Padé Approximations of e^{Ah} and preservation of quadratic Lyapunov functions

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We investigate whether or not quadratic Lyapunov functions are preserved under Padé approximations.

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1 Introduction

Determining whether or not a finite set of linear time-invariant (LTI) systems have a common quadratic Lyapunov function (CQLF) is a problem that arises frequently [1]. In converting a switching system that is constructed from a set of LTI systems into a discrete counterpart, or in transforming a set of matrices into a new set in which testing for the existence of a CQLF is more convenient, one is often interested in developing transformations that preserve the existence of a CQLF. One class of transformations is given by the diagonal Padé approximation of order p to the matrix exponential e^{Ah} [2]:

$$e^{Ah} \approx Q_p(Ah)Q_p^{-1}(-Ah) \quad \text{with} \quad Q_p(Ah) = \sum_{k=0}^{p} c_k(Ah)^k \quad \text{and} \quad c_k = \frac{(2p-k)!p!}{(2p)!k!(p-k)!}\ , \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ and $h > 0$ is a discrete time step. It is well known that such transformations preserve the stability of LTI systems, and furthermore, that the first order transformation $(I + A\frac{h}{2})(I - A\frac{h}{2})^{-1}$, known as the bilinear or Tustin transform, also preserves a CQLF for finite set of systems. Of principle interest in this note is to establish whether this latter property is shared by higher order Padé approximations.

Comment : Properties that are invariant under Padé approximations have recently been the subject of attention in the control engineering literature. For example, even for a single LTI system, it was recently shown that non-quadratic Lyapunov functions may not be preserved under the bilinear transform. This fact was first demonstrated in [3], where it was proven that unlike quadratic Lyapunov functions (QLFs), ∞-norm and 1-norm type Lyapunov functions are not necessarily preserved under the bilinear mapping. Furthermore, it has been shown recently in [4] that for certain linear time varying (LTV) systems that are stable, but not quadratically stable, stability may not even be preserved under the bilinear transform.

Notation : In the following discussion we say that a real matrix $P = P^T > 0$ is a Lyapunov matrix for $A \in \mathbb{R}^{n\times n}$ if it satisfies the Lyapunov inequality $A^T P + PA < 0$ for A and P is a common Lyapunov matrix (CLM) for $\mathscr{A} = \{A_1, \dots, A_m\}$ if it satisfies the Lyapunov inequality for all $A_i \in \mathscr{A}$. Moreover, we say that a real matrix $P_d = P_d^T > 0$ is a Stein matrix for $A_d \in \mathbb{R}^{n\times n}$ if it satisfies the Stein inequality $A_d^T P_d A_d - P_d < 0$ for A_d and P_d is a common Stein matrix (CSM) for $\mathscr{A}_d = \{A_{d1}, \dots, A_{dm}\}$ if it satisfies the Stein inequality for all $A_{di} \in \mathscr{A}_d$.

2 Quadratic stability and higher order Padé approximations of e^{Ah}

The question we want to answer here is whether a given quadratic Lyapunov function for an LTI system is preserved under Padé approximations.

Lemma 2.1 *Suppose that $A \in \mathbb{R}^{n\times n}$, $h \in \mathbb{R}$ is positive and A_d is the 1^{st} order diagonal Padé approximation of e^{Ah} (that is, the bilinear transform of A). Then, P is a Lyapunov matrix for A if and only if P is a Stein matrix for A_d.*

P r o o f. The proof is obtained by substituting the expression for $A_d = (I + \frac{h}{2}A)(I - \frac{h}{2}A)^{-1}$ into the Stein inequality for A_d. Post- and pre-multiplying this inequality by $I - \frac{h}{2}A$ and its transpose, respectively, and simplifying, results in the equivalent inequality $A^T P + PA < 0$ which is the Lyapunov inequality for A. □

An immediate consequence of Lemma 2.1 is that CQLFs are preserved under first order Padé approximations.

