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A class of SU(2) instanton configurations in curved space–times

B P Dolan†

Mathematics Department, University of Durham, South Road, Durham, DH1 3LE, UK

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Abstract. A class of SU(2), classical, n instanton configurations is constructed over space–times which consist of S^4 wrapped round itself n times.

1. Introduction

Solutions to classical SU(2) Yang–Mills coupled to gravity have been found by a number of authors (Cho and Freund 1975, Charap and Duff 1977, de Alfaro *et al* 1979, Gürsey *et al* 1979, Jafarizadeh *et al* 1980). In constructing multi-meron SU(2) Yang–Mills configurations, de Alfaro *et al* (1979) found a single instanton solution of SU(2) Yang–Mills coupled to gravity. This was extended by Gürsey *et al* (1979) and Jafarizadeh *et al* (1980) to an O(4) symmetric n instanton solution in a space–time consisting of S^4 wrapped round itself n times. This was done in order to fit an n instanton solution into $\mathbb{H}P^1$ (for a review of $\mathbb{H}P^n$ models, see Gürsey and Tze 1979). Previous attempts to do this had only resulted in two instanton configurations in which the instantons were infinitely far apart, or had zero size (Neinast and Stack 1980, Felzager and Leinaas 1980). Such configurations have been called 'virtual stationary points' by Nahm (1980).

In this paper, the O(4) symmetric solutions of Gürsey *et al* (1979) and Jafarizadeh *et al* (1980) are extended to a more general class of solutions. The first part of the paper sets up notation and conventions for SU(2) Yang–Mills coupled to gravity. Then an ansatz for a class of (anti)self-dual configurations is developed. Thirdly it is shown how this ansatz relates to the solutions of Gürsey *et al*, and how it extends them beyond the O(4) spherically symmetric case. Finally the main results are summarised.

2. General solutions

SU(2) Yang–Mills coupled to gravity, with a cosmological constant Λ, has the Lagrangian

$$\mathscr{L} = -\frac{1}{4\kappa}\sqrt{g}(R+2\Lambda) - \frac{1}{2e^2}\sqrt{g}\,\mathrm{Tr}\{g^{\mu\nu}g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma}\} \tag{1}$$

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where R is the curvature scalar obtained from the metric $g_{\mu\nu}$, $g = \det g_{\mu\nu}$ (the conventions are those of Weinberg (1972)), $\kappa = 4\pi G$ where G is the gravitational constant, and e is the Yang–Mills coupling constant, which has been scaled out of A_μ.

$$F_{\mu\nu} = \frac{1}{2\mathrm{i}}\sigma_a F^a_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \tag{2}$$

$$A_\mu = \frac{1}{2\mathrm{i}}\sigma_a A^a_\mu \tag{3}$$

σ_a, $a = 1, 2, 3$, are the Pauli matrices.

The Euler–Lagrange equations of motion for $F_{\mu\nu}$ are

$$\partial_\mu\{\sqrt{g}F^{\mu\nu}\} = \sqrt{g}[F^{\mu\nu}, A_\mu] \tag{4}$$

($g_{\mu\nu}$ has signature $(++++)$), and for R they are

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}(R + 2\Lambda) = -2\kappa T_{\mu\nu} \tag{5}$$

where

$$\begin{aligned} T_{\mu\nu} &= (1/e^2)\{F^a_{\mu\rho}F^a_{\nu\lambda}g^{\rho\lambda} - \tfrac{1}{4}g_{\mu\nu}g^{\rho\tau}g^{\sigma\lambda}F^a_{\rho\sigma}F^a_{\tau\lambda}\} \\ &= -\frac{2}{e^2}\operatorname{Tr}\{F_{\mu\rho}F_{\nu\lambda}g^{\rho\lambda} - \tfrac{1}{4}g_{\mu\nu}g^{\alpha\beta}g^{\rho\sigma}F_{\alpha\rho}F_{\beta\sigma}\}. \end{aligned} \tag{6}$$

Since $g^{\mu\nu}T_{\mu\nu} = 0$, it is necessary that $R = -4\Lambda$ and in order to satisfy (5), the allowable space–times must be restricted to those of constant scalar curvature.

Points in space–time are labelled by the quaternion

$$x = x_0 - \mathrm{i}\boldsymbol{\sigma}\cdot\boldsymbol{x} = x_j e_j$$

where $e_j = (1_{2\times 2}, -\mathrm{i}\boldsymbol{\sigma})$ form a basis for the quaternions.

