \dots$$

where  $T_n^*$  is  $n^{\text{th}}$  order in the small parameter, giving

$$\frac{dT_1^*}{dt} = \frac{i}{\hbar} V^*, \quad \frac{dT_2^*}{dt} = -\frac{i}{\hbar} V^* T_1^*, \quad \text{etc}$$

Consider the first order equation

$$\frac{dT_1^*}{dt} = \frac{i}{\hbar} V^* \Rightarrow T_1^* = -\frac{i}{\hbar} \int_0^t V^*(t') dt'$$

Suppose that, in the eigenbasis of the unperturbed Hamiltonian,  $V$  has the form

$$V(t) = \begin{pmatrix} v_{00}(t) & v_{01}(t) \\ \bar{v}_{01}(t) & v_{11}(t) \end{pmatrix} \Rightarrow V^*(t) = \begin{pmatrix} v_{00}(t) & [e^{i(E_0 - E_1)t/\hbar} \bar{v}_{01}(t)] \\ [e^{i(E_1 - E_0)t/\hbar} v_{01}(t)]_0 & v_{11}(t) \end{pmatrix}$$

with  $v_{00}$  and  $v_{11}$  real. Then, to first order in  $v_{ij}$

$$\begin{pmatrix} z_0^{(1)}(t) \\ z_1^{(1)}(t) \end{pmatrix} = \left\{ \exp\left(-\frac{iHt}{\hbar}\right) \right\} \{1_{2 \times 2} + T_1^*\} \begin{pmatrix} z_0(0) \\ z_1(0) \end{pmatrix} = \begin{pmatrix} e^{-iE_0 t/\hbar} & 0 \\ 0 & e^{-iE_1 t/\hbar} \end{pmatrix} \times \\ \begin{pmatrix} 1 - \frac{i}{\hbar} \int_0^t v_{00}(t') dt' & -\frac{i}{\hbar} \int_0^t e^{i(E_0 - E_1)t'/\hbar} v_{01}(t') dt' \\ -\frac{i}{\hbar} \int_0^t e^{i(E_1 - E_0)t'/\hbar} \bar{v}_{01}(t') dt' & 1 - \frac{i}{\hbar} \int_0^t v_{11}(t') dt' \end{pmatrix} \begin{pmatrix} z_0(0) \\ z_1(0) \end{pmatrix}$$

where the superscript on  $z_i^{(1)}$  means first order in  $v_{ij}$ .

This gives

$$z_0^{(1)}(t) = \left\{ 1 - \frac{i}{\hbar} \int_0^t v_{00}(t') dt' \right\} e^{-iE_0 t/\hbar} z_0(0) - \frac{i}{\hbar} e^{-iE_0 t/\hbar} z_1(0) \int_0^t e^{i(E_0 - E_1)t'/\hbar} v_{01}(t') dt'$$

$$z_1^{(1)}(t) = \left\{ 1 - \frac{i}{\hbar} \int_0^t v_{11}(t') dt' \right\} e^{-iE_1 t/\hbar} z_1(0) - \frac{i}{\hbar} e^{-iE_1 t/\hbar} z_0(0) \int_0^t e^{i(E_1 - E_0)t'/\hbar} \bar{v}_{01}(t') dt'$$

$$\Rightarrow \xi_0^{(1)}(t) = \frac{z_0^{(1)}(t)}{z_1^{(1)}(t)} = \frac{z_0(0)}{z_1(0)} e^{-i(E_0 - E_1)t/\hbar} \left\{ 1 - \frac{i}{\hbar} \int_0^t v_{00}(t') dt' + \frac{i}{\hbar} \int_0^t v_{11}(t') dt' + \frac{i}{\hbar} \frac{z_0(0)}{z_1(0)} \int_0^t e^{i(E_1 - E_0)t'/\hbar} \bar{v}_{01}(t') dt' - \frac{i}{\hbar} \frac{z_1(0)}{z_0(0)} \int_0^t e^{i(E_0 - E_1)t'/\hbar} v_{01}(t') dt' \right\}$$

and

$$\xi_1^{(1)}(t) = \frac{z_1^{(1)}(t)}{z_0^{(1)}(t)} = \frac{z_1(0)}{z_0(0)} e^{-i(E_1 - E_0)t/\hbar} \left\{ 1 + \frac{i}{\hbar} \int_0^t v_{00}(t') dt' - \frac{i}{\hbar} \int_0^t v_{11}(t') dt' - \frac{i}{\hbar} \frac{z_0(0)}{z_1(0)} \int_0^t e^{i(E_1 - E_0)t'/\hbar} \bar{v}_{01}(t') dt' + \frac{i}{\hbar} \frac{z_1(0)}{z_0(0)} \int_0^t e^{i(E_0 - E_1)t'/\hbar} v_{01}(t') dt' \right\}.$$

As an example of the use of these formulae, consider the case where  $v_{ij} = 0$  for  $t < 0$  and  $v_{ij}$  are constant for  $t > 0$ , with  $v_{01} = v_{10} = v > 0$  real.

