Event-Based Transmission Scheduling and LQG Control Over a Packet Dropping Link

Alex S. Leong * **Daniel E. Quevedo** * **Takashi Tanaka** **
Subhrakanti Dey *** **Anders Ahlén** ***

* Department of Electrical Engineering (EIM-E), Paderborn
University, Paderborn, Germany (e-mail: alex.leong@upb.de,
dquevedo@ieee.org).
** Department of Automatic Control, School of Electrical Engineering,
Royal Institute of Technology, Stockholm, Sweden (e-mail:
ttanaka@kth.se)
*** Department of Engineering Science, Uppsala University, Uppsala,
Sweden (e-mail: Subhra.Dey@signal.uu.se, anders.ahlen@signal.uu.se)

Abstract: This paper studies a joint transmission scheduling and controller design problem, which minimizes a linear combination of the control cost and expected energy usage of the sensor. Assuming that the sensor transmission decisions are event-based and determined using the random estimation error covariance information available to the controller, we show a separation in the design of the transmission scheduler and controller. The optimal controller is given as the solution to an LQG-type problem, while the optimal transmission policy is a threshold policy on the estimation error covariance at the controller.

1. INTRODUCTION

In event triggered estimation and control, sensors and/or controllers will communicate only when certain events occur, see e.g. Li et al. (2010), Wu et al. (2013), Trimpe and D'Andrea (2014), Tabuada (2007), Heemels et al. (2012), Ramesh et al. (2012), Quevedo et al. (2014). Using such event triggering mechanisms can often lead to a reduction in communication and energy usage, important in resource constrained environments such as wireless control systems, while still maintaining a certain level of performance. Various different event triggering mechanisms have been proposed, such as thresholding type policies which trigger a transmission if and only if certain quantities such as the estimation error, error in predicted output, or estimation error covariance, exceed a given threshold.

The motivations for using event triggering rules are often based on heuristics, although some structural results on optimal triggering rules have also been derived. In estimation, for the case of noiseless measurements and no packet drops, Lipsa and Martins (2011) and Nayyar et al. (2013) showed a threshold behaviour in the difference between the current state and most recently transmitted state. For variance based triggering (where transmit decisions depend on the estimation error covariance with packet drops, it was shown in Leong et al. (2017) that a threshold policy is optimal, in the sense that it minimizes a linear combination of the expected estimation error covariance and expected energy usage. In event triggered control, the optimality of certainty equivalence in the control law was shown in Molin and Hirche (2013), but with noisy measurements the conditional expectations are in general difficult to evaluate (Molin and Hirche (2010)).

In the present work we study a joint transmission scheduling and controller design problem which minimizes a linear combination of the control cost and expected energy usage of the sensor. The sensor transmits local state estimates over an i.i.d. packet dropping link to the controller. Sensor transmissions are scheduled at the controller, which is assumed to have more computational capabilities, based on the randomly time-varying estimation error covariances at the controller. Under this setup, we show that a separation of the transmission scheduling and controller design problems holds. The controller design problem is a LQG-type problem, and the transmission scheduling problem is similar to a problem previously studied in Leong et al. (2017), with the optimal transmission policy being a threshold policy in the estimation error covariance. In the infinite horizon case, simple analytical expressions for the performance can also be derived.

The paper is organized as follows. The system model is presented in Section 2, and the problem formulation in Section 3. The separation of transmission scheduling and controller design is shown in Section 4, with the optimal controller also given. The transmission scheduling problem is analyzed in Section 5. The infinite horizon case is considered in Section 6. Numerical results are presented in Section 7.

2. SYSTEM MODEL

A diagram of the system model is shown in Fig. 1. Consider a discrete time process
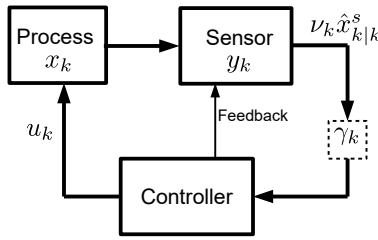
$$x_{k+1} = Ax_k + Bu_k + w_k \qquad (1)$$

Fig. 1. System Model

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^p$, and w_k is i.i.d. Gaussian with zero mean and covariance $Q > 0$.[1] The initial state x_0 is Gaussian with zero mean and covariance P_0. The sensor has measurements

$$y_k = Cx_k + v_k, \qquad (2)$$

where $y_k \in \mathbb{R}^m$ and v_k is Gaussian with zero mean and covariance $R > 0$. The noise processes $\{w_k\}$ and $\{v_k\}$ are assumed to be mutually independent.