The topological charge for the Yang–Mills field is

$$\begin{aligned} k &= -\frac{1}{16\pi^2}\int \mathrm{d}^4x\,\sqrt{g}\operatorname{Tr}({}^*F^{\mu\nu}F_{\mu\nu}) \\ &= \frac{1}{64\pi^2}\int \mathrm{d}^4x\,\varepsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma} \end{aligned} \tag{7}$$

while the action is

$$S_{\text{Y-M}} = \frac{1}{4e^2}\int \mathrm{d}^4x\,\sqrt{g}g^{\mu\rho}g^{\nu\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}; \tag{8}$$

$$\varepsilon^{\mu\nu\alpha\beta} = \begin{cases} +1 & \text{even permutations of 0123} \\ -1 & \text{odd permutations of 0123} \\ 0 & \text{otherwise} \end{cases}$$

is a tensor density of weight -1 while $(1/\sqrt{g})\varepsilon^{\mu\nu\alpha\beta}$ is a tensor.

To look for simultaneous solutions of (4) and (5), consider the ansatz (Gürsey *et al* 1979, Jafarizadeh *et al* 1980)

$$A_\mu = \frac{1}{2}\,\frac{u\partial_\mu u^+ - (\partial_\mu u)u^+}{(1 + u_i u_i)} \tag{9}$$

(i.e. A_μ is restricted to lie in the $\mathbb{H}P^1$ sector of SU(2) Yang–Mills) where

$$u = u_0 - \mathrm{i}\boldsymbol{\sigma}\cdot\boldsymbol{u} = u_i e_i$$

is a quaternionic function of x. $(u_i u_i = \frac{1}{2}\operatorname{Tr}(uu^+))$.

This gives the Yang–Mills field tensor

$$F_{\mu\nu} = \frac{\partial_\mu u \partial_\nu u^+ - \partial_\nu u \partial_\mu u^+}{(1+u_i u_i)^2}. \tag{10}$$

The metric is taken to be

$$g_{\mu\nu} = \frac{\lambda}{4}\operatorname{Tr}\frac{\partial_\mu u(\partial_\nu u^+) + \partial_\nu u(\partial_\mu u^+)}{(1+u_i u_i)^2}$$

$$= \lambda\frac{\partial_\mu u_i \partial_\nu u_i}{(1+u_j u_j)^2} \tag{11}$$

(where λ is a real, positive constant).

One can always choose local coordinates $x' = u$ in which the metric takes the form of that of S^4, but the nature of the space–time depends on the range of the x'_i.

Provided $u(x)$ is such that $g_{\mu\nu}$ is non-singular (this excludes the single meron configuration $u = x/(x_i x_i)^{1/2}$), then $F_{\mu\nu}$ is automatically (anti)self-dual.

To see this, define four quaternions

$$h_\mu = \sqrt{\lambda}\,\partial_\mu u/(1+u_i u_i). \tag{12}$$

Then

$$g_{\mu\nu} = h_{i\mu} h_{i\nu} \tag{13}$$

and $h_{i\mu}$ can be thought of as vierbein fields with i labelling the locally flat coordinates and μ the curvilinear coordinates. Then

$$h_{i\mu} h_j^\mu = \delta_{ij} \tag{14}$$

(i, j can be either upper or lower since in flat Euclidean space–time there is no distinction between covariant and contravariant indices).

Using (14) and the ansatz (9), (10), (11) it can be shown, after some algebra, that the equations of motion (4) reduce to

$$\partial_\mu\{\sqrt{g}\, g^{\mu\rho} g^{\nu\sigma}(\partial_\rho u \partial_\sigma u^+ - \partial_\sigma u \partial_\rho u^+)\} = 0. \tag{15}$$

(In proving this, one uses the fact that $h_\mu q h^\mu = -2q^+$ and $h_\mu q h^{+\mu} = 4q_0$ for any quaternion q.)

Choosing local coordinates $x' = u$ it is easily seen that (15) is automatically satisfied for any $u(x)$, provided it gives a non-singular, continuously differentiable metric $g_{\mu\nu}$.