Then if  $E_0 \neq E_1$ ,

$$\xi_0^{(1)}(t) = \frac{z_0(0)}{z_1(0)} e^{-i(E_0 - E_1)t/\hbar} \left\{ 1 + \frac{it}{\hbar} (v_{11} - v_{00}) - \frac{v}{E_0 - E_1} \frac{z_1(0)}{z_0(0)} [e^{i(E_0 - E_1)t/\hbar} - 1] + \frac{v}{E_1 - E_0} \frac{z_0(0)}{z_1(0)} [e^{i(E_1 - E_0)t/\hbar} - 1] \right\}$$

with a similar expression for  $\xi_1^{(1)}(t)$ .

To picture the evolution of the system consider the following cases,

i)  $z_0(0) = 0, E_0 = -E_1 = E > 0$  (i. e. the system sits at the north pole until  $V$  is switched on). Then  $\xi_0(0)$  is well defined and

$$\xi_0^{(1)}(t) = \left( \frac{v}{E} \sin Et/\hbar \right) e^{-iEt/\hbar - i\pi/2}.$$

Thus

$$\arg \xi_0^{(1)}(t) = \phi(t) = -Et/\hbar - \pi/2 \text{ for } \sin Et/\hbar > 0 \\ = -Et/\hbar + \pi/2 \text{ for } \sin Et/\hbar < 0$$



and

$$|\xi_0^{(1)}(t)| = \frac{v}{E} |\sin Et/\hbar| = \tan \frac{\theta}{2}.$$

For  $\frac{v}{E} \ll 1$ ,

$$\theta \approx \frac{2v}{E} |\sin Et/\hbar| = -\frac{2v}{E} \cos \phi.$$

Hence the system describes a small circle, radius  $\frac{v}{E}$ , tangent to the north pole (see fig. 3). The period is  $\hbar/2E$  since the circle is covered twice for  $0 \leq t \leq 2\pi\hbar/E$ .

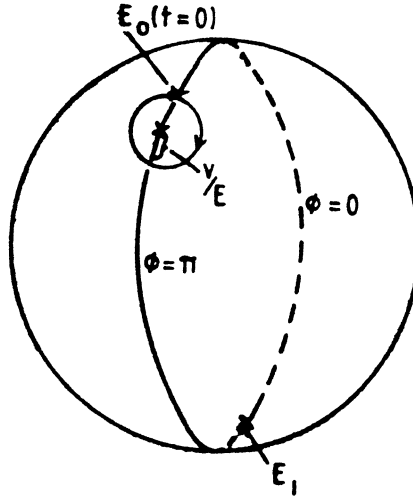


FIG. 3.

ii)  $z_0(0) = 1, z_1(0) = i, E_0 = -E_1 = E > 0, v_{11} = v_{00} > 0$ .

Either  $\xi_0$  or  $\xi_1$  will do as a co-ordinate. Take  $\xi_0$

$$\xi_0^{(1)}(t) = \left(1 + \frac{v}{E} \sin 2Et/\hbar\right) e^{-2iEt/\hbar - i\pi/2}$$

Thus

$$\arg \xi_0^{(1)}(t) = -2Et - \pi/2$$

$$|\xi_0^{(1)}(t)| = \left(1 + \frac{v}{E} \sin 2Et/\hbar\right) = \tan \frac{\theta}{2}.$$

For  $\frac{v}{E} \ll 1$ , the system circles round the equator until  $V$  is switched on at  $t = 0$ . Thereafter the system follows the great circle given by

$$\begin{aligned} \theta &= \frac{\pi}{2} + \frac{v}{E} \sin 2Et/\hbar \\ \phi &= -2Et - \pi/2 \end{aligned}$$

which is a circle tilted through an angle  $\frac{v}{E}$  (see fig. 4).