The sensor transmits local state estimates $\hat{x}^s_{k|k}$ to the controller, which generally gives better performance than transmitting measurements (Xu and Hespanha (2005)). Let $\nu_k \in \{0, 1\}$ be decision variables such that $\nu_k = 1$ if and only if $\hat{x}^s_{k|k}$ is to be transmitted to the controller at time k. Define the information set available to the sensor at time k as:[2]

$$\mathcal{I}^s_k = \{y_0, \ldots, y_k, u_0, \ldots, u_{k-1}, \nu_0, \ldots, \nu_k\} \qquad (3)$$

and the local state estimates and error covariances by:

$$\hat{x}^s_{k|k-1} \triangleq \mathbb{E}[x_k | \mathcal{I}^s_{k-1}], \quad \hat{x}^s_{k|k} \triangleq \mathbb{E}[x_k | \mathcal{I}^s_k]$$
$$P^s_{k|k-1} \triangleq \mathbb{E}[(x_k - \hat{x}^s_{k|k-1})(x_k - \hat{x}^s_{k|k-1})^T | \mathcal{I}^s_{k-1}]$$
$$P^s_{k|k} \triangleq \mathbb{E}[(x_k - \hat{x}^s_{k|k})(x_k - \hat{x}^s_{k|k})^T | \mathcal{I}^s_k].$$

At time instances when $\nu_k = 1$, the sensor transmits its local state estimate $\hat{x}^s_{k|k}$ over a packet dropping channel to the controller. To take into account energy usage, we will assume that each transmission will require an energy of E, while non-transmissions do not consume energy. Let γ_k be random variables such that $\gamma_k = 1$ if the sensor transmission at time k is successfully received, and $\gamma_k = 0$ otherwise. We will assume that $\{\gamma_k\}$ is i.i.d. Bernoulli with

$$\mathbb{P}(\gamma_k = 1) = \lambda \in (0, 1).$$

At instances where $\nu_k = 1$, it is assumed that the controller knows whether the transmission was successful or not, with dropped packets discarded. Define the information set available to the controller at time k as

$$\mathcal{I}^c_k \triangleq \{\nu_0, \ldots, \nu_k, \nu_0 \gamma_0, \ldots, \nu_k \gamma_k, \nu_0 \gamma_0 \hat{x}^s_{0|0}, \ldots, \nu_k \gamma_k \hat{x}^s_{k|k},$$
$$u_0, \ldots, u_{k-1}\} \qquad (4)$$

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[1] For a symmetric matrix X, we say that $X > 0$ if X is positive definite, and $X \geq 0$ if X is positive semi-definite. For symmetric matrices X and Y, we say that $X > Y$ if $X - Y$ is positive definite, and $X \geq Y$ if $X - Y$ is positive semi-definite.
[2] We assume that u_{k-1} is fed back to the sensor before transmission at time k, and is needed in order to form state estimates at the sensor. Instead of u_{k-1}, it will turn out that feeding back γ_{k-1} is also sufficient for reconstructing u_{k-1} at the sensor, see Remark 1.

and the state estimates and error covariances at the controller by:

$$\hat{x}^c_{k|k-1} \triangleq \mathbb{E}[x_k | \mathcal{I}^c_{k-1}], \quad \hat{x}^c_{k|k} \triangleq \mathbb{E}[x_k | \mathcal{I}^c_k]$$
$$P^c_{k|k-1} \triangleq \mathbb{E}[(x_k - \hat{x}^c_{k|k-1})(x_k - \hat{x}^c_{k|k-1})^T | \mathcal{I}^c_{k-1}]$$
$$P^c_{k|k} \triangleq \mathbb{E}[(x_k - \hat{x}^c_{k|k})(x_k - \hat{x}^c_{k|k})^T | \mathcal{I}^c_k].$$

The decision variables ν_k are computed at the controller (based on information available at time $k-1$), and fed back to the sensor before transmission at time k. This can be done at the same time as the feedback of u_{k-1}, see Fig. 1.

3. PROBLEM STATEMENT

We wish to jointly design the transmission decisions $\{\nu_k\}$ and control signals $\{u_k\}$ to solve the following problem:

$$\min_{\{\nu_k, u_k\}} \mathbb{E}\left[\sum_{k=0}^{N-1}(x_k^T W x_k + u_k^T U u_k + \beta \nu_k E) + x_N^T W x_N\right] \quad (5)$$

where the matrices $W \geq 0$ and $U > 0$, and the scalar parameter $\beta \geq 0$ is a design parameter weighting the tradeoff between the control cost and energy usage.

