Classical and Quantum Gravity

The Kahler 2-form in D=11 supergravity

Class. Quantum Grav. 2 (1985) 309-321. Printed in Great Britain

The Kahler 2-form in $D = 11$ supergravity

Brian P Dolan
Department of Natural Philosophy, University of Glasgow, Glasgow G12 8QQ, UK

Received 28 September 1984

Abstract. Solutions of $D=11$ supergravity are presented, in which the internal space consists of Kahler manifolds and S^1. The 4-form field is constructed from bilinears in the Kahler 2-forms as well as the volume element of four-dimensional spacetime. The group of isometries of the internal metric is $(\mathrm{SU}(3)/Z_3)\times(\mathrm{SU}(2)/Z_2)\times\mathrm{U}(1)$. Four-dimensional spacetime is either anti-de-Sitter or Robertson-Walker.

1. Introduction

There is much current interest in non-Abelian Kaluza-Klein theories in which the gauge group of internal symmetry acts on a compact, n-dimensional manifold, which is a submanifold of a larger $(4+n)$ dimensional spacetime [1-4]. Supergravity supplies an attractive method of incorporating fermions into Kaluza-Klein theories. In particular, eleven-dimensional supergravity seems to have the smallest number of dimensions in which one can hope to fit a gauge group with the algebra of $\mathrm{SU}(3)\times \mathrm{SU}(2)\times\mathrm{U}(1)$.

There are now many known solutions of the bosonic equations of motion of $D=11$ supergravity [5]. All of these solutions rely on the Freund-Rubin ansatz [6], in which the 4-form field, $\mathcal{F}$, consists of a term which is a constant multiple of the volume element of four-dimensional anti-de-Sitter spacetime, possibly with additional terms on the internal space. This results in a manifold which is a direct product $M_4\times B$, where M_4 is anti-de-Sitter space and B a compact, seven-dimensional manifold.

In this paper some other solutions will be explored, in which the internal space is constructed from Kahler manifolds and S^1, and the internal part of $\mathcal{F}$ is quadratic in the Kahler 2-forms. Abbreviated versions of some of the results have already appeared in [7] and [8]. Here more details will be presented, along with some new results. Kahler manifolds and Kahler 2-forms have been considered in the context of ten-dimensional Einstein-Maxwell systems by Watamura [9], [10] and in the dimensionally reduced $N=2$, non-chiral $D=10$ supergravity in references [11] and [12].

In § 2 the bosonic Lagrangian and equations of motion are presented in differential form language. In § 3 solutions are presented in which the internal space is $CP^2\times S^2\times S^1$ and four-dimensional spacetime is of Robertson-Walker type. In § 4 the problem of spinor structures on CP^2 is discussed and modified metrics for fibre bundles of S^1 and S^2 over CP^2 are presented. In § 5 and § 6 these 'twisted' metrics are used to construct more solutions. Section 7 contains a summary and conclusions. Notation and conventions are summarised in appendix 1 and the properties of orthonormal

1-forms for the Fubini-study metrics and Kahler 2-forms on CP^2 and $CP^1 \approx S^2$ are presented in appendix 2. Appendix 3 contains some details of calculations needed in § 5 and § 6.

2. Equations of motion

The bosonic part of eleven-dimensional supergravity has the following action density 11-form [13]

$$\Lambda = \mathbb{R}_{AB} \wedge \overset{*}{_{11}} E^{AB} - \mathscr{F} \wedge \overset{*}{_{11}} \mathscr{F} + \tfrac{2}{3}\lambda^{-1} \mathscr{F} \wedge \mathscr{F} \wedge \mathscr{A} \tag{1}$$

where $\mathbb{R}_{AB}$ are the curvature 2-forms obtained from the orthonormal 1-forms E^A via the torsion-free condition (see appendix 1). $\mathscr{F}$ is a 4-form derived, at least locally, from the 3-form potential $\mathscr{A}$, $\mathscr{F} = d\mathscr{A}$, whose presence in the full Lagrangian is necessary to balance the bosonic and fermionic degrees of freedom of eleven-dimensional supergravity [13]. λ is a constant and $\overset{*}{_{11}}$ the eleven-dimensional Hodge duality operator.

The Einstein equations obtained from (1) by varying the metric are

$$\mathbb{R}_{AB} \wedge \overset{*}{_{11}} E^{AB}{}_C = \mathscr{F} \wedge i_C(\overset{*}{_{11}} \mathscr{F}) - (i_C \mathscr{F}) \wedge \overset{*}{_{11}} \mathscr{F}. \tag{2}$$

For the definition of i_C, see appendix 1. Note that the third term of (1) does not contribute to (2) since it does not contain any information about the metric.

The matter field equations, obtained by varying the 3-form $\mathscr{A}$ in (1) are

$$d \overset{*}{_{11}} \mathscr{F} = \mathscr{F} \wedge \mathscr{F} / \lambda. \tag{3}$$

The potential $\mathscr{A}$ does not appear in the equations of motion (2) and (3) and in the following $\mathscr{F}$ will be such that no global potential exists (though one will always exist locally, on each coordinate patch, and on the coordinate overlaps two potentials can always be matched by a gauge transformation). Hence a third condition is imposed on $\mathscr{F}$

$$d\mathscr{F} = 0 \tag{4}$$

(2), (3) and (4) are the bosonic equations of $D = 11$ supergravity.