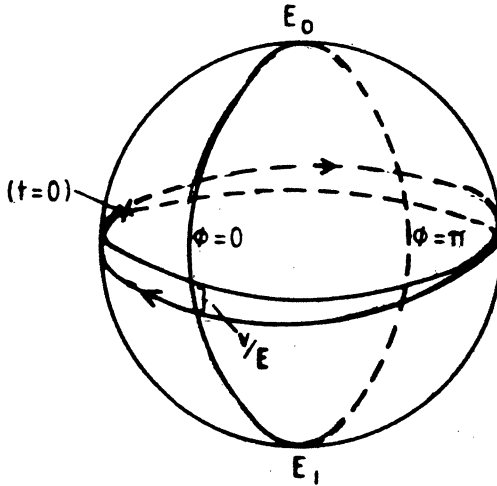


FIG. 4.

Finally consider the case of degeneracy  $E_0 = E_1 = E > 0$ . Then, with  $v_{00} = v_{11} > 0$  and  $v_{01} = v_{10}$  real ( $> 0$ )

$$\xi_0^{(1)}(t) = \frac{z_0(0)}{z_1(0)} \left\{ 1 + \frac{ivt}{\hbar} \left( \frac{z_0(0)}{z_1(0)} - \frac{z_1(0)}{z_0(0)} \right) \right\}.$$

For example  $z_0(0) = 0$ . Then the system sits at the north pole until  $t = 0$ , when  $V$  is switched on. Subsequently

$$\xi_0^{(1)}(t) = -\frac{ivt}{\hbar} \Rightarrow \arg \xi_0^{(1)}(t) = -\pi/2, \quad |\xi_0^{(1)}(t)| = \frac{vt}{\hbar}.$$

Thus, provided  $\frac{vt}{\hbar} \ll 1$ ,  $\theta$  grows linearly  $\theta = \frac{2vt}{\hbar}$ , and  $\phi$  remains constant.

Hence the system begins to roll down a great circle of constant longitude (see fig. 5).

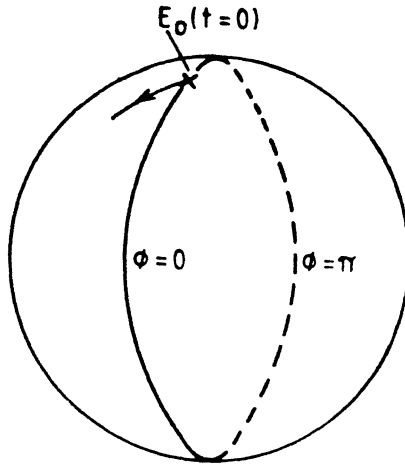


FIG. 5.

**c) Non-commuting Hermitian operators.**

Given a time independent Hamiltonian, H, with eigenvalues  $E_0$  and  $E_1$ , consider any other Hermitian operator

$$A = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \text{ } a \text{ and } c \text{ real}$$

in the Hamiltonian eigenbasis. Let  $\begin{pmatrix} 1 \\ n_+ \end{pmatrix}$  and  $\begin{pmatrix} 1 \\ n_- \end{pmatrix}$  be eigenvectors of A (this assumes that  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  is not an eigenvector) with eigenvalues  $\lambda_+$  and  $\lambda_-$  respectively.

Then

$$\lambda_{\pm} = \frac{1}{2} \{ a + c \pm [(a - c)^2 + 4|b|^2]^{\frac{1}{2}} \}$$

$$n_{\pm} = \eta \left\{ 1 \pm \left[ 1 + \frac{1}{|\eta|^2} \right]^{\frac{1}{2}} \right\}$$

where  $\eta = \frac{c - a}{2b}$ ,  $c \neq a$ . If  $c = a$ , then

$$n_{\pm} = \pm \bar{b}/|b|.$$

The two eigenstates must correspond to two points on  $S^2$ . These are given by

$$\xi_1^+ = n_+ \quad \text{and} \quad \xi_1^- = n_-$$

Hence

$$\begin{aligned} \arg \xi_1^+ &= \arg \eta = \phi_+ & \arg \xi_1^- &= \arg \eta + \pi = \phi_- \\ |\xi_1^+| &= |\eta| + [1 + |\eta|^2]^{\frac{1}{2}} = \tan(\theta_+/2) \\ |\xi_1^-| &= [1 + |\eta|^2]^{\frac{1}{2}} - |\eta| = \tan(\theta_-/2) \end{aligned}$$

Thus

$$\begin{aligned} \tan(\theta_+/2) &= \frac{1}{\tan(\theta_-/2)} = -\tan\left(\frac{\theta_- - \pi}{2}\right) \\ &\Rightarrow \theta_+ = \pi - \theta_- \quad \phi_+ = \pi + \phi_- \end{aligned}$$

(if  $c = a$ , then  $\theta_+ = \theta_- = \pi/2$ ) and the eigenstates of  $A$  lie diametrically opposite one another on  $S^2$ . To visualise the situation, consider the following two cases.