As stated before, the decision variables ν_k are determined at the controller and fed back to the sensor. Assuming that the transmit decisions ν_k depends only on $(P^c_{0|0}, \ldots, P^c_{k-1|k-1})$,[3] in the next section we will show that the design of $\{\nu_k\}$ and $\{u_k\}$ can be "separated", in the sense that we can rewrite (5) as

$$\min_{\{\nu_k\}} \left[\min_{\{u_k\}} \mathbb{E}\left[\sum_{k=0}^{N-1}(x_k^T W x_k + u_k^T U u_k) + x_N^T W x_N\right] \right.$$
$$\left. + \mathbb{E}\left[\sum_{k=0}^{N-1} \beta \nu_k E\right] \right], \qquad (6)$$

with the inner optimization over $\{u_k\}$ being the solution to an LQG-type control problem. In a similar spirit, separation-type results have been derived for various different problems such as joint sensor and controller design for information regularized LQG control (Tanaka and Sandberg (2015)), power management for wireless control systems (Gatsis et al. (2014)), and joint transmission energy allocation and control for energy harvesting sensors (Knorn and Dey (2015)).

4. SEPARATION OF TRANSMISSION SCHEDULER AND CONTROLLER DESIGN

In this section, we will prove the following:
Theorem 1. For transmit decisions ν_k dependent only on $(P^c_{0|0}, \ldots, P^c_{k-1|k-1})$, problem (5) is equivalent to problem (6). Furthermore, the optimal solution to the problem:

$$\min_{\{u_k\}} \mathbb{E}\left[\sum_{k=0}^{N-1}(x_k^T W x_k + u_k^T U u_k) + x_N^T W x_N\right] \qquad (7)$$

is of the form

$$u_k^* = -(B^T S_{k+1} B + U)^{-1} B^T S_{k+1} A \hat{x}^c_{k|k} \triangleq L_k \hat{x}^c_{k|k}, \quad (8)$$

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[3] This is in fact equivalent to letting ν_k depend only on $P^c_{k-1|k-1}$, see Corollary 1. Allowing transmission decisions to depend on $P^c_{k|k-1}$ is similar to the variance based event triggering scheme of Trimpe and D'Andrea (2014), although random packet drops were not considered there.

with $S_N = W$, $S_k = A^T S_{k+1} A + W - A^T S_{k+1} B (B^T S_{k+1} B + U)^{-1} B^T S_{k+1} A$, $k = N - 1, \ldots, 0$, and optimal cost

$$\text{tr}(S_0 P_0) + \sum_{k=0}^{N-1} \text{tr}(S_{k+1} Q)$$
$$+ \sum_{k=0}^{N-1} \text{tr}\big((A^T S_{k+1} A + W - S_k) \mathbb{E}[P_{k|k}^c]\big). \tag{9}$$

Proof Let us first assume that $\{\nu_k\}$ is a deterministic sequence. Then $\mathbb{E}[x_k | \mathcal{I}_k^s] = \mathbb{E}[x_k | y_0, \ldots, y_k, u_0, \ldots, u_{k-1}]$, and the local state estimates and error covariances at the sensor can be computed using the standard Kalman filtering equations:

$$\hat{x}_{k|k}^s = \hat{x}_{k|k-1}^s + K_k (y_k - C \hat{x}_{k|k-1}^s)$$
$$\hat{x}_{k+1|k}^s = A \hat{x}_{k|k}^s + B u_k$$
$$P_{k|k}^s = P_{k|k-1}^s - P_{k|k-1}^s C^T (C P_{k|k-1}^s C^T + R)^{-1} C P_{k|k-1}^s$$
$$P_{k+1|k}^s = A P_{k|k}^s A^T + Q, \tag{10}$$

where $K_k = P_{k|k-1}^s C^T (C P_{k|k-1}^s C^T + R)^{-1}$. In addition, one can easily show that the state estimates and error covariances at the controller can be computed by:

$$\hat{x}_{k|k}^c = \begin{cases} A \hat{x}_{k-1|k-1}^c + B u_{k-1} &, \nu_k \gamma_k = 0 \\ \hat{x}_{k|k}^s &, \nu_k \gamma_k = 1 \end{cases}$$
$$P_{k|k}^c = \begin{cases} A P_{k-1|k-1}^c A^T + Q &, \nu_k \gamma_k = 0 \\ P_{k|k}^s &, \nu_k \gamma_k = 1. \end{cases} \tag{11}$$