Furthermore, $F_{\mu\nu}$ is automatically (anti)self-dual since

$$F_{\mu\nu} = (h_\mu h_\nu^+ - h_\nu h_\mu^+)/\lambda = h_{i\mu} h_{j\nu}(e_i e_j^+ - e_j e_i^+)/\lambda$$

$$= 2\mathrm{i}\sigma_a \bar{\eta}^a_{ij} h_{i\mu} h_{j\nu}/\lambda \tag{16}$$

where $\bar{\eta}^a_{ij}$ is the symbol introduced by 't Hooft (1976). Thus

$$*F^{\mu\nu} = \tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}/\sqrt{g}$$

$$= (1/\lambda)\mathrm{i}\sigma_a \bar{\eta}^a_{ij}(\varepsilon^{\mu\nu\rho\sigma}/\sqrt{g}) h_{i\rho} h_{j\sigma}. \tag{17}$$

Now

$$\varepsilon^{\mu\nu\rho\sigma}/\sqrt{g} = \pm\varepsilon^{\mu\nu\rho\sigma}/h = \pm h_i^{\mu} h_j^{\nu} h_k^{\rho} h_l^{\sigma}(\varepsilon_{ijkl}) \tag{18}$$

where

$$h = \det(h_{i\mu}) = [\det(h_i^{\mu})]^{-1} = \pm\sqrt{g}. \tag{19}$$

Therefore

$$\begin{aligned} {}^*F^{\mu\nu} &= \pm \mathrm{i}\sigma_a \bar{\eta}^a_{ij} \varepsilon_{rskl} h_r^{\mu} h_s^{\nu} h_k^{\rho} h_l^{\sigma} h_{i\rho} h_{j\sigma}/\lambda \\ &= \pm F^{\mu\nu} \end{aligned} \tag{20}$$

using (14) and the self-duality properties of $\bar{\eta}^a_{ij}$.

Note that this proof of (anti)self-duality works for *any* h_μ, not just those of the form (12). Thus given *any* non-singular metric $g_{\mu\nu}$ the vierbeins can be used to construct an (anti)self-dual Yang–Mills field configuration as above. However, since $T_{\mu\nu}$ vanishes for a self-dual Yang–Mills configuration, equation (5) restricts $g_{\mu\nu}$ to that of a space of constant scalar curvature. For h_μ of the form (12), A_μ is given by (9).

The Yang–Mills action for the ansatz (9), (10), (11) is

$$\begin{aligned} S_{\text{Y-M}} &= -\frac{1}{2e^2\lambda^2}\int \sqrt{g}\,\mathrm{Tr}(h_\mu h_\nu^+ - h_\nu h_\mu^+)(h^\mu h^{\nu+} - h^\nu h^{\mu+})\,\mathrm{d}^4x \\ &= (48/e^2\lambda^2)\int \sqrt{g}\,\mathrm{d}^4x \\ &= (48/e^2\lambda^2)\times(\text{volume of the space–time}) \end{aligned} \tag{21}$$

and the topological charge of the Yang–Mills field is

$$k = \pm(e^2/8\pi^2)S_{\text{Y-M}} = \pm(6/\pi^2\lambda^2)\int \sqrt{g}\,\mathrm{d}^4x. \tag{22}$$

The sign of k depends on the sign of $\det(h_{i\mu})$.

Gürsey *et al* (1979) and Jafarizadeh *et al* (1980) have shown that $u(x) = x^n$ gives an n instanton solution. This can be seen most readily by going to spherical polars in 4D

$$x = r(\cos\theta + \hat{r}\sin\theta) \tag{23}$$

where

$$\hat{r} = \cos\varphi\, e_1 + \sin\varphi\cos\psi\, e_2 + \sin\varphi\sin\psi\, e_3 \tag{24}$$

$$\hat{r}^2 = -1 \tag{25}$$

r is a radial coordinate, and

$$\mathrm{d}^4x = \mathrm{d}r\, d\theta\, \mathrm{d}\varphi\, \mathrm{d}\psi$$

$$0 \leqslant \theta \leqslant \pi, \qquad 0 \leqslant \varphi \leqslant \pi, \qquad 0 \leqslant \psi \leqslant 2\pi, \qquad 0 \leqslant r < \infty.$$

Then

$$x^n = r^n(\cos n\theta + \hat{r}\sin n\theta) \tag{26}$$

and

$$g_{\mu\nu}=\frac{\lambda r^{2n-2}}{(1+r^{2n})^2}\begin{pmatrix} n^2 & & & 0 \\ & n^2r^2 & & \\ & & r^2\sin^2 n\theta & \\ 0 & & & r^2\sin^2 n\theta\sin^2\varphi \end{pmatrix} \tag{27}$$

is diagonal with

$$\sqrt{g}=\frac{\lambda^2 n^2 r^{4n-1}}{(1+r^{2n})^4}\sin^2 n\theta\sin\varphi. \tag{28}$$

Again, choosing coordinates $x'=u=x^n$, the metric is seen to be that of S^4, except that now

$$0\leqslant r'<\infty, \qquad 0\leqslant\theta'\leqslant n\pi, \qquad 0\leqslant\varphi'\leqslant\pi, \qquad 0\leqslant\psi'\leqslant 2\pi,$$

so that S^4 is wrapped around itself n times.