3. $B = CP^2 \times S^2 \times S^1$

Let the seven-dimensional internal space be $CP^2 \times S^2 \times S^1$ and take $E^a = e^a$, $E^m = e^m$ where e^a and e^m are orthonormal 1-forms for the Fubini-study metrics on CP^2 and S^2 respectively (see appendix 2). The gauge symmetry is $(\mathrm{SU}(3)/Z_3) \times (\mathrm{SU}(2)/Z_2) \times \mathrm{U}(1)$. With K_2 and K_1 the Kahler 2-forms on CP^2 and S^2 respectively, take the following ansatz for $\mathscr{F}$

$$\mathscr{F} = pE^{1230} + (q/2) K_2 \wedge K_2 + (r/\sqrt{2}) K_1 \wedge K_2 \tag{5}$$

with p, q and r constants. Then

$$\overset{*}{_{11}} \mathscr{F} = -(p/2) E^4 \wedge K_1 \wedge K_2 \wedge K_2 + q E^{12304} \wedge K_1 + (r/\sqrt{2}) E^{12304} \wedge K_2. \tag{6}$$

If the volume form of four-dimensional spacetime and E^4 are independent of position on $CP^2 \times S^2$ ($dE^{12304} = 0$) then

$$d\mathscr{F} = 0, \qquad d\overset{*}{_{11}}\mathscr{F} = -\frac{p}{2}\, dE^4_{\wedge} K_{1\wedge} K_{2\wedge} K_2, \tag{7}$$

since the Kahler 2-forms are closed.

By inspection $d\overset{*}{_{11}}\mathscr{F} = \lambda^{-1}\mathscr{F}_{\wedge}\mathscr{F} \neq 0$ is impossible, hence (3) and (4) are satisfied provided

$$d\overset{*}{_{11}}\mathscr{F} = \mathscr{F}_{\wedge}\mathscr{F} = 0 \Leftrightarrow pr = 0,\ pq = 0 \text{ and } pdE^4 = 0. \tag{8}$$

Therefore, either $p \neq 0$, $r = q = 0$ and $dE^4 = 0$ or $p = 0$. The case $p \neq 0$ is the usual Freund-Rubin ansatz, but $CP^2 \times S^2 \times S^1$ will not give a solution in this case since it does not admit an Einstein metric, due to the S^1 factor. The second case, $p = 0$, is different from the usual Freund-Rubin ansatz since $\mathscr{F}$ then contains no term on four-dimensional spacetime. For the remainder of § 3 it will be assumed that $p = 0$.

Now consider the Einstein equations (2) with $p = 0$:

$$\begin{aligned}
&\mathbb{R}_{AB\wedge}\overset{*}{_{11}}(E^{AB\alpha}) = (q^2 + r^2)\overset{*}{_{11}}E^{\alpha},\\
&\mathbb{R}_{AB\wedge}\overset{*}{_{11}}(E^{ABa}) = -q^2\overset{*}{_{11}}E^{a},\\
&\mathbb{R}_{AB\wedge}\overset{*}{_{11}}(E^{ABm}) = (q^2 - r^2)\overset{*}{_{11}}E^{m},\\
\Rightarrow\ &\mathbb{R}_{\beta\gamma\wedge}\overset{*}{_5}(E^{\beta\gamma\alpha}) = (q^2 + r^2 - R_{CP^2} - R_{S^2})\overset{*}{_5}E^{\alpha},\\
&\mathbb{R}_{bc\wedge}\hat{\overset{*}{_4}}(e^{bca}) = -(q^2 + R_5 + R_{S^2})\hat{\overset{*}{_4}}e^{a},\\
&0 = (q^2 - r^2 - R_5 - R_{CP^2})\hat{\overset{*}{_2}}e^{m}.
\end{aligned} \tag{9}$$

(For index conventions and definitions see appendices 1 and 2). Using (A1.1) and (A1.2), equations (9) give (with R_5 the curvature scalar for E^{α})

$$\begin{aligned}
&R_5 = \tfrac{5}{3}(q^2 + r^2 - R_{CP^2} - R_{S^2}),\\
&R_{CP^2} = -2(q^2 + R_5 + R_{S^2}), \qquad q^2 - r^2 = R_5 + R_{CP^2}\\
\Rightarrow\ &R_{CP^2} = \tfrac{2}{3}(r^2 + 4q^2), \qquad R_{S^2} = \tfrac{2}{3}(2r^2 - q^2), \qquad R_5 = -\tfrac{5}{3}(r^2 + q^2).
\end{aligned} \tag{10}$$

Thus (9) are solved if E^{α} is an orthonormal basis of 1-forms for a five-dimensional Einstein metric with negative scalar curvature

$$\mathbb{R}_{\beta\gamma\wedge}\overset{*}{_5}E^{\beta\gamma\alpha} = -(r^2 + q^2)\overset{*}{_5}E^{\alpha} \tag{11}$$

and the curvature scalars on CP^2 and S^2 are $\tfrac{2}{3}(r^2 + 4q^2)$ and $\tfrac{2}{3}(2r^2 - q^2)$ respectively (note that this requires $q^2 < 2r^2$).

All that remains to obtain a complete solution is to solve (11) with the fifth dimension topologically S^1. This problem has been considered in [7] and [14].

With the ansatz

$$g_{(5)} = \sum_{\alpha=0}^{4} E^{\alpha} \otimes E^{\alpha} = -dt \otimes dt + R^2(t) g^k_{(3)} + \Phi^2(t)\, d\xi \otimes d\xi \tag{12}$$

$0 \leq \xi < 2\pi$ and $g^k_{(3)}$ a metric for a maximally symmetric 3-space with $k = (+1, -1, 0)$ for positive, negative or zero 3-space curvature respectively, equations (11) reduce to

$(\cdot = \mathrm{d}/\mathrm{d}t)$

$$\begin{aligned} &\ddot{R}/R + (\dot{R}/R)^2 + k/R^2 = -(r^2+q^2)/6 \\ &(\dot{R}/R)^2 + \dot{R}\dot{\Phi}/R\Phi + k/R^2 = -(r^2+q^2)/6 \\ &2\ddot{R}/R + (\dot{R}/R)^2 + k/R^2 + \ddot{\Phi}/\Phi + 2\dot{R}\dot{\Phi}/R\Phi = -(r^2+q^2)/2 \end{aligned} \tag{13}$$

which have general solution† $\{\omega = [\frac{1}{3}(r^2+q^2)]^{1/2}\}$

$$\begin{aligned} &R(t) = [P\sin(\omega t+\delta) - 2k/\omega^2]^{1/2} \\ &\Phi(t) = \frac{Qw}{2}\frac{\cos(\omega t+\delta)}{[P\sin(\omega t+\delta) - 2k/\omega^2]^{1/2}} \end{aligned} \tag{14}$$

where P, Q and δ are constants. The special case $k=0$ and $r^2=2q^2$ was presented in [11] ($r^2=2q^2$ makes the metric on $CP^2\times S^2$ Einstein).