Following similar arguments as in Schenato et al. (2007), we can show that problem (7) has the solution given by (8). Furthermore, the value functions

$$V_N(\mathcal{I}_N^c) \triangleq \mathbb{E}[x_N^T W x_N | \mathcal{I}_N^c]$$
$$V_k(\mathcal{I}_k^c) \triangleq \min_{u_k} \mathbb{E}[x_k^T W x_k + u_k^T U u_k + V_{k+1}(\mathcal{I}_{k+1}^c) | \mathcal{I}_k^c] \tag{12}$$

can be expressed as

$$V_k(\mathcal{I}_k^c) = \mathbb{E}[x_k^T S_k x_k | \mathcal{I}_k^c] + c_k$$
$$c_N = 0$$
$$c_k = \text{tr}\big((A^T S_{k+1} A + W - S_k) P_{k|k}^c\big) + \text{tr}(S_{k+1} Q)$$
$$\quad + \mathbb{E}[c_{k+1} | \mathcal{I}_k^c], \quad k = N - 1, \ldots, 0. \tag{13}$$

Problem (7) then has optimal cost $V_0(\mathcal{I}_0^c)$ given by (9).

Thus, the results (8)-(9) hold when the sequence $\{\nu_k\}$ is deterministic. We will next show that problem (5) is equivalent to problem (6), and that (8)-(9) also hold when each ν_k depends only on $(P_{0|0}^c, \ldots, P_{k-1|k-1}^c)$. Firstly, ν_k being dependent only on $(P_{0|0}^c, \ldots, P_{k-1|k-1}^c)$ means that it is a function of $\mathcal{I}_{k-1}^c$. Hence, by comparing (3) and (4), one can see that the decision ν_k does not provide any additional information about x_k. Therefore,

$$\mathbb{E}[x_k | \mathcal{I}_k^s] = \mathbb{E}[x_k | y_0, \ldots, y_k, u_0, \ldots, u_{k-1}, \nu_0, \ldots, \nu_k]$$
$$= \mathbb{E}[x_k | y_0, \ldots, y_k, u_0, \ldots, u_{k-1}],$$

and (10) will still hold. To show that (11) also holds when ν_k depends only on $(P_{0|0}^c, \ldots, P_{k-1|k-1}^c)$, we will use induction. The claim is clearly true for $k = 0$. Suppose that (11) holds for $k = 0, \ldots, l$. Since ν_{l+1} depends only

on $(P_{0|0}^c, \ldots, P_{l|l}^c)$, which by the induction hypothesis (11) means that ν_{l+1} does not depend on x_l, we then have

$$\hat{x}_{l+1|l+1}^c = \begin{cases} A \hat{x}_{l|l}^c + B u_l &, \nu_{l+1} \gamma_{l+1} = 0 \\ \hat{x}_{l+1|l+1}^s &, \nu_{l+1} \gamma_{l+1} = 1 \end{cases}$$
$$P_{l+1|l+1}^c = \begin{cases} A P_{l|l}^c A^T + Q &, \nu_{l+1} \gamma_{l+1} = 0 \\ P_{l+1|l+1}^s &, \nu_{l+1} \gamma_{l+1} = 1. \end{cases}$$

Thus by induction (11) holds for all k. Since the recursion (11) for $P_{k|k}^c$ does not depend on the control signals $\{u_k\}$, ν_k also does not depend on $\{u_k\}$, and therefore problem (5) is equivalent to (6). As the control does not affect the estimation error covariance, we also have separation of the estimator and controller (Bertsekas (2005)). Finally, we may use the fact that ν_k does not depend on the control signals, and similar arguments as in Schenato et al. (2007), to verify that (8)-(9) will also hold. ∎

By the above results, we have that

$$\min_{\{\nu_k, u_k\}} \mathbb{E}\Big[\sum_{k=0}^{N-1} (x_k^T W x_k + u_k^T U u_k + \beta \nu_k E) + x_N^T W x_N\Big]$$
$$= \text{tr}(S_0 P_0) + \sum_{k=0}^{N-1} \text{tr}(S_{k+1} Q) + \min_{\{\nu_k\}} \Big[\sum_{k=0}^{N-1} \text{tr}\big((A^T S_{k+1} A$$
$$+ W - S_k) \mathbb{E}[P_{k|k}^c]\big) + \sum_{k=0}^{N-1} \mathbb{E}[\beta \nu_k E]\Big]$$

with the optimal control u_k^* given by (8). It then remains to solve the problem

$$\min_{\{\nu_k\}} \Big[\sum_{k=0}^{N-1} \text{tr}\big((A^T S_{k+1} A + W - S_k) \mathbb{E}[P_{k|k}^c]\big) + \sum_{k=0}^{N-1} \mathbb{E}[\beta \nu_k E]\Big] \tag{14}$$

to determine the optimal transmission scheduling. Now by (11), $P_{k|k}^c$ is a function of only $P_{k-1|k-1}^c$, ν_k, and γ_k, with $\{\gamma_k\}$ being i.i.d. By regarding $P_{k-1|k-1}^c$ as the "state", ν_k as the "decision", and γ_k as the "disturbance", from the basic assumptions of dynamic programming (Bertsekas, 2005, pp. 13, 17) we can conclude the following:

Corollary 1. In problem (14), restricting ν_k to be a function of only $P_{k-1|k-1}^c$ is equivalent to letting ν_k be a function of $(P_{0|0}^c, \ldots, P_{k-1|k-1}^c)$.

