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LETTER TO THE EDITOR

An $S^2 \times S^1$ bundle over CP^2—a solution of $d = 11$ supergravity with isometry group $(\mathrm{SU}(3)/Z_3) \times \mathrm{U}(2)$

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Abstract. By considering the dimensional reduction of $d = 11$ supergravity to the $N = 2$, $d = 10$ non-chiral theory, and using the torsion free connection on CP^2 to construct an S^2 bundle over CP^2, a new solution of $d = 11$ supergravity is presented with internal symmetry group $(\mathrm{SU}(3)/Z_3) \times \mathrm{U}(2)$.

There are now many solutions known of the $d = 11$ supergravity equations of motion [1, 2]. In particular, the dimensional reduction to $N = 2$, $d = 10$, non-chiral supergravity has led to the investigation of Hopf fibrations of compact, seven-dimensional manifolds [3, 4]. In this letter, a new solution of the latter type is presented in which the manifold is topologically a twisted product of $(S^2 \times S^1)$ over CP^2. The isometry group is $(\mathrm{SU}(3)/Z_3) \times \mathrm{U}(2)$.

The dimensional reduction from $d = 11$ to $d = 10$ is effected in the following manner for the bosonic field equations. Let e^M $(M = 0, 1, \ldots, 10)$ be the orthonormal 1-forms for the metric in eleven dimensions, and make the ansatz

$$e^{10} = \exp(\sigma)(\mathrm{d}x^{10} + \tfrac{1}{3}\lambda A) \qquad \lambda = \text{constant} \tag{1}$$

where A is a 1-form in ten dimensions—all fields are assumed independent of the eleventh dimension (the factor of $\frac{1}{3}$ is for later convenience). For the 4-form field, $\mathcal{F}$, let

$$\mathcal{F} = f + g_\wedge e^{10}$$

where f is a 4-form in ten dimensions and g is a 3-form in ten dimensions.

Then the bosonic field equations in ten dimensions are (with $F = \mathrm{d}A$ and $A, B, C, \ldots = 0, 1, \ldots, 9$) [3]

$$R_{AB\wedge} * e^{ABC} = \tau_\sigma^C + \tau_F^C + \tau_f^C + \tau_g^C = \tau^C$$

$$\tfrac{1}{3}\lambda\, \mathrm{d}[(\exp(9\sigma/4) * F] = -2\exp(3\sigma/4) g_\wedge * f$$

$$\mathrm{d} * \mathrm{d}\sigma = \tfrac{2}{3}\exp(-3\sigma/2) g_\wedge * g - \tfrac{1}{3}\exp(3\sigma/4) f_\wedge * f - \tfrac{1}{18}\lambda^2 \exp(9\sigma/4) F_\wedge * F$$

$$\mathrm{d}[\exp(-3\sigma/2) * g] - \tfrac{1}{3}\lambda \exp(3\sigma/4) F_\wedge * f = -f_\wedge f$$

$$\mathrm{d}[\exp(3\sigma/4) * f] = 2 f_\wedge g$$

$$\mathrm{d}F = \mathrm{d}g = 0$$

$$\mathrm{d}f = \tfrac{1}{3}\lambda g_\wedge F$$

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τ^C are the energy–momentum 9-forms for the matter fields

$$\tau^C_\sigma = -\tfrac{9}{8}(i^C \mathrm{d}\sigma_\wedge * \mathrm{d}\sigma + \mathrm{d}\sigma_\wedge i^C * \mathrm{d}\sigma)$$
$$\tau^C_F = -\tfrac{1}{18}\lambda^2 \exp(9\sigma/4)(i^C F_\wedge * F - F_\wedge i^C * F)$$
$$\tau^C_f = -\exp(3\sigma/4)(i^C f_\wedge * f - f_\wedge i^C * f)$$
$$\tau^C_g = -\exp(-3\sigma/2)(i^C g_\wedge * g + g_\wedge i^C * g)$$

i^C is the interior derivative (contraction with the vector orthonormal to e^C), $*$ is the ten-dimensional Hodge duality operator. Indices are raised and lowered with $\eta_{AB} = \mathrm{diag}(-+\ldots+)$. R_{AB} are the curvature 2-forms. The shorthand notation $e^{ABC\ldots} = e^A_\wedge e^B_\wedge e^C_\wedge \ldots$ has been used.

To construct a solution, first employ the Freund–Rubin ansatz [5], and assume that four-dimensional spacetime is anti-de Sitter space (AdS). Take

$$f = \varepsilon \times (\text{volume form of AdS}), \qquad \varepsilon = \text{constant}$$

next, let $\sigma = 0$ and $g = 0$ ($\sigma = \text{constant} \neq 0$ merely requires a redefinition of ε).

A metric on the seven-dimensional internal space (or, equivalently, F and the metric on the six-dimensional space) corresponding to an $(S^2 \times S^1)$ bundle over CP^2 can be constructed in the following way.

Let b^α, $\alpha = 4, 5, 6, 7$ be orthonormal 1-forms for the Fubini-Study metric on CP^2 and b^m, $m = 8, 9$ be orthonormal 1-forms for the standard metric on S^2, radius $a = \text{const.}$ Let L^m_j, $m = 8, 9$; $j = 1, 2, 3$ be the components in the b^m basis of the three Killing 1-forms on S^2.