The limits of t must be chosen so that $(P\sin(\omega t+\delta) - 2k/\omega^2) \geq 0$ at all times. For $k=-1$ (3-space hyperbolic), if $|P| \leq 2/\omega^2$ then t can run over many cycles and R^2 will always be positive. However, if $k=0$, $+1$ or $|P| > 2/\omega^2$ then t must be restricted. For example, the case of flat 3-space, $k=0$, with $\delta=0$, must have $0 \leq t < \pi/\omega$ and it is not possible to take multiple copies of this range, as the metric would not be differential everywhere.

Since in Kaluza–Klein theories it is usually assumed that the size of the internal dimensions is of the order of a few tens or hundreds of the Planck length, ω^{-1} will be a few tens or hundreds of the Planck time. Hence solutions with $k=0$, $+1$ or $|P| > 2/\omega^2$ would correspond to universes lasting only $\sim 10^{-41}$ s: Thus the flat 3-space solution must be rejected on physical grounds, unless some way can be found to patch together many cycles in such a way as to make the joins smooth.

The solutions with $k=-1$ and $|P| < 2/\omega^2$ can be patched together over many cycles in a smooth fashion, but still suffer from the pathology of high spatial curvature, in common with the usual anti-de-Sitter solutions.

A further difficulty with these solutions is that of the non-existence of a spin structure on CP^2 [15], [18]. Hence, without further modification, it does not seem possible to introduce spinors into these solutions. This problem is addressed in the next section.

Solutions can be obtained using Kahler manifolds other than CP^2 in the same way e.g. CP^3 or $S^2\times S^2\times S^2$, but these have the wrong gauge symmetry [7].

4. Spin^c structures on CP^2

A possible way round the problem of the lack of spin structure on CP^2 is to introduce a spin^c structure [15], [16]. This means introducing a gauge field and using minimal coupling in such a way that spinors can be well defined. Rather than introducing *ad hoc* gauge fields, which would be contrary to the basic tenet of supergravity, it is possible to use some of the remaining components of the internal, seven-dimensional, metric to construct such gauge fields. The metric on $CP^2\times S^2\times S^1$, which up to now has been the direct product of the Fubini-study metrics on CP^2 and S^2 and a time dependent metric on S^1, can be modified to be a metric for a six-dimensional S^2 bundle

† I wish to thank I M Barbour for pointing out the most general solution to me.

over CP^2 direct producted with S^1 or, a five-dimensional S^1 bundle over CP^2, direct producted with S^2.

Such metrics can be constructed by starting from the Fubini-study metric on CP^2 and considering the torsion free, metric compatible connection, which is an SO(4) Lie algebra valued 1-form. Since the Lie algebra of SO(4) is isomorphic to that of $SU(2)\times SU(2)$, these 1-forms can be split into two SU(2) Lie algebra valued 1-forms, one self-dual and one anti-self dual. These 1-forms are constructed from the SO(4) Lie algebra valued connection 1-forms on CP^2 (see appendix 2) as follows

$$A^1_\pm=\tfrac{1}{2}(\omega_{10,7}\pm\omega_{8,9}),\qquad A^2_\pm=\tfrac{1}{2}(\omega_{10,8}\pm\omega_{9,7}),\qquad A^3_\pm=\tfrac{1}{2}(\omega_{10,9}\pm\omega_{7,8}). \tag{15}$$

Using the explicit expressions for ω_{ab} in terms of e^a, these are

$$\begin{aligned}&A^1_+=A^2_+=0, && A^3_+=(3/2b)re^9\equiv A_+,\\ &A^1_-=-e^7/br, && A^2_-=-e^8/br, \qquad A^3_-=[-(2+r^2)/2br]e^9.\end{aligned} \tag{16}$$

For the self-dual case, A^i_+ $(i=1,2,3)$ the SU(2) Lie algebra valued 1-forms reduce to the Lie algebra of U(1), while the anti-self-dual 1-forms remain in the full SU(2) Lie algebra. This reflects the fact that the holonomy group of CP^2 is $SU(2)\times U(1)$. Of course the A_+ and A^i_- are only defined on a single coordinate patch of CP^2 and a 'string' singularity results if this coordinate patch is extended to cover the whole of CP^2, just as in the Dirac monopole. To avoid this, A_+ and A^i_- must be defined differently on different coordinate patches and related by gauge transformations on the overlaps.

These SU(2) and U(1) connections can now be considered as gauge connections on CP^2 and introduced to the metric in the following way. Let

$$E^4=cd\xi+cA_+,\qquad E^a=e^a, \tag{17}$$

with $c=$ constant, be a five-dimensional metric on an S^1 bundle over CP^2, with isometry group $SU(3)/Z_3\times U(1)$, modulo U(1) gauge transformations of A_+. Alternatively, let

$$E^m=e^m-2aA^j_-L^m_j,\qquad E^a=e^a, \tag{18}$$

be a six-dimensional metric on an S^2 bundle over CP^2, with isometry group $SU(3)/Z_3\times SU(2)/Z_2$, modulo SU(2) gauge transformations on A^i_-. Here L^m_j are the components, in the e^m basis, of the three Killing vectors on S^2 $(j=1,2,3)$.

It is possible to use a combination of both (17) and (18) to obtain an $(S^2\times S^1)$ bundle over CP^2 which yields a solution of $D=11$ supergravity [17], but this will not be considered here as $\mathscr{F}$ does not involve the use of Kahler 2-forms.