In Leong et al. (2017) a related problem, namely transmission scheduling for state estimation,

$$\min_{\{\nu_k\}} \Big[\sum_{k=0}^{N-1} \text{tr}\big(\mathbb{E}[P_{k|k}^c]\big) + \sum_{k=0}^{N-1} \mathbb{E}[\beta \nu_k E]\Big],$$

was studied. In the next section we will briefly describe how problem (14) can be solved, by extending the techniques used in Leong et al. (2017).

Remark 1. From (11) and (8), we see that if γ_{k-1} is fed back to the sensor before transmission at time k, then the sensor can reconstruct $\hat{x}_{k-1|k-1}^c$ and hence u_{k-1}. From the communication viewpoint this may be advantageous, since γ_{k-1} is binary valued, rather than the real valued (and possibly vector) control signal u_{k-1}. Feeding back γ_{k-1} can also be used by the sensor to reconstruct $P_{k-1|k-1}^c$, thus also allowing the scheduling of $\{\nu_k\}$ to be done at the

sensor (provided the sensor has sufficient computational capabilities).

5. SOLUTION OF TRANSMISSION SCHEDULING PROBLEM

To simplify the notation, let us denote
$$G_k \triangleq A^T S_{k+1} A + W - S_k.$$
We wish to solve the problem
$$\min_{\{\nu_k\}} \Big[\sum_{k=0}^{N-1} \mathrm{tr}\big(G_k \mathbb{E}[P_{k|k}^c]\big) + \sum_{k=0}^{N-1} \mathbb{E}[\beta \nu_k E] \Big] \quad (15)$$
where ν_k depends only on $P_{k-1|k-1}^c$, see Corollary 1. Problem (15) can be further rewritten as

$$\min_{\{\nu_k\}} \mathbb{E}\Big[\sum_{k=0}^{N-1} \mathrm{tr}\big(G_k \mathbb{E}[P_{k|k}^c | \mathcal{I}_{k-1}^c, \nu_k]\big) + \sum_{k=0}^{N-1} \mathbb{E}[\beta \nu_k E | \mathcal{I}_{k-1}^c, \nu_k] \Big]$$

$$= \min_{\{\nu_k\}} \mathbb{E}\Big[\sum_{k=0}^{N-1} \mathrm{tr}\big(G_k \mathbb{E}[P_{k|k}^c | P_{k-1|k-1}^c, \nu_k]\big)$$
$$+ \sum_{k=0}^{N-1} \mathbb{E}[\beta \nu_k E | P_{k-1|k-1}^c, \nu_k] \Big]$$

$$= \min_{\{\nu_k\}} \mathbb{E}\Big[\sum_{k=0}^{N-1} \nu_k \lambda \mathrm{tr}(G_k P_{k|k}^s) + (1 - \nu_k \lambda) \mathrm{tr}(G_k f(P_{k-1|k-1}^c))$$
$$+ \sum_{k=0}^{N-1} \beta \nu_k E \Big]. \quad (16)$$

The first equality in (16) holds since $P_{k-1|k-1}^c$ is a deterministic function of $\mathcal{I}_{k-1}^c$, and $P_{k|k}^c$ is a function of only $P_{k-1|k-1}^c$, ν_k, and γ_k. The second equality holds since

$$\mathrm{tr}\big(G_k \mathbb{E}[P_{k|k}^c | P_{k-1|k-1}^c, \nu_k]\big)$$
$$= \mathrm{tr}\big(G_k (\nu_k \lambda P_{k|k}^s + (1 - \nu_k \lambda) f(P_{k-1|k-1}^c))\big)$$
$$= \nu_k \lambda \mathrm{tr}(G_k P_{k|k}^s) + (1 - \nu_k \lambda) \mathrm{tr}(G_k f(P_{k-1|k-1}^c)).$$

Define the set
$$\mathcal{S} \triangleq \{f^n(P_{k|k}^s) | n = 0, 1, \dots, \quad k = 0, 1, 2, \dots\}, \quad (17)$$
where
$$f(X) \triangleq A X A^T + Q,$$
and $f^n(.)$ denotes the n-fold composition of $f(.)$, with the convention $f^0(X) = X$. From (11) we see that $\mathcal{S}$ consists of all possible values of $P_{k|k}^c$.