The topological charge and action of the Yang–Mills field are, from (22),

$$k=n, \qquad S_{\text{Y-M}}=(8\pi^2/e^2)n$$

since space–time has the volume $\lambda^2\pi^2 n/6$ and $\det(h_{i\mu})>0$. In x' coordinates, the metric (27) is conformal to the flat space metric, with conformal factor $\Omega^2=\lambda(1+x'_ix'_i)^{-2}$. The curvature scalar is (with $\Box'=\partial/\partial x'_i\,\partial/\partial x'_i$, the flat space Laplacian)

$$\begin{aligned} R&=6\Omega^{-3}\Box'\Omega \\ &=6\{\partial'_i\partial'_i(1+x'_jx'_j)^{-1}\}\{1+x'_kx'_k\}^3/\lambda \\ &=-48/\lambda. \end{aligned} \tag{29}$$

However, any function $u(x)$ giving a continuous, non-singular metric via (11) gives rise to an (anti)self-dual solution. In particular, n instanton configurations will be given by polynomials in x with quaternion coefficients. These are homotopic to x^n as has been shown by Eilenberg and Niven (1944).

For example

$$u(x)=\prod_{i=1}^{n}(x-b_i) \tag{30}$$

could be thought of as describing n instantons at arbitrary positions b_i.

Consider the case $n=2$. Without loss of generality, the origin can be moved to lie halfway between b_1 and b_2. Then the time axis (real axis) can be rotated so as to pass through b_1 and b_2. Thus

$$u(x)=(x+b)(x-b) \tag{31}$$

with b real.

Then, using coordinates (t,ρ,φ,ψ) where $\rho^2=(x_1^2+x_2^2+x_3^2)$ with $-\infty<t<\infty$, $0\leqslant\rho<\infty$, $0\leqslant\varphi\leqslant\pi$, $0\leqslant\psi\leqslant 2\pi$, $d^4x=dt\,d\rho\,d\varphi\,d\psi\,x=(t+\hat{r}\rho)$, $g_{\mu\nu}$ is diagonal

$$g_{\mu\nu}=\frac{4\lambda}{\{1+[(t+b)^2+\rho^2][(t-b)^2+\rho^2]\}^2}\begin{pmatrix} (t^2+\rho^2) & & & 0 \\ & (t^2+\rho^2) & & \\ & & t^2\rho^2 & \\ 0 & & & t^2\rho^2\sin^2\varphi \end{pmatrix} \tag{32}$$

and

$$\sqrt{g}=\frac{16\lambda^2 t^2\rho^2(t^2+\rho^2)\sin\varphi}{\{1+[(t+b)^2+\rho^2][(t-b)^2+\rho^2]\}^4}. \tag{33}$$

Then the volume of space–time is

$$\begin{aligned} v &= 128\pi\lambda^2\int_0^\infty \mathrm{d}t\int_0^\infty \mathrm{d}\rho\,\frac{t^2\rho^2(t^2+\rho^2)}{\{1+[(t+b)^2+\rho^2][(t-b)^2+\rho^2]\}^4} \\ &= \pi^2\lambda^2/3 \end{aligned} \tag{34}$$

and (22) gives $k=2$, $S_{\text{Y-M}}=16\pi^2/e^2$.

3. Conclusions

Given any continuous, non-singular, differentiable metric, $g_{\mu\nu}$, form vierbeins $h_{i\mu}$ and thus construct four quaternions h_μ. Then forming $K_{\mu\nu}=h_\mu h_\nu^+$ take the pure quaternionic part of $K_{\mu\nu}$ as $(\lambda/2)F_{\mu\nu}$ and the real part of $K_{\mu\nu}$ as $g_{\mu\nu}$; then $F_{\mu\nu}$ is automatically (anti)self-dual, in the space–time described by $g_{\mu\nu}$. Only h_μ of the form

$$h_\mu=\sqrt{\lambda}\,\frac{\partial_\mu u}{(1+u_i u_i)}$$

are considered in this paper since for these it is easy to find A_μ.

Further, in order that Einstein's field equations be satisfied, $g_{\mu\nu}$ must describe a space–time of constant scalar curvature. The cosmological constant has the value $\Lambda=-R/4$. There is only one degree of freedom between $g_{\mu\nu}$ and $F_{\mu\nu}$, that of λ.

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