Then take

$$e^\alpha = b^\alpha, \qquad e^m = b^m - 2aL^m_j A^j \tag{2}$$

where A^j are three 1-forms on CP^2. Everything has now been specified, except the four 1-forms A and A^j ($j = 1, 2, 3$). 1-forms which satisfy the equations of motion can be constructed in the following way. Consider the connection on CP^2 obtained from b^α via the zero torsion structure equation. For a general four-dimensional manifold, this would be an SO(4) Lie algebra valued 1-form, and can be split into two SU(2) Lie algebra valued 1-forms by taking the self-dual and anti-self-dual parts in the SO(4) indices. For CP^2, however, the holonomy group is U(2), rather than the full SO(4) and one finds that the anti-self-dual part gives an SU(2) Lie algebra valued 1-form, while the self-dual part reduces to a U(1) connection (the conventions are such that the volume element of CP^2 is $b^4_\wedge b^5_\wedge b^7_\wedge b^6$ in the basis given below). A^j are identified with the anti-self-dual SU(2) connection and A with the self-dual U(1) connection.

The quantisation condition for such a configuration is automatically satisfied, since on any coordinate overlap two sets of orthonormal 1-forms are defined, differing by a well defined U(2) gauge transformation, which also gives the gauge transformation on A and A^j between the two coordinate patches.

Explicitly, with σ^j left invariant 1-forms on S^3 and orthonormal 1-forms [6]

$$b^4 = cr\sigma^1/(1+r^2)^{1/2} \qquad b^5 = cr\sigma^2/(1+r^2)^{-1/2}$$
$$b^6 = cr\sigma^3/(1+r^2) \qquad b^7 = cdr/(1+r^2)$$

$0 \leqslant r < \infty$, $c = \text{constant}$, then A and A^j are given by

$$A^2 = -b^4/cr \qquad A^2 = -b^5/cr \qquad A^3 = -[(2+r^2)/2cr]b^6 \qquad A = (3/2c)re^6.$$

This gives F proportional to the Kähler 2-form,

$$F = (3/c^2)(b^7_\wedge b^6 + b^4_\wedge b^5)$$

and the 2-forms on CP^2 obtained from A^j are

$$G^j = \mathrm{d}A^j + \varepsilon^j{}_{kl} A^k_\wedge A^l.$$

Explicitly

$$G^1 = (1/c^2)(e^7_\wedge e^6 - e^4_\wedge e^5) \qquad G^2 = (1/c^2)(e^7_\wedge e^5 - e^6_\wedge e^4)$$

$$G^3 = (1/c^2)(e^7_\wedge e^4 - e^5_\wedge e^6).$$

With this ansatz one finds

$$\mathrm{d}^* F = \mathrm{d}^* f = 0$$

and the remaining equations of motion reduce to ($\mu = 0, 1, 2, 3$ label Ads)

$$R_{AB\wedge} * e^{AB\mu} = (\varepsilon^2 + \lambda^2) * e^\mu, \qquad R_{AB\wedge} * e^{AB\alpha} = -\lambda^2 * e^\alpha,$$

$$R_{AB\wedge} * e^{ABm} = (\varepsilon^2 - \lambda^2) * e^m$$

with the constraint $e^2 = 3\lambda^2$.

With $R_{S^2} = 2/a^2$ the curvature scalar for S^2 (obtained from b^m), R_{CP^2} and R_{Ads} the curvature scalars for CP^2 (obtained from b^α) and anti-de Sitter space respectively, these equations can be solved (this involves the calculation of the curvature 2-forms obtained from (2) via the torsion free structure equation). The result is

$$R_{\text{Ads}} = 2(4\varepsilon^2 + 8a^2 - R_{CP^2} - R_{S^2})$$

$$= -3\varepsilon^2 - \tfrac{1}{2}R_{CP^2} - R_{S^2}$$

$$= -2\varepsilon^2 - R_{CP^2} + 16a^2.$$

There is a one-parameter family of solutions. Let a, the radius of S^2, parametrise the solutions. Then

$$\varepsilon^2 = 4a^2 + 1/a^2$$

$$R_{S^2} = 2/a^2$$

$$R_{CP^2} = 2(3/a^2 + 20a^2)$$

$$R_{\text{Ads}} = -8(1/a^2 + 4a^2)$$

with $\varepsilon^2 \geqslant 4$ for consistency.

To determine the symmetries of this solution, consider the way in which it is constructed. The ansatz for the internal metric retains the symmetry of CP^2, $\mathrm{SU}(3)/Z_3$, since Lie transport along the flow lines of the isometries of CP^2 merely results in a gauge transformation of A^j and A, because A^j and A are constructed directly from b^α. However, the quantity which appears in the full metric, e.g., $L^m_j A^j$ in (2), is an SU(2) scalar, since it is contracted over j, and so is invariant under Lie transport along the flow lines of the isometries of CP^2. Similarly, the isometries of the fibres, $S^2 \times S^1(\mathrm{U}(2))$, are unchanged by the introduction of A^j and A, just as for the standard, non-Abelian, Kaluza-Klein ansatz (except that here the base space is CP^2 rather than spacetime).

Hence the full group of isometries of the seven-dimensional internal space is $SU(3)/Z_3 \times U(2)$. The full U(2) connection is nothing more than the connection, obtained from the zero torsion structure equation, on CP^2. The space has the topology of an $(S^2 \times S^1)$ bundle over CP^2 with structure group U(2).

The situation is similar to that of the BPST instanton over S^4 [7]. There the holonomy group is SO(4) and the zero torsion connection on S^4 lies in the full SO(4) Lie algebra. By isolating one of the two SU(2) sub-algebras (self-dual and anti-self dual parts), one can construct an $SU(2) \approx S^3$ principal bundle over S^4, which is topologically S^7. Using an ansatz similar to (2) one obtains a squashed 7-sphere metric [2]. There are differences, however, in the CP^2 construction above, in that the seven-dimensional manifold is not a principal bundle. A more exact parallel with S^4 (at least for the six-dimensional S^2 bundle over CP^2 obtained by using A^j alone, without A) is the scheme of Barr [8] for SU(2) instantons in six-dimensional Kaluza-Klein theories.

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