Now define the functions $J_k(.) : \mathcal{S} \to \mathbb{R}$ by:
$$J_N(P) \triangleq 0$$
$$J_k(P) \triangleq \min_{\nu_k} \Big\{ \nu_k \lambda \mathrm{tr}(G_k P_{k|k}^s) + (1 - \nu_k \lambda) \mathrm{tr}(G_k f(P))$$
$$+ \beta \nu_k E + \nu_k \lambda J_{k+1}(P_{k|k}^s) + (1 - \nu_k \lambda) J_{k+1}(f(P)) \Big\}$$
for $k = N - 1, \dots, 1, 0$, where we note that $\{P_{k|k}^s\}$ is a deterministic sequence given the initial error covariance. Numerically, problem (16) can then be solved by using dynamic programming, by computing $J_k(P_{k-1|k-1}^c)$ for $k = N - 1, \dots, 1, 0$.

To provide further insight, we will next derive some structural results on the optimal solution to problem (16).

Since ν_k takes on either the values 0 or 1, $J_k(P)$ can be rewritten as
$$J_k(P) = \min \Big\{ \mathrm{tr}(G_k f(P)) + J_{k+1}(f(P)),$$
$$\lambda \mathrm{tr}(G_k P_{k|k}^s) + (1 - \lambda)\mathrm{tr}(G_k f(P)) + \beta E$$
$$+ \lambda J_{k+1}(P_{k|k}^s) + (1 - \lambda) J_{k+1}(f(P)) \Big\}$$
with the two terms in the minimization corresponding to the cases $\nu_k = 0$ and $\nu_k = 1$. Let
$$\phi_k(P) \triangleq \lambda \mathrm{tr}(G_k f(P)) - \lambda \mathrm{tr}(G_k P_{k|k}^s) - \beta E + \lambda J_{k+1}(f(P))$$
$$- \lambda J_{k+1}(P_{k|k}^s), \quad (18)$$
which denotes the difference between the two terms, considered as a function of P. Note that if $\phi_k(P) < 0$ then the sensor will not transmit, while if $\phi_k(P) > 0$ then the sensor will transmit.

Definition 1. A function $f(.) : \mathcal{S} \to \mathbb{R}$ is *increasing* if $X \leq Y \Rightarrow f(X) \leq f(Y)$.

Theorem 2. (i) The function $\phi_k(P)$ is an increasing function of P, for $k = 0, \dots, N - 1$.
(ii) Suppose that the pair (A, C) is observable, the pair $(A, Q^{1/2})$ is controllable, and that the Kalman filter at the sensor is operating in steady state. For unstable systems the optimal solution to problem (16) is a threshold policy of the form
$$\nu_k^*(P_{k-1|k-1}^c) = \begin{cases} 0, & P_{k-1|k-1}^c \leq P_k^* \\ 1, & \text{otherwise}, \end{cases}$$
where the threshold $P_k^* \in \mathcal{S}$ in general depends on k.

Proof (i) Let us first verify that $G_k \geq 0$. From the recursion for S_k we have
$$G_k = A^T S_{k+1} A + W - S_k$$
$$= A^T S_{k+1} B (B^T S_{k+1} B + U)^{-1} B S_{k+1} A,$$
which is positive semi-definite since S_{k+1} is positive semi-definite, and U is positive definite by assumption. Next, since $G_k \geq 0$, we can easily show by using Lemma 8.4.12 of Bernstein (2009) that $\mathrm{tr}(G_k f(P))$ is increasing in P. Finally, a simple induction argument shows that $J_{k+1}(f(P))$ is increasing in P. Hence the function $\phi_k(P)$ defined by (18) is increasing in P.
(ii) Since the pair (A, C) is observable and the pair $(A, Q^{1/2})$ is controllable, $P_{k|k}^s$ converges to a steady state value $\bar{P}$ as $k \to \infty$. If the Kalman filter at the sensor is operating in steady state, then the set $\mathcal{S}$ defined in (17) simplifies to $\mathcal{S} = \{\bar{P}, f(\bar{P}), f^2(\bar{P}), \dots\}$, and the elements of $\mathcal{S}$ satisfy the total ordering (see e.g. Shi and Zhang (2012))
$$\bar{P} \leq f(\bar{P}) \leq f^2(\bar{P}) \leq \dots.$$
Furthermore, if the system is unstable, then $\mathrm{tr}(G_k f^n(\bar{P})) \to \infty$ as $n \to \infty$, and so $\phi_k(P) > 0$ for sufficiently large P, i.e. the sensor will transmit for sufficiently large P. Combining this fact with (i) gives the result. ∎

6. INFINITE HORIZON FORMULATION

The infinite horizon counterpart of problem (5) can also be studied. In this section, we will assume that the pairs (A, B) and $(A, Q^{1/2})$ are controllable, and that the pairs (A, C) and $(A, W^{1/2})$ are observable.