i) If  $b$  is real and  $b \ll 1, \eta \gg 1, c \neq a$ . Then

$$\theta_+ = \pi - \frac{1}{2\eta} \quad \theta_- = \frac{1}{2\eta} \quad \phi_+ = 0 \quad \phi_- = \pi$$

(see fig. 6) Lines of constant mixing angle for the eigenstates of  $A$  are circles of constant latitude relative to the eigenpoles of  $A$ . These intersect the circles of constant  $\theta$ , since the eigenpoles of  $A$  are tilted relative to the eigenpoles of the Hamiltonian through the angle  $\theta_- = 1/2\eta$ .

If  $A$  is a perturbed Hamiltonian, then this shows that the results

i) and ii) of part b) for the time dependent perturbation theory are exact for small  $\frac{v}{E}$  i. e. the orbits are correct for all time.

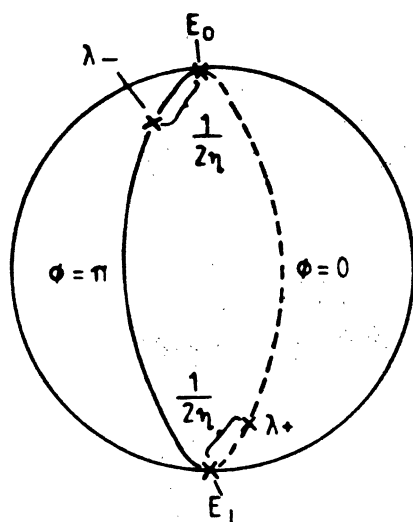


FIG. 6.

ii) If  $a = c$ , then the eigenpoles of  $A$  lie on the equator of the Hamiltonian. If the state is set up at  $t = 0$  with an equal mixture of eigenstates of the Hamiltonian, then its subsequent evolution will take it directly through both eigenstates of  $A$ , with period  $\hbar/(E_0 - E_1)$ .

### THE THREE STATE QUANTUM SYSTEM

Consider a quantum system with three discrete eigenstates and a time independent Hamiltonian. Topologically the space of all possible quantum states in  $\mathbb{C}P^2$ . The Schrödinger equation for a Hamiltonian with eigenvalues  $E_0, E_1$  and  $E_2$  is

$$\frac{d}{dt} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix} = -\frac{i}{\hbar} \begin{pmatrix} E_0 & 0 & 0 \\ 0 & E_1 & 0 \\ 0 & 0 & E_2 \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix} \Rightarrow z_j(t) = z_j(0) e^{-iE_j t/\hbar}.$$

If  $z_0(0) \neq 0$ , let  $\xi_1(t) = z_1(t)/z_0(t)$  and  $\xi_2(t) = z_2(t)/z_0(t)$ . (The case  $z_0(0) = 0$  can be treated in an obvious way, by choosing a different co-ordinate chart in  $\mathbb{C}P^2$ )  $\xi_1$  and  $\xi_2$  are complex co-ordinates on  $\mathbb{C}P^2$ , and the system follows a one dimensional curve in  $\mathbb{C}P^2$ , parameterised by  $t$  and given explicitly by

$$\begin{aligned} \xi_1(t) &= \xi_1(0) e^{-i(E_1 - E_0)t/\hbar} \\ \xi_2(t) &= \xi_2(0) e^{-i(E_2 - E_0)t/\hbar} \end{aligned}$$

Unlike the two eigenstate system, this curve is not closed unless

$$n(E_2 - E_0) = m(E_1 - E_0)$$

for some integers  $n$  and  $m$  (if either  $E_1 = E_0$  or  $E_2 = E_0$ , one co-ordinate remains fixed and the motion is restrained to  $\mathbb{C}P^1$ ).

Let

$$\begin{aligned} \xi_1 &= r_1 e^{i\phi_1} & 0 \leq r_1 < \infty, & & 0 \leq \phi_1 < 2\pi \\ \xi_2 &= r_2 e^{i\phi_2} & 0 \leq r_2 < \infty, & & 0 \leq \phi_2 < 2\pi \end{aligned}$$

then the time evolution of the system is such that  $r_1$  and  $r_2$  are fixed (by initial conditions) but  $\phi_1$  and  $\phi_2$  vary with time. Thus the curve described by the system lies on a two dimensional torus embedded in  $\mathbb{C}P^2$ .

Explicitly

$$\begin{aligned} \phi_1(t) &= \phi_1(0) + (E_0 - E_1)t/\hbar \\ \phi_2(t) &= \phi_2(0) + (E_0 - E_2)t/\hbar \end{aligned}$$

The situation is depicted figs. 7 and 8.