To see that A_+ or A^j_- allow spinor structures to be defined on CP^2, one can use the index theorem for the Dirac operator in these background fields [15], [18]. First, construct the curvature 2-forms associated with these connections by taking the self-dual and anti-self-dual combinations of the curvature 2-forms (A2.9) in a manner similar to (15)

$$F_+=dA_+=(3/b^2)(e^{10}_\wedge e^9+e^7_\wedge e^8)=(3/b^2)K_2 \tag{19}$$

and

$$\begin{aligned}&F^j_-=dA^j_-+\varepsilon^j_{kl}A^k_{-\wedge}A^l_-,\\ &F^1_-=(1/b^2)(e^{10}_\wedge e^7-e^8_\wedge e^9),\\ &F^2_-=(1/b^2)(e^{10}_\wedge e^8-e^9_\wedge e^7),\qquad F^3_-=(1/b^2)(e^{10}_\wedge e^9-e^7_\wedge e^8).\end{aligned} \tag{20}$$

For the U(1) case, the index theorem gives

$$\nu_+ - \nu_- = -\frac{1}{8} + \frac{1}{8\pi^2}\int_{CP^2} F_{+\wedge} F_+ = -\frac{1}{8} + \frac{9}{8} = 1 \tag{21}$$

where ν_+ is the number of right-handed and ν_- the number of left-handed spinors on CP^2. That this is an integer reflects the fact that these spinors are well defined. $F_+ = 0$ in (21) gives $\nu_+ - \nu_- = -\frac{1}{8}$, an obvious absurdity which indicates the lack of a spin structure on CP^2 without gauge fields. In general, multiples of F_+ would also give a spinor structure e.g. nF_+ in (21) would yield

$$\nu_+ - \nu_- = -\tfrac{1}{8} + \tfrac{9}{8} n^2. \tag{22}$$

So, provided $(9n^2 - 1)$ is divisible by 8, nF_+ (derivable from nA_+) would give a spinor structure.

For the SU(2) case the index theorem gives

$$\nu_+ - \nu_- = -\frac{2}{8} + \frac{2}{8\pi^2}\int_{CP^2} F^j_{-\wedge} F^k_- \delta_{jk} = -\frac{1}{4} - \frac{3}{4} = -1, \tag{23}$$

showing that a spinor structure is again well defined in this case. This construction was developed, though not applied to $D = 11$ supergravity in [8].

Thus (17) and (18) allow spinor structures to be defined on CP^2, but it is still necessary to check the equations of motion.

5. $B = (CP^2 \times S^1)_{\text{twisted}} \times S^2$

To look for solutions of the bosonic equations of eleven-dimensional supergravity, consider an ansatz similar to (5) for $\mathcal{F}$, but using (17) instead of a product metric for CP^2 and S^1, and $E^m = e^m$ $(m = 5, 6)$

$$\begin{aligned}\mathcal{F} &= pE^1_\wedge E^2_\wedge E^3_\wedge E^0 + qE^7_\wedge E^8_\wedge E^{10}_\wedge E^9 + \tfrac{1}{2}\sqrt{2}r(E^{10}_\wedge E^7 + E^8_\wedge E^9)_\wedge E^5_\wedge E^6 \\ &= pE^{1230} + \tfrac{1}{2}qK_{2\wedge} K_2 + \tfrac{1}{2}\sqrt{2}rK_{1\wedge} K_2.\end{aligned} \tag{24}$$

Consider first of all equations (3), under the assumption that four-dimensional spacetime is independent of the internal coordinates $(dE^{1230} = 0)$.

$$\begin{aligned} \overset{*}{_{11}}\mathcal{F} &= pE^{4\ldots10} + qE^{1230}_\wedge E^{456} + \tfrac{1}{2}\sqrt{2}rE^{12304}{}_\wedge K_2 \\ \Rightarrow d\overset{*}{_{11}}\mathcal{F} &= qE^{1230}{}_\wedge dE^4_\wedge K_1 + \tfrac{1}{2}\sqrt{2}rE^{1230}_\wedge dE^4_\wedge K_2 \\ &= 3q(c/b^2)E^{1230}{}_\wedge K_{2\wedge} K_1 + (3rc/\sqrt{2}b^2)E^{1230}_\wedge K_{2\wedge} K_2 \end{aligned} \tag{25}$$

since $dE^4 = cF_+ = 3cK_1/b^2$.

Now

$$\mathcal{F}_\wedge \mathcal{F} = pqE^{1230}{}_\wedge K_{2\wedge} K_2 + \sqrt{2}prE^{1230}_\wedge K_{2\wedge} K_1. \tag{26}$$

Hence equation (3) gives

$$\begin{aligned} &3qc/\sqrt{2}b^2 = pr/\lambda \qquad \text{and} \qquad 3rc/\sqrt{2}b^2 = pq/\lambda \\ &\Rightarrow r = q \qquad \text{and} \qquad p = 3c\lambda/\sqrt{2}b^2, \end{aligned} \tag{27}$$

except when $q = r = 0$. Obviously $d\mathcal{F} = 0$.

Since p is non-zero in this case, four-dimensional spacetime will be taken to be anti-de-Sitter spacetime and, in contrast to the $B = CP^2 \times S^2 \times S^1$ solution, E^4 is spacetime independent.

The Einstein equations (2), with $\mathscr{F}$ as in (24) reduce to:

$$\begin{aligned}
&\mathbb{R}_{AB\wedge} \overset{*}{_{11}} E^{AB\mu} = (p^2+q^2+r^2) \overset{*}{_{11}} E^{\mu},\\
&\mathbb{R}_{AB\wedge} \overset{*}{_{11}} (E^{ABm}) = (-p^2+q^2-r^2) \overset{*}{_{11}} E^{m},\\
&\mathbb{R}_{AB\wedge} \overset{*}{_{11}} (E^{ABa}) = -(p^2+q^2) \overset{*}{_{11}} E^{a},\\
&\mathbb{R}_{AB\wedge} \overset{*}{_{11}} (E^{AB4}) = (-p^2+q^2+r^2) \overset{*}{_{11}} E^{4}.
\end{aligned} \tag{28}$$

With E^4 as in (17) and R_{AdS} the curvature scalar of anti-de-Sitter spacetime, one finds (see appendix 3)

$$\begin{aligned}
&R_{\text{AdS}} = 2(p^2+q^2+r^2-R_{S^2}-R_{CP^2}+9c^2/b^4),\\
&R_{\text{AdS}} = q^2-p^2-r^2-R_{CP^2}+9c^2/b^4,\\
&9c^2/b^4 = -\tfrac{1}{3}(p^2+3q^2+r^2+R_{\text{AdS}}+R_{S^2}),\\
&R_{CP^2} = -2(p^2+q^2+R_{\text{AdS}}+R_{S^2}).
\end{aligned} \tag{29}$$

