An Analysis of University Mathematics Teaching using the Knowledge Quartet

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We analyse accounts written by three mathematics lecturers on their practice using the Knowledge Quartet framework. This framework has been used to study how a teacher's knowledge of mathematics and mathematics pedagogy influences his/her actions in the classroom at both the primary and secondary level. We consider how the framework could be used to study university level teaching, and we report on the dimensions of teacher knowledge that were made visible by this framework.
Keywords: Knowledge Quartet, teacher knowledge, university mathematics teaching.

INTRODUCTION

The first three authors of this paper are mathematics lecturers at three universities in Ireland, who also engage in mathematics education research. Over the course of two years, they wrote accounts of incidents which occurred during their teaching as part of a professional development project using the Discipline of Noticing (Mason, 2002). In this paper, we report on our more recent use of a different theoretical framework, the Knowledge Quartet framework (Rowland, Huckstep & Thwaites, 2005), to analyse these accounts. The Knowledge Quartet categorizes situations from classrooms where mathematical knowledge surfaces in teaching. There has been one previous attempt to use the framework to analyse university mathematics teaching (Rowland, 2009). The focus of that paper was the knowledge-grounded foundation beliefs of the university lecturer, about mathematics and about teaching and learning.

The purpose of our current study is twofold. Firstly, we are interested in whether the Knowledge Quartet framework could be applied to study teaching at university level. Secondly, we would like to know what features of university teaching are highlighted when our set of accounts are analysed using the Knowledge Quartet. Previously, the first three authors had analysed their accounts to study the many decision points that arose while teaching and in O'Shea, Breen and Meehan (2017) these decision points and their triggers were categorised. We were interested to see if using the Knowledge Quartet framework would draw attention to other aspects of teaching in the accounts.

In this article, we will first of all consider the literature on teacher knowledge, especially at university level. We will then expand on the Knowledge Quartet framework, and give some results from our analysis using this lens. Finally, we will discuss our findings and suggest some future avenues for research.

LITERATURE REVIEW

The Knowledge Quartet is a theoretical tool for observing, analysing and reflecting on actual mathematics teaching. Ball, Thames and Phelps (2008) also studied mathematics classrooms to develop a theory of mathematical knowledge for teaching that built on the work of Shulman (1987). This resulted in the identification of an important subdomain of content knowledge - ‘specialized content knowledge’. This is distinct from ‘common content knowledge’ and is unique to the work of teaching.

Independently, Ainley and Luntley (2007) suggested that experienced teachers draw on ‘attention-dependent knowledge’ in addition to subject knowledge and pedagogical knowledge (both general and subject-specific). Few research studies have been concerned with the knowledge employed in university mathematics teaching. McAlpine and Weston (2000) conducted a research study with six professors considered exemplary in their teaching and found that all the professors drew on pedagogical knowledge, pedagogical content knowledge, content knowledge and knowledge of learners (following Shulman (1987)) while monitoring their own actions and making decisions during lectures. This was despite the fact that three of the professors were mathematicians who had no pedagogical training (while the remaining three were mathematics educators or trained teachers). McAlpine and Weston (2000) hypothesised that the mathematicians constructed this knowledge largely through experience and reflection, and that their lack of training led them to depend more on their experience than the mathematics educators did.

On the other hand, Wagner, Speer and Rossa (2007) examined the knowledge, other than content knowledge, required by a mathematician teaching an undergraduate course. They reported that he was unable to anticipate how students would respond to particular activities and how the content or sequence of individual classes contributed to the instructional goals of the entire course. The authors claim these findings lend support to the assertion that there is knowledge particular to teaching that is distinct from, and not easily constructed from, knowledge of content.

Speer and Wagner (2009) focussed on whole-class discussions and examined the nature of the knowledge that a mathematician could employ to make effective use of undergraduates' mathematical contributions in a way that furthered the goals for the class. Their analysis focussed on the role of (a lack of) pedagogical content knowledge and specialized content knowledge in the difficulties experienced by the instructor in scaffolding student learning while orchestrating such discussions.

THEORETICAL FRAMEWORK

The Knowledge Quartet

The Knowledge Quartet is a ‘theory’ in the sense that it proposes a way of thinking about mathematics teaching in the usual institutional settings (lessons/classes), with a focus on the disciplinary content (mathematics) of the lesson.

The Knowledge Quartet (KQ) was the outcome of empirical research at the University of Cambridge, UK (Rowland et al., 2005), in which 24 mathematics lessons were videotaped and scrutinised. The research team identified aspects of the teachers' actions in the classroom that could be construed as being informed by their mathematics subject matter knowledge or pedagogical content knowledge. This inductive process initially generated a set of 18 codes (later expanded to 21), subsequently grouped into four broad, super-ordinate categories or dimensions.

The first dimension of the KQ, *foundation*, consists of teachers' mathematics-related knowledge, beliefs and understanding. The second dimension, *transformation*, concerns knowledge-in-action as demonstrated both in planning to teach and in the act of teaching itself. The third dimension, *connection*, concerns the ways by which the teacher achieves coherence within and between sessions. The final dimension, *contingency*, is witnessed in classroom events that were not envisaged in the teachers' planning. Essentially, it is the ability to "think on one's feet".

Conceptualising the Knowledge Quartet

The concise conceptualisation of the KQ which now follows is a synthesis of the characteristics of its four dimensions.

Foundation

The first member of the KQ is rooted in the foundation of the teacher's theoretical background and beliefs. It concerns their knowledge, understanding and ready recourse to what was learned in preparation (intentionally or otherwise) for their role in the classroom. The key components of this theoretical background are: knowledge and understanding of mathematics *per se*; knowledge of significant tracts of the literature and thinking which has resulted from systematic enquiry into the teaching and learning of mathematics; and espoused beliefs about mathematics, including beliefs about why and how it is learnt. The remaining three categories focus on knowledge-in-action as demonstrated both in planning to teach and in the act