We wish to solve the following problem:

$$\min_{\{\nu_k, u_k\}} \limsup_{N \to \infty} \frac{1}{N} \mathbb{E}\Big[\sum_{k=0}^{N-1} (x_k^T W x_k + u_k^T U u_k + \beta \nu_k E) \Big] \quad (19)$$

Theorem 3. (i) For transmit decisions ν_k dependent only on $P_{k-1|k-1}^c$, problem (19) is equivalent to the problem:

$$\min_{\{\nu_k\}} \Big[\min_{\{u_k\}} \limsup_{N \to \infty} \frac{1}{N} \mathbb{E}\Big[\sum_{k=0}^{N-1} (x_k^T W x_k + u_k^T U u_k) \Big]$$
$$+ \limsup_{N \to \infty} \frac{1}{N} \mathbb{E}\Big[\sum_{k=0}^{N-1} \beta \nu_k E \Big] \Big].$$

(ii) In steady state, the optimal solution to:

$$\min_{\{u_k\}} \limsup_{N \to \infty} \frac{1}{N} \mathbb{E}\Big[\sum_{k=0}^{N-1} (x_k^T W x_k + u_k^T U u_k) \Big]$$

is of the form

$$u_k^* = -(B^T S_\infty B + U)^{-1} B^T S_\infty A \hat{x}_{k|k}^c \triangleq L_\infty \hat{x}_{k|k}^c,$$

where S_∞ satisfies $S_\infty = A^T S_\infty A + W - A^T S_\infty B (B^T S_\infty B + U)^{-1} B^T S_\infty A$, with optimal cost

$$\text{tr}(S_\infty Q) + \limsup_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \text{tr}\big((A^T S_\infty A + W - S_\infty) \mathbb{E}[P_{k|k}^c]\big). \quad (20)$$

(iii) Consider the problem:

$$\min_{\{\nu_k\}} \limsup_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \Big(\text{tr}\big((A^T S_\infty A + W - S_\infty) \mathbb{E}[P_{k|k}^c]\big)$$
$$+ \mathbb{E}[\beta \nu_k E] \Big). \quad (21)$$

Suppose $\lambda > 1 - \frac{1}{\max_i |\sigma_i(A)|^2}$, where $\sigma_i(A)$ is an eigenvalue of A. Then there exists a stationary solution to problem (21) which is a threshold policy of the form

$$\nu_k^*(P_{k-1|k-1}^c) = \begin{cases} 0 & , \ P_{k-1|k-1}^c \leq P^* \\ 1 & , \ \text{otherwise}, \end{cases} \quad (22)$$

where the threshold $P^* \in \mathcal{S}$ does not depend on k.

Proof Theorem 3 can be proved by using similar arguments as in Sections 4 and 5, and taking limits as $N \to \infty$. For the existence of stationary solutions to problem (21) in part (iii), this is shown by verifying conditions in Sennott (1999) for the existence of solutions to average cost problems with countably infinite state space. Under the condition $\lambda > 1 - \frac{1}{\max_i |\sigma_i(A)|^2}$, this verification can be accomplished by using similar arguments as in Theorem III.1 of Leong et al. (2017). ∎

Under the threshold policy (22), simple analytical[4] expressions for $\mathbb{E}[\nu_k E]$ and $\mathbb{E}[P_{k|k}^c]$ can also be derived. We will only state the results, the derivations can be found in Leong et al. (2017). Let $t \in \mathbb{N}$ represent the threshold such that $P^* = f^t(\bar{P})$. Note that t will depend on the value of β chosen in problem (19). Also let

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[4] The threshold parameter t where $P^* = f^t(\bar{P})$ will in general still need to be found numerically

$$\pi_j = \begin{cases} \dfrac{\lambda}{\lambda t + 1} & , \ j = 0, \ldots, t \\ \dfrac{(1-\lambda)^{j-t} \lambda}{\lambda t + 1} & , \ j = t+1, t+2, \ldots. \end{cases}$$