Define $\alpha = (c^2/b^2)^{1/2}$ the 'squashing' parameter, giving the ratio of the length scale of S^1 to that of CP^2 ($R_{CP^2} = 24/b^2$, see appendix 2). Then (29) are four equations in the four unknowns R_{AdS}, R_{CP^2}, R_{S^2} and α^2 which can be solved (using $q^2 = r^2$ from (27)) to give

$$\begin{aligned}
&R_{\text{AdS}} = -\tfrac{8}{3}(p^2+q^2), \qquad R_{CP^2} = 2(p^2+q^2),\\
&R_{S^2} = \tfrac{2}{3}(p^2+q^2), \qquad \alpha^2 = \tfrac{4}{9}(p^2-2q^2)/(p^2+q^2)
\end{aligned} \tag{30}$$

(obviously this requires $2q^2 < p^2$).

Equation (27) provides one further constraint between p and q

$$6p^2/(p^2-2q^2) = \lambda^2,$$

except when $q = r = 0$.

Thus, for fixed λ, there is only one degree of freedom in the solution. The case $q = r = 0$ is a special case of the $CP^2 \times S^2$ solution of [11].

6. $B = (CP^2 \times S^2)_{\text{twisted}} \times S^1$

To find solutions with this topology, consider an ansatz similar to (5) for $\mathscr{F}$, but using (18) for the metric on $(CP^2 \times S^2)_{\text{twisted}}$.

$$\begin{aligned}
\mathscr{F} &= pE^1{}_\wedge E^2{}_\wedge E^3{}_\wedge E^0 + qE^7{}_\wedge E^8{}_\wedge E^{10}{}_\wedge E^9 + \tfrac{1}{2}\sqrt{2}r(E^{10}{}_\wedge E^7 + E^8{}_\wedge E^9)_\wedge E^5{}_\wedge E^6\\
&= pE^{1230} + \tfrac{1}{2}qK_{2\wedge}K_2 + \tfrac{1}{2}\sqrt{2}rK_{2\wedge}E^{56}.
\end{aligned} \tag{31}$$

Note that with (18) E^{56} is no longer the Kahler 2-form on S^2, though it does contain the Kahler 2-form. As in § 3, it will be assumed that E^{1230} is independent of the internal dimensions ($dE^{1230} = 0$), and E^4 is independent of the other six internal dimensions.

First the matter equations

$$d\mathscr{F}=0 \Rightarrow K_{2\wedge} dE^{56}=0$$
$$\Rightarrow K_{2\wedge} F^{j}_{-\wedge} \hat{*}_{2} L_{j}=0, \tag{32}$$

which is satisfied by F^j_- in (20), since $K_{2\wedge} F^j_- = 0$ for $j=1,2,3$. (For the definition of $\hat{*}_2$ see appendix 2.)

$$d \overset{*}{_{11}} \mathscr{F}=\mathscr{F}_{\wedge} \mathscr{F}/\lambda$$
$$\Rightarrow -\tfrac{1}{2} p dE^4_{\wedge} K_{1\wedge} K_{2\wedge} K_2 - qE^{12304}{}_{\wedge} dE^{56} \tag{33}$$
$$=(1/\lambda)(pqE^{1230}{}_{\wedge} K_{2\wedge} K_2+\sqrt{2}\, prE^{1230}_{\wedge} E^{56}_{\wedge} K_2).$$

This can only be solved if $p=q=0$ when $d\overset{*}{_{11}}\mathscr{F}=\mathscr{F}_{\wedge}\mathscr{F}=0$. Then (31) is

$$\mathscr{F}=rK_{2\wedge} E^{56}/\sqrt{2}, \tag{34}$$

which will split the eleven dimensions into five + six.

With the ansatz (34) for $\mathscr{F}$, Einstein's equations (2) reduce to

$$\mathbb{R}_{AB\wedge} \overset{*}{_{11}} E^{AB\alpha}=r^2 \overset{*}{_{11}} E^{\alpha}, \qquad \mathbb{R}_{AB\wedge} \overset{*}{_{11}} E^{ABa}=0,$$
$$\mathbb{R}_{AB\wedge} \overset{*}{_{11}} E^{ABm}=-r^2 \overset{*}{_{11}} E^{m}. \tag{35}$$

With E^m as in (18) and R_5 the curvature scalar of the metric described by E^α, one finds (see appendix 3):

$$3R_5=5(r^2-R_{CP^2}-R_{S^2}+8a^2/b^4),$$
$$R_{CP^2}=-2(R_5+R_{S^2}), \qquad 16a^2/b^4=R_{CP^2}+R_5+r^2. \tag{36}$$

Let $\beta=(a^2/b^2)^{1/2}$ be the 'squashing' parameter giving the ratio of the scales for S^2 and CP^2. Then, using $R_{CP^2}=24/b^2$, $R_{S^2}=2/a^2$, (36) gives

$$R_{CP^2}=2r^2/(3-2\beta^2), \qquad R_{S^2}=\tfrac{2}{3}r^2(6-5\beta^2)/(3-2\beta^2), \qquad R_5=-\tfrac{5}{3}r^2. \tag{37}$$

The relation $R_{CP^2}/R_{S^2}=12\beta^2$ gives a quadratic equation for β^2

$$20\beta^4-24\beta^2+1=0, \tag{38}$$

which has solutions $\beta^2=1.16$ and $\beta^2=0.0432$.

The remaining five-dimensional Einstein equations with negative scalar curvature

$$\mathbb{R}_{\beta\gamma\wedge} \overset{*}{_5} E^{\beta\gamma\alpha}=-r^2 \overset{*}{_5} E^{\alpha} \tag{39}$$

can be solved as in § 3 to give Robertson–Walker type spacetimes with a time-dependent S^1.