Corollary 2.1 *$P = P^T > 0$ is a CLM for a set of matrices $\mathscr{A} = \{A_1, \dots, A_m\}$ if and only if P is CSM for the set of matrices $\mathscr{A}_d = \{A_{d1}, \dots, A_{dm}\}$, where A_{di} is the first order diagonal Padé approximation of A_i.*

Now we have the following result for second order diagonal Padé approximations.

Lemma 2.2 *Suppose that $A \in \mathbb{R}^{n\times n}$, $h \in \mathbb{R}$ is positive and A_d is the 2^{nd} order diagonal Padé approximation of e^{Ah}.*

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(i) *If P is a Lyapunov matrix for A then, P is also a Stein matrix for A_d.*

(ii) *If P is a Stein matrix for A_d then, $P_c = P + \frac{h^2}{12}A^T PA$ is a Lyapunov matrix for A.*

Proof. Substitute $A_d = (I + \frac{h}{2}A + \frac{h^2}{12}A^2)(I - \frac{h}{2}A + \frac{h^2}{12}A^2)^{-1}$ into the Stein inequality for A_d. Post- and pre-multiplying this inequality by $I - \frac{h}{2}A + \frac{h^2}{12}A^2$ and its transpose, respectively, and simplifying, results in the equivalent inequality

$$A^T P + PA + \frac{h^2}{12}A^T(A^T P + PA)A < 0. \tag{2}$$

Suppose that P is a Lyapunov matrix for A. Then $A^T P + PA < 0$ and inequality (2) is satisfied. This implies that $A_d^T PA_d - P < 0$, that is, P is a Stein matrix for A_d, proving (i). Suppose now that P is a Stein matrix for A_d. Then it satisfies inequality (2) which is equivalent to

$$A^T(P + \frac{h^2}{12}A^T PA)^T + (P + \frac{h^2}{12}A^T PA)A < 0. \tag{3}$$

Defining $P_c = P + \frac{h^2}{12}A^T PA$, and noting $P_c = P_c^T > 0$, the last expression becomes a Lyapunov inequality for A, proving (ii). □

The last lemma shows that when a matrix A is Hurwitz stable with a Lyapunov matrix P, the 2^{nd} order Padé approximation of e^{Ah} is Schur stable with the Stein matrix P. However, it does not state that the converse is true. In fact the converse is not true. If it were true then, as is the case for the bilinear transform, CQLFs would be preserved under the 2^{nd} order Padé approximation. However, the following example shows that this need not be the case.

Example 2.1 Consider the Hurwitz matrices:

$$A_1 = \begin{bmatrix} 1.56 & -100 \\ 0.1 & -4.44 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -1 & 0 \\ 0 & -0.1 \end{bmatrix}.$$

Since the matrix product A_1A_2 has negative real eigenvalues it follows that there is no CLM [1]. Now consider the matrices A_{d1}, A_{d2} obtained under the 2^{nd} order diagonal Padé approximation of e^{2A_i} with the discrete time step $h = 2$:

$$A_{d1} = \begin{bmatrix} -0.039 & 0.4205 \\ -0.0004 & -0.0138 \end{bmatrix}, \qquad A_{d2} = \begin{bmatrix} 0.1429 & 0 \\ 0 & 0.8187 \end{bmatrix}.$$

These matrices have a CSM

$$P_d = \begin{bmatrix} 2.3294 & -0.0138 \\ -0.0138 & 2.7492 \end{bmatrix}.$$

Comment : Example 2.1, together with Lemma 2.2, illustrate the following facts. Let $\mathscr{A} = \{A_1, \dots, A_m\}$ be a set of Hurwitz matrices and $\mathscr{A}_d = \{A_{d1}, \dots, A_{dm}\}$ the corresponding set of Schur stable matrices obtained under the 2^{nd} order diagonal Padé approximation. If P is a CLM for $\mathscr{A}$ then P is a CSM for $\mathscr{A}_d$. However, as the example demonstrates, the existence of a CSM for the $\mathscr{A}_d$ does not imply the existence of a CLM for $\mathscr{A}$.

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