Then we have

$$\mathbb{E}[\nu_k E] = \frac{E \pi_0}{\lambda} = \frac{E}{\lambda t + 1},$$

and

$$\mathbb{E}[P_{k|k}^c] = \sum_{j=0}^{\infty} \pi_j f^j(\bar{P}).$$

It can be shown that the infinite series above converges if $\lambda > 1 - \frac{1}{\max_i |\sigma_i(A)|^2}$, and can then be computed numerically. Problem (19) thus has optimal cost

$$\text{tr}(S_\infty Q) + \text{tr}\Big((A^T S_\infty A + W - S_\infty) \sum_{j=0}^{\infty} \pi_j f^j(\bar{P})\Big) + \beta \frac{E}{\lambda t + 1},$$

with expected energy usage per-stage

$$\mathbb{E}[\nu_k E] = \frac{E}{\lambda t + 1} \quad (23)$$

and expected control cost per-stage given by

$$\mathbb{E}\Big[x_k^T W x_k + u_k^T U u_k \Big]$$
$$= \text{tr}(S_\infty Q) + \text{tr}\Big((A^T S_\infty A + W - S_\infty) \sum_{j=0}^{\infty} \pi_j f^j(\bar{P})\Big). \quad (24)$$

7. NUMERICAL RESULTS

We consider a system with parameters

$$A = \begin{bmatrix} 1.3 & 0.5 \\ 0.2 & 0.9 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ C = [\,1 \ 1\,], \ Q = I, \ R = 1.$$

The weighting matrices for the control cost are $W = I$ and $U = 1$. The packet reception probability $\lambda = 0.7$, and transmission energy is here taken as $E = 1$.

7.1 Finite Horizon

We use the finite horizon $N = 10$. We assume that the Kalman filter at the sensor is in steady state with $P_{k|k}^s = \bar{P}, \forall k$, and the initial covariance of x_0 is $P_0 = f(\bar{P})$. Fig. 2 plots the expected energy usage $\mathbb{E}\Big[\sum_{k=0}^{N-1} \nu_k E \Big]$ vs the expected control cost

$$\mathbb{E}\Big[\sum_{k=0}^{N-1} (x_k^T W x_k + u_k^T U u_k) + x_N^T W x_N \Big],$$

obtained by solving problem (5) for different values of β. Each of the points is obtained by Monte Carlo averaging over 100000 different iterations. One can observe a tradeoff between the energy usage and the control performance.

7.2 Infinite Horizon

Next we consider the infinite horizon problem. Fig. 3 plots the per-stage expected energy usage $\mathbb{E}\Big[\nu_k E \Big]$ vs the per-stage expected control cost $\mathbb{E}\Big[x_k^T W x_k + u_k^T U u_k \Big]$, obtained by solving problem (19) for different values of β. Each of
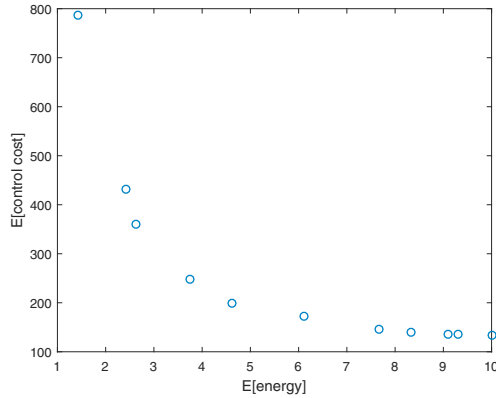
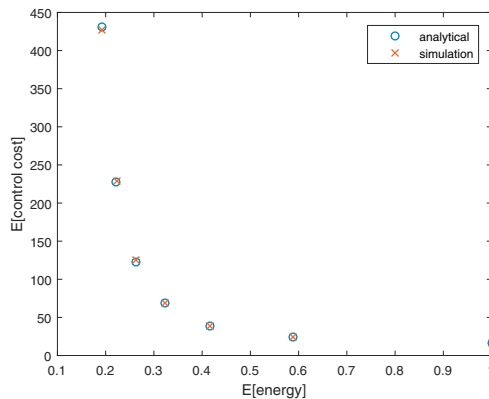Fig. 2. Finite horizon. Expected energy usage vs Expected control cost.

Fig. 3. Infinite horizon. Expected energy usage per-stage vs Expected control cost per-stage.

the points is obtained by time averaging a Monte Carlo iteration of length 1000000. Also plotted in Fig. 3 are the analytical expressions (23) and (24). We see that there is very close agreement between the simulation results and analytical expressions (23)-(24).

8. CONCLUSION

A joint transmission scheduling and control design problem has been studied in this paper, which minimizes a linear combination of the control cost and expected energy usage of the sensor. We have shown a separation in the design of the event-based transmission scheduler and controller, and presented the solutions for the optimal policies.

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