7. Conclusions

By looking for solutions of $D=11$ supergravity involving CP^2 and S^2 in the internal space and constructing the 4-form $\mathscr{F}$ out of a combination of Kahler 2-forms on CP^2 and S^2 and the volume form of four-dimensional spacetime, three different kinds of solution with isometry group of the internal space $SU(3)/Z_3 \times SU(2)/Z_2 \times U(1)$ have been constructed.

(1) $\mathscr{F} = \frac{1}{2}qK_{2\wedge}K_2 + \frac{1}{2}\sqrt{2}rK_{1\wedge}K_2$, $B = CP^2 \times S^2 \times S^1$ and the spacetime is Robertson-Walker with $R^2(t) = (P\sin(\omega t + \delta) - 2k/\omega^2)$, where P and δ are constants, $k = \pm 1, 0$ gives the 3-space curvature and $\omega = [\frac{1}{3}(r^2 + q^2)]^{1/2}$ with $q^2 < 2r$. In addition, the scale of the S^1 factor oscillates.

(2) $\mathscr{F} = pE^{1230} + \frac{1}{2}qK_{2\wedge}K_2 + \frac{1}{2}\sqrt{2}rK_{1\wedge}K_2$, $B = (CP^2 \times S^1)_{\text{twisted}} \times S^2$ and spacetime is anti-de-Sitter. The ratio of the scale sizes of S^1 to CP^2 is fixed by the parameter λ in the Lagrangian, except for the case of $q = r = 0$ when $\alpha = \frac{2}{3}$.

(3) $\mathscr{F} = \frac{1}{2}\sqrt{2}rK_{2\wedge}E^{56}$, $B = (CP^2 \times S^2)_{\text{twisted}} \times S^1$ with E^{56} as in (18). Spacetime is Robertson–Walker as in § 1 and the ratio of the scale on S^2 to that of CP^2 has two discrete values, $\beta^2 = 1.16$ and $\beta^2 = 0.043$. Solution (1) will not admit a spinor structure, while (2) and (3) do.

In common with all other known solutions of $D = 11$ supergravity these suffer from three major difficulties.

(a) Spacetime has a very high, negative curvature not observed in nature.

(b) The isometry group is $[SU(3) \times SU(2) \times U(1)]/Z_3 \times Z_2$ rather than $SU(3) \times SU(2) \times U(1)$ and so it is difficult to see how coloured quarks can be fitted in.

(c) When the topology is $M_4 \times B$, it does not seem possible to obtain chiral fermions in four dimensions, starting from eleven.

Appendix 1.

Capital Roman letters label all eleven dimensions:

$$A, B, C = 0, 1, \ldots, 10.$$

Greek letters near the beginning of the alphabet label four-dimensional spacetime and S^1:

$$\alpha, \beta, \gamma = 0, 1, 2, 3, 4.$$

Greek letters near the middle of the alphabet label four-dimensional spacetime only:

$$\mu, \nu, \rho = 0, 1, 2, 3.$$

The index 4 labels S^1.

Small Roman letters near the beginning of the alphabet label CP^2:

$$a, b, c = 7, 8, 9, 10.$$

Small Roman letters near the middle of the alphabet label S^2:

$$m, n = 5, 6.$$

Orthonormal 1-forms are represented by E^A(e^a and e^m are reserved for the Fubini-study metrics on CP^2 and S^2—appendix 2). When there is no risk of confusion $E^{ABC\ldots} = E^A_\wedge E^B_\wedge E^C_\wedge \ldots$ will be used for wedge products.

$\overset{*}{_{11}}$ is the eleven-dimensional Hodge duality operator, sending p-forms to $(11-p)$ forms. The orientation is given by

$$\overset{*}{_{11}}1 = E^1_\wedge E^2_\wedge E^3_\wedge E^0 E^4_\wedge E^5_\wedge E^6_\wedge E^7_\wedge E^8_\wedge E^{10}_\wedge E^9.$$

It is convenient also to define five- and four-dimensional metrics given by E^α and E^μ

respectively and orientations

$$\overset{*}{_5}1 = E^{12304},$$
$$\overset{*}{_4}1 = E^{1230}.$$

The eleven-dimensional metric has signature $(-+\ldots+)$ and $\eta_{AB} = \mathrm{diag}(-+\ldots+)$ raises and lowers indices.

i_A is the interior derivative (being contraction with the orthonormal vector metric dual to E^A) sending p-forms to $(p-1)$-forms. It satisfies the following identities

$$i_A(E^B) = \delta^B_A, \qquad i_A(E^{BC}) = \delta^B_A E^C - \delta^C_A E^B, \qquad \text{(A1.1)}$$
$$i_A \overset{*}{_{11}} E^C = \overset{*}{_{11}}(E^C_\wedge E_A).$$

Further useful identities are

$$E^A_{\wedge}\overset{*}{_{11}} E^B = \eta^{AB} \overset{*}{_{11}} 1, \qquad E^A_\wedge \overset{*}{_{11}} E^{BC} = \eta^{AC} \overset{*}{_{11}} E^B - \eta^{AB} \overset{*}{_{11}} E^C$$

and

$$E^A_\wedge i_A w = pw \qquad \text{for any } p\text{-form } w. \qquad \text{(A1.2)}$$

Curvature 2-forms, $\mathbb{R}_{AB}$, are derivable from E^A via the torsion-free condition and the structure equations

$$\Omega_{AB} = \tfrac{1}{2}(E^C i_A i_B dE_C - i_A dE_B + i_B dE_A) \qquad \text{(A1.3)}$$
$$\mathbb{R}_{AB} = d\Omega_{AB} + \Omega_{AC\wedge}\Omega^C{}_B$$

where d is the exterior derivative.

Appendix 2.