The point  $\xi_1 = \xi_2 = 0$  does not move with time. It is a stationary state,

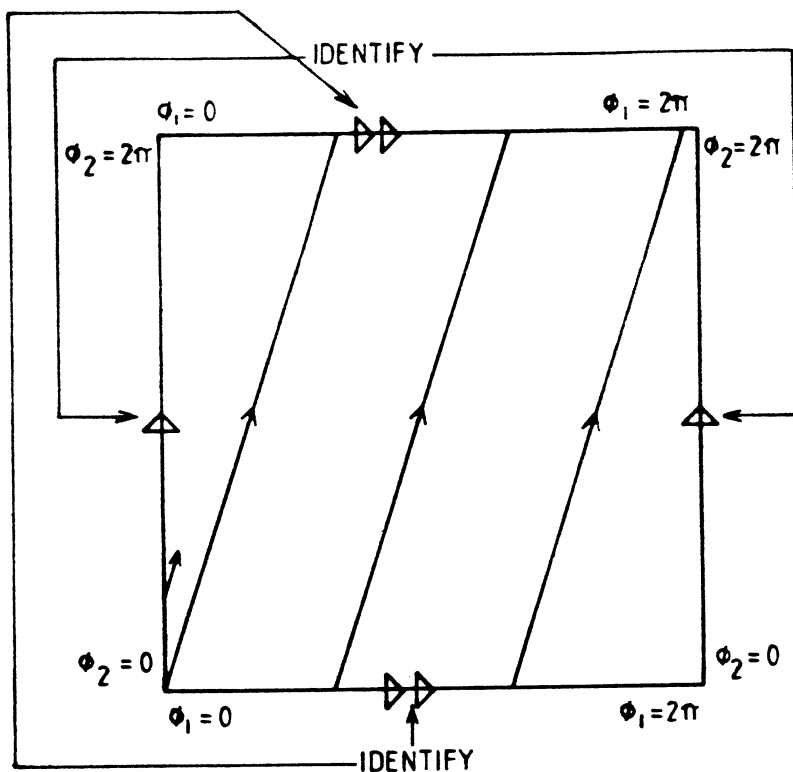


FIG. 7. —  $E_0 - E_2 > E_0 - E_1 > 0$ .

the eigenstate of the Hamiltonian associated with the eigenvalue  $E_0$ . There are two other stationary points (which require different co-ordinate charts) given by

$$\begin{aligned} \xi'_1 = z_0/z_1 = 0 & \quad \text{for } z_1 \neq 0 \text{ and} & \quad \xi''_1 = z_0/z_2 = 0 & \quad \text{for } z_2 \neq 0. \\ \xi'_2 = z_2/z_1 = 0 & & \quad \xi''_2 = z_1/z_2 = 0 & \end{aligned}$$

These three stationary points correspond to the three zeros of the Hamiltonian vector field on  $\mathbb{C}P^2$ . That there are three zeros of any vector field on  $\mathbb{C}P^2$  follows from the fact that the Euler characteristic,  $\chi$ , of  $\mathbb{C}P^2$  is 3.

Now consider the following change in co-ordinates

$$\begin{aligned} \xi_1 &= \rho \cos \frac{\theta}{2} \exp \left[ \frac{i}{2}(\psi + \phi) \right] \\ \xi_2 &= \rho \sin \frac{\theta}{2} \exp \left[ \frac{i}{2}(\psi - \phi) \right] \end{aligned}$$

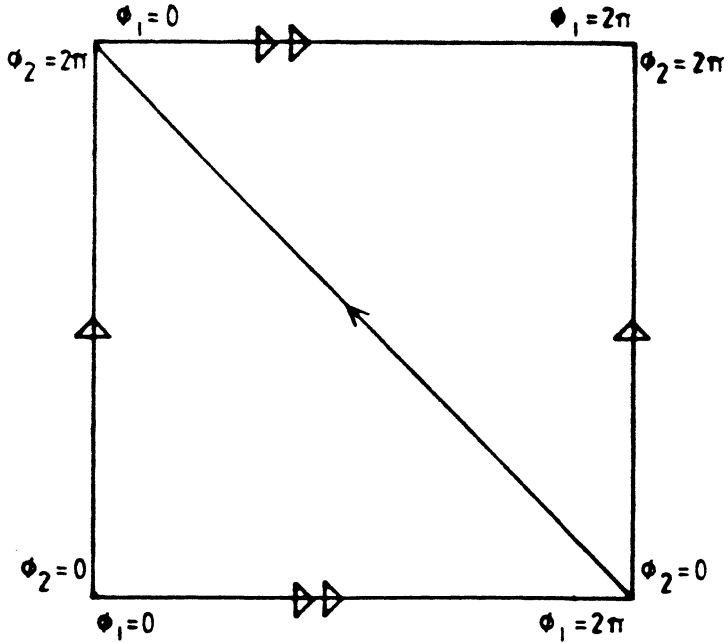


FIG. 8. —  $E_0 = 0, E_1 = -E_2 > 0$ .