2.1. $S^2 \approx CP^1$

Orthonormal 1-forms are

$$e^5 = a d\theta, \qquad e^6 = a \sin\theta d\phi, \qquad 0 \leq \theta < \pi, \qquad 0 \leq \phi < 2\pi, \qquad \text{(A2.1)}$$

where a is a constant, giving

$$R_{56} = e^{56}/a^2. \qquad \text{(A2.2)}$$

The curvature scalar is thus $R_{S^2} = 2/a^2$. The Kahler 2-form is simply the volume element $K_1 = e^{56}$. Killing 1-forms (metric dual to Killing vectors) are

$$L_1 = \sin\phi e^5 + \cos\phi \cos\theta e^6,$$
$$L_2 = \cos\phi e^5 - \sin\phi \cos\theta e^6, \qquad \text{(A2.3)}$$
$$L_3 = \sin\theta e^6.$$

These satisfy

$$dL_j = (1/a)\varepsilon_j{}^{kl} L_{k\wedge} L_l = 0. \qquad \text{(A2.4)}$$

Define

$$h_{ij} = L^5_i L^5_j + L^6_i L^6_j,$$

where $L_i = L_i{}^m e^n \eta_{mn}$.

The following relations are used in appendix 3.

$$\delta^{ij}h_{ij}=2, \qquad \delta^{ij}L_{i\wedge}\hat{*}_2(L_{j\wedge}e^m)=-\hat{*}_2 e^m, \tag{A2.5}$$

where $\hat{*}_2$ is the Hodge duality operator on S^2 with respect to e^m. Contraction with the orthonormal vector metric dual to e^m will be represented by $\hat{i}_m$. The group of isometries of this metric is $SO(3)\approx SU(2)/Z_2$.

2.2. CP^2

Orthonormal 1-forms are:

$$\begin{aligned} &e^{10}=bdr/(1+r^2), && e^9=br\sigma_3/(1+r^2),\\ &e^7=br\sigma_1/(1+r^2)^{1/2}, && e^8=br\sigma_2/(1+r^2)^{1/2}, \end{aligned} \tag{A2.6}$$

with b constant and $0\leq r<\infty$. σ_1, σ_2 and σ_3 are left-invariant 1-forms on S^3:

$$\begin{aligned} &\sigma_1=\tfrac{1}{2}(\sin\gamma d\alpha-\sin\alpha\cos\gamma d\beta),\\ &\sigma_2=-\tfrac{1}{2}(\cos\gamma d\alpha+\sin\alpha\sin\gamma d\beta),\\ &\sigma_3=\tfrac{1}{2}(\cos\alpha d\beta+d\gamma),\\ &0\leq\alpha<\pi, \qquad 0\leq\beta<2\pi, \qquad 0\leq\gamma<4\pi. \end{aligned} \tag{A2.7}$$

These lead to connection 1-forms:

$$\begin{aligned} &\omega_{7,10}=e^7/br, && \omega_{8,9}=e^7/br,\\ &\omega_{8,10}=e^8/br, && \omega_{9,7}=e^8/br,\\ &\omega_{9,10}=e^9(1-r^2)/br && \omega_{7,8}=e^9(1+2r^2)/br, \end{aligned} \tag{A2.8}$$

and curvature 2-forms:

$$\begin{aligned} &R_{7,10}=R_{8,9}=(1/b^2)(e^7_\wedge e^{10}+e^8_\wedge e^9),\\ &R_{8,10}=R_{9,7}=(1/b^2)(e^8_\wedge e^{10}+e^9_\wedge e^7),\\ &R_{9,10}=(1/b^2)(4e^9_\wedge e^{10}-2e^7_\wedge e^8), \qquad R_{7,8}=\frac{1}{b^2}(4e^7_\wedge e^8-2e^9_\wedge e^{10}). \end{aligned} \tag{A2.9}$$

Thus the curvature scalar is

$$R_{CP^2}=24/b^2, \tag{A2.10}$$

and the Einstein condition

$$R_{ab\wedge}\hat{*}_4 e^{abc}=(12/b^2)\hat{*}_4 e^c, \tag{A2.11}$$

where $\hat{*}_4$ is the Hodge duality operator for the Fubini-study metric. ($\hat{*}_4 1=e^7_\wedge e^8_\wedge e^{10}_\wedge e^9$). $\hat{i}_c$ denotes contraction with the orthonormal vector metric dual to e^c. The group of isometries of this metric is $SU(3)/Z_3$.

The Kahler 2-form is

$$K_2=e^{10}_\wedge e^7+e^8_\wedge e^9, \tag{A2.12}$$

and is self-dual under $\hat{*}_4$. It squares to twice the volume element of CP^2.

$$K_{2\wedge}K_2=2e^7_\wedge e^8_\wedge e^{10}_\wedge e^9. \tag{A2.13}$$

Appendix 3.

The details of the reduction of Einstein's equations from (28) to (29) for metrics of type (17) (and from (35) to (36) for metrics of the type (18)) are standard in Kaluza-Klein theories, though perhaps unfamiliar in differential form language (see [4]).

3.1. Equation (29)

Evaluating $\mathbb{R}_{AB}$ using (A1.3), and (17), equations (28) reduce to

$$\mathbb{R}_{\rho\sigma\wedge}{*_4} E^{\rho\sigma\mu}{}_{\wedge}\hat{*}_4 1 = (p^2+q^2+r^2-R_{S^2}-R_{CP^2}){*_4} E^{\mu}_{\wedge}\hat{*}_4 1+\tfrac{1}{2}c^2 F_{+\wedge}\hat{*}_4 F_{+\wedge}{*_4} E^{\mu}, \tag{A3.1}$$

$$0=(q^2-p^2-r^2-R_{\mathrm{AdS}}-R_{CP^2})\hat{*}_4 1_{\wedge}\hat{*}_2 e^m+\tfrac{1}{2}c^2 F_{+\wedge}\hat{*}_4 F_{+\wedge}\hat{*}_2 e^m, \tag{A3.2}$$

$$c^2 F_{+\wedge}\hat{*}_4 F_+ = -\tfrac{2}{3}(p^2+3q^2+r^2+R_{\mathrm{AdS}}+R_{S^2})\hat{*}_4 1, \tag{A3.3}$$

$$R_{ab\wedge}\hat{*}_4 e^{abc} = -(p^2+q^2+R_{\mathrm{AdS}}+R_{S^2})\hat{*}_4 e^c+\tfrac{1}{2}c^2(F_{+\wedge}\hat{i}^c\hat{*}_4 F_+ - \hat{i}^c F_{+\wedge}\hat{*}_4 F_+), \tag{A3.4}$$

$$d\hat{*}_4 F_+ = 0. \tag{A3.5}$$

Using (19) for F_+ and the fact that K_1 is self-dual and closed, (A3.5) is satisfied and

$$F_{+\wedge}\hat{i}_c\hat{*}_4 F_+ - \hat{i}_c F_{+\wedge}\hat{*}_4 F_+ = 0. \tag{A3.6}$$

Thus (A3.1-4) reduce to

$$\mathbb{R}_{\rho\sigma\wedge}{*_4} E^{\rho\sigma\mu} = (p^2+q^2+r^2-R_{S^2}-R_{CP^2}+9c^2/b^4){*_4} E^{\mu}, \tag{A3.7}$$

$$0=q^2-p^2-r^2-R_{\mathrm{AdS}}-R_{CP^2}+9c^2/b^4, \tag{A3.8}$$

$$9c^2/b^4 = -\tfrac{1}{3}(p^2+3q^2+r^2+R_{\mathrm{AdS}}+R_{S^2}), \tag{A3.9}$$

$$R_{ab\wedge}\hat{*}_4 e^{abc} = -(p^2+q^2+R_{\mathrm{AdS}}+R_{S^2})\hat{*}_4 e^c. \tag{A3.10}$$

3.2. Equation (36)

Evaluating $\mathbb{R}_{AB}$ using (A1.3) and (18), equations (35) reduce to

$$\mathbb{R}_{\beta\gamma\wedge}{*_5} E^{\beta\gamma\alpha}{}_{\wedge}\hat{*}_4 1 = (r^2-R_{CP^2}-R_{S^2}){*_5} E^{\alpha}_{\wedge}\hat{*}_4 1+(2a^2)F^j_{-\wedge}\hat{*}_4 F^k_- h_{jk\wedge}{*_5} E^{\alpha}, \tag{A3.11}$$

$$R_{ab\wedge}\hat{*}_4 e^{abc} = -(R_5+R_{S^2})\hat{*}_4 e^c+(2a^2)(F^j_{-\wedge}\hat{i}^c\hat{*}_4 F^k_- - \hat{i}^c F^j_{-\wedge}\hat{*}_4 F^k_-)h_{jk}, \tag{A3.12}$$

$$\begin{aligned}(2a^2)F^j_{-\wedge}\hat{*}_4 F^k_{-\wedge}(3h_{jk}\hat{*}_2 e^m + L_{j\wedge}\hat{i}^m\hat{*}_2 L_k + L_{k\wedge}\hat{i}^m\hat{*}_2 L_j)\\ = (R_{CP^2}+R_5+r^2)\hat{*}_4 1_{\wedge}\hat{*}_2 e^m,\end{aligned} \tag{A3.13}$$

$$D_{A-}(\hat{*}_4 F^j_-) = d(\hat{*}_4 F^j_-)+2\varepsilon^j{}_{kl}A^k_{-\wedge}\hat{*}_4 F^l_- = 0. \tag{A3.14}$$

Using (20) for F^j_- and the fact that F^j_- is anti-self-dual under $\hat{*}_4$, (A3.14) is the Bianchi identity for F^j_-. Similarly

$$(F^j_{\wedge}\hat{i}_c\hat{*}_4 F^k - \hat{i}^c F^j_{\wedge}\hat{*}_4 F^k)h_{jk} = 0,$$

since h_{jk} is symmetric. Using (A2.5), (A3.11-13) reduce to

$$\mathbb{R}_{\beta\gamma\wedge}\overset{*}{_5} E^{\beta\gamma\alpha} = (r^2 - R_{CP^2} - R_{S^2} + 8a^2/b^4)\overset{*}{_5} E^{\alpha}, \tag{A3.15}$$

$$R_{ab\wedge}\hat{\overset{*}{_4}} e^{abc} = -(R_5 + R_{S^2})\hat{\overset{*}{_4}} e^c, \tag{A3.16}$$

$$16a^2/b^4 = (R_{CP^2} + R_5 + r^2). \tag{A3.17}$$

Wedging (A3.15) with E_α and (A3.16) with e^c and using (A1.2) gives (36).

References

[1] Kaluza T 1921 *Sitzungsber. Preuss. Akad. Wiss.* 966
[2] Klein O 1925 *Z. Phys.* **37** 895; 1928 *Z. Phys.* **46** 188
[3] Witten E 1981 *Nucl. Phys.* B **186** 412
[4] Benn I M and Tucker R W 1983 *A Riemannian Description of Non-Abelian Gauge Interactions* Lancaster University preprint
[5] Castellani L, Romans L J and Warner N P 1983 *A Classification of Compactifying Solutions for D = 11 Supergravity, CALTECH Preprint CALT-68-1055*
[6] Freund P G O and Rubin M A 1980 *Phys. Lett.* **97B** 233
[7] Dolan B 1984 *Phys. Lett.* **140B** 304
[8] Dolan B 1984 *Spinors on CP^2 in $d \geqslant 9$ Supergravity* (1985 *Ann. Inst. H Poincaré* to appear)
[9] Watamura S 1983 *Phys. Lett.* **129B** 188
[10] Watamura S 1984 *Phys. Lett.* **136B** 245
[11] Campbell I C G and West P C 1984 *Nucl. Phys.* B **243** 112
[12] Nilsson B E W and Pope C N *Hopf Fibration of Eleven Dimensional Supergravity, Impèrial College preprint, Imperial/TP/83-84/41*
[13] Cremmer E, Julia B and Scherk J 1978 *Phys. Lett.* **76B** 409
[14] Appelquist T and Chodos A 1983 *Phys. Rev.* D **28** 772
[15] Hawking S W and Pope C N 1978 *Phys. Lett.* **73B** 42
[16] Atiyah M F, Bott R and Shapiro A 1964 *Topology* **3** 3
[17] Dolan B 1985 *Class. Quantum Grav.* submitted
[18] Eguchi T, Gilkey P B and Hanson A J 1980 *Phys. Rep.* **66** 213