where  $0 \leq \rho < \infty, 0 \leq \theta < \pi, 0 \leq \phi < 2\pi, 0 \leq \psi < 4\pi$ ,  $\theta, \phi$  and  $\psi$  are Euler angles on  $S^3$  ( $\phi$  is a different co-ordinate from  $\phi_1$  and  $\phi_2$  used earlier). In this co-ordinate system, the Fubini-Study metric on  $CP^2$  is given by [1]

$$ds^2 = (1 + \rho^2)^{-2} \left\{ d\rho^2 + \frac{1}{4} \rho^2 (\cos \theta d\phi + d\psi)^2 \right\} + \frac{1}{4} (1 + \rho^2)^{-1} \rho^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$

The tangent vectors to the torus on which a curve, obeying the Schrödinger equation, lies are  $\frac{\partial}{\partial \psi}$  and  $\frac{\partial}{\partial \phi}$ , for fixed  $r$  and  $\theta$ . These two vector fields are Killing vectors for the metric,  $g$ , and obviously commute. The group of isometries of  $g$  is  $SU(3)/\mathbb{Z}_3$ , which has rank 2. Thus  $\frac{\partial}{\partial \psi}$  and  $\frac{\partial}{\partial \phi}$  actually span the Cartan subalgebra (i. e. the maximal Abelian subalgebra) of the algebra of Killing symmetries for  $g$  (in the standard representation of

the  $SU(3)$  algebra,  $\frac{\partial}{\partial\psi}$  and  $\frac{\partial}{\partial\phi}$  would be linear combinations of  $\lambda_3$  and  $\lambda_8$ ).

This is a generalisation of the  $CP^1$  case, where the Cartan subalgebra was only one dimensional and so the flow lines lay on  $T^1 \approx S^1$  rather than  $T^2$ .

Thus we are led to the interesting interpretation of the torus on which the curve for a quantum system lies — it is the surface generated by the flow lines of the Killing vectors in the Cartan subalgebra of the group of isometries of the Fubini-Study metric on  $CP^2$ . The size of the torus is specified by the initial conditions  $\rho(0)$  and  $\theta(0)$ , and the curve followed on the torus is specified by the initial conditions  $\psi(0)$  and  $\phi(0)$  and the Hamiltonian eigenvalues. As before, if the Hamiltonian is time independent, the curve must not intersect itself, but just winds round the torus indefinitely (fig. 7).

Of course, all this is only valid if the system is left undisturbed. If a measurement is made, the system immediately jumps to a point in  $CP^2$  corresponding to an eigenvalue of measured operator.

### THE $(n + 1)$ -STATE QUANTUM SYSTEM

A system with  $(n + 1)$  eigenstates is topologically  $CP^n$ . Generalising the results of the previous two sections, the time evolution of such a system (if left undisturbed) will be described by a one dimensional curve on an  $n$ -dimensional torus,  $T^n$ , the direction of the curve being specified by the Hamiltonian eigenvalues  $E_0, \dots, E_n$ . The Hamiltonian, considered as a vector field on  $CP^n$ , will have  $(n + 1)$  zeros since the Euler characteristic for  $CP^n$  is  $n + 1$  and so there will be  $n + 1$  stationary states, the eigenstates of the Hamiltonian (corresponding to the limit as the torus shrinks to zero size).

The group of isometries of the Fubini-Study metric on  $CP^n$  is  $SU(n + 1)/Z_{n+1}$  which has rank  $n$  and therefore the Cartan subalgebra will have dimension  $n$ . For given initial conditions, the torus will be generated by the flow lines of the Killing vector fields for the metric which lie in the Cartan subalgebra.

In the limit of  $n \mapsto \infty$  (e. g. the Harmonic Oscillator) the manifold  $CP^\infty$  is well defined and so can be used to describe such a discrete system. A continuum system (i. e. one with an eigenvalue spectrum which is continuous) however, cannot be described in this way since the infinity of eigenvalues are dense in the real line and it is not possible to set up a one to one correspondence between the eigenvalues and the integers which label the co-ordinates in  $CP^\infty$ .



## POSITION DEPENDENT HAMILTONIANS

So far only abstract Hamiltonians have been considered, with given eigenvalues (possibly time dependent). Now consider the case where the Hamiltonian depends on position (it will be assumed that the number of eigenvalues does not depend on position). When the Hamiltonian depends on position, the situation is more complicated. In a full quantum treatment, the eigenvalues of the Hamiltonian are only obtained after integrating bilinears in eigenfunctions over all space. In general, the system will have both discrete and continuous sectors. One approach would be to put the system in a finite box, which would make the spectrum totally discrete.

Here a simpler, semi-classical treatment will be applied. The position of the particle will be treated classically, while the internal degrees of freedom (*e. g.* spin) will be treated quantum mechanically. This situation is, mathematically, a  $\mathbb{C}P^n$  bundle over three dimensional space, with a different Hamiltonian vector field at each point in three space. As we move about in three space, the zeros of the vector field will move about on  $\mathbb{C}P^n$ . Choosing one such zero gives a unique point on  $\mathbb{C}P^n$  associated with each point in space and so gives a section of the  $\mathbb{C}P^n$  bundle over the three space (this assumes that the eigenvalues of the Hamiltonian are non-degenerate). The fibre group for the bundle will be  $SU(n+1)/\mathbb{Z}_{n+1}$  since the Hamiltonian vector field can be moved about by Lie transport along the flow lines of the action of this group on  $\mathbb{C}P^n$ .

For example, consider the case of an electron moving in the classical background of a charged particle with, possibly external, magnetic fields. The electron's spin is a two state quantum system and so can be described by  $\mathbb{C}P^1 \approx S^2$ . Since the field of the particle is infinite at the origin, this point must effectively be removed from space, giving space the topology  $\mathbb{R}^+ \times S^2$  ( $\mathbb{R}^+$  = positive real numbers). The system can therefore be described by an  $S^2$  bundle over  $\mathbb{R}^+ \times S^2$ , the structure group of which will be  $SO(3) \approx SU(2)/\mathbb{Z}_2$ .

If the Hamiltonian for the system is given, then we know how the zeros of the Hamiltonian vector field on the quantum  $S^2$  move as we move about in three space. This associates an element of  $SO(3)$  (a rotation of the quantum  $S^2$ ) with every one dimensional path in space. Taking the limit as the paths shrink to zero gives one element of the Lie algebra of  $SO(3)$  associated with every direction in three space, *i. e.* an  $SO(3)$  Lie algebra valued connection on the bundle. Thus the Hamiltonian determines the connection and hence the global topology of the bundle.

Since  $\mathbb{R}^+$  is contractible, bundles over  $\mathbb{R}^+ \times S^2$  are in one to one correspondence with bundles over  $S^2$ . For any bundle with structure group,  $G$ , over a sphere,  $S^n$ , the bundle is classified by an element of  $\pi_{n-1}(G)$  [3].

Now  $\pi_1(\text{SO}(3)) = \mathbb{Z}_2$ , therefore we know that there are two inequivalent bundles for this system. One is obvious, the trivial bundle, where a constant, external magnetic field is applied to *e. g.* a Hydrogen atom, and the electron's spin is aligned parallel or anti-parallel, as in the Zeeman effect. Then the zeros of the Hamiltonian vector field do not move on the quantum  $S^2$  at all as we move about the spatial  $S^2$ , and the  $\text{SO}(3)$  connection is everywhere zero.

The other, non-trivial, bundle is also of interest, however. To see what physical system this bundle corresponds to, consider the elements of  $\pi_1(\text{SO}(3))$ . If, in moving round a closed path in  $\text{SO}(3)$ , starting from the identity, we arrive back at the identity via a rotation of  $4\pi n$ ,  $n$  an integer, then this corresponds to the identity element of  $\pi_1(\text{SO}(3))$ . If however, we arrive back after a rotation of  $(4n + 2)\pi$ , then this corresponds to the non-trivial element of  $\pi_1(\text{SO}(3))$ . Suppose  $n = 0$ . If we follow a closed path on the spatial  $S^2$ , *e. g.* round the equator, then this will correspond to a closed path in  $\text{SO}(3)$ , given by the movement of the zeros of the Hamiltonian vector field (spin up and spin down in the magnetic field). If the zeros of the Hamiltonian vector field rotate through  $2\pi$  on the quantum  $S^2$ , then we have a quantum system corresponding to the non-trivial element of  $\pi_1(\text{SO}(3))$ . (Remember that the zeros of the Hamiltonian vector field are always diametrically opposite one another).

Such a configuration is supplied by an electron moving in a radial magnetic field. Then the eigenstates of the Hamiltonian correspond to the spin pointing radially inwards or radially outwards, and they rotate through  $2\pi$  on the quantum  $S^2$  as we go once round the equator of the spatial  $S^2$ .

Let  $0 \leq \alpha < \pi$  and  $0 \leq \beta < 2\pi$  be polar co-ordinate on the spatial  $S^2$ , radius  $r$  ( $\theta$  and  $\phi$  are reserved for polar co-ordinates on the quantum  $S^2$ ) and

$$\begin{aligned} B_1 &= B \sin \alpha \cos \beta \\ B_2 &= B \sin \alpha \sin \beta \\ B_3 &= B \cos \alpha \end{aligned}$$

where  $B = g/r^2$  with  $g$  the monopole strength. The Hamiltonian is

$$H = \underline{\sigma} \cdot \underline{B} = \begin{pmatrix} B_3 & B_1 - iB_2 \\ B_1 + iB_2 & -B_3 \end{pmatrix}.$$

Then, on the quantum  $S^2$ , with co-ordinate  $\xi_1$  as in section 2c, the zeros of  $H$  are given by

$$\begin{aligned} \xi_1^+ &= \frac{B_1 - iB_2}{B_3 + B} = \tan \frac{\alpha}{2} e^{i\beta} \\ \xi_1^- &= \frac{B_1 - iB_2}{B_3 + B} = -\cot \frac{\alpha}{2} e^{i\beta} = \cot \frac{\alpha}{2} e^{i\beta + i\pi}. \end{aligned}$$

So

$$\phi^+ = \beta \quad |\xi_1^+| = \tan \frac{\alpha}{2} \Rightarrow \theta^+ = \alpha$$

and

$$\phi^- = \beta + \pi \quad |\xi_1^-| = \cot \frac{\alpha}{2} \Rightarrow \theta^- = \pi - \alpha$$

showing that  $\phi^+$  rotates through  $2\pi$  as  $\beta$  does, as claimed.

Classically, there are no stable orbits for an electron in a monopole field. However, if the central charged particle has electric charge as well as magnetic charge, then stable orbits exist, and such a system could be realised as an electron orbiting a dyon.

More generally, for an  $n + 1$  state quantum system in the classical background of a charged particle, the relevant topology would be that of a  $\mathbb{C}\mathbb{P}^n$  bundle over  $\mathbb{R}^+ \times \mathbb{S}^2$ , with structure group  $\text{SU}(n + 1)/\mathbb{Z}_{(n+1)}$ . These bundles are characterised by  $\pi_1(\text{SU}(n + 1)/\mathbb{Z}_{n+1}) \approx \mathbb{Z}_{n+1}$ . Hence there are  $n + 1$  such inequivalent bundles.

## CONCLUSIONS

It has been shown how an  $(n + 1)$  state quantum system evolves, according to the Schrödinger picture, as a curve in  $\mathbb{C}\mathbb{P}^n$ . The curve lies on an  $n$ -dimensional torus,  $\mathbb{T}^n$ , which is generated by the flow lines of the Killing vectors spanning the Cartan subalgebra of the isometry group for the Fubini-Study metric on  $\mathbb{C}\mathbb{P}^n$ . The size of the torus is determined by the initial conditions of the quantum state. The one dimensional curve, and the rate at which it evolves, is determined by the Hamiltonian, which is viewed as a tangent vector to the curve. The Hamiltonian is a vector field on  $\mathbb{C}\mathbb{P}^n$  with  $n + 1$  zeros (corresponding to the fact that the Euler characteristic of  $\mathbb{C}\mathbb{P}^n$  is  $n + 1$ ). The points of  $\mathbb{C}\mathbb{P}^n$  at which the Hamiltonian vector field vanishes correspond to the  $n + 1$  eigenstates of the Hamiltonian.

When the Hamiltonian depends on position, the whole system can be viewed semi-classically as a  $\mathbb{C}\mathbb{P}^n$  bundle over three space, with fibre group  $\text{SU}(n + 1)/\mathbb{Z}_{n+1}$ . The Hamiltonian determines the connection. The specific case of an electron in a dyon field (corresponding to an  $\mathbb{S}^2$  bundle over  $\mathbb{S}^2$ ) has been considered in detail.

It is interesting to note that, if the connection is given dynamical status, we are naturally led to the notion of a non-relativistic  $\text{SU}(n + 1)$  gauge theory associable with  $n + 1$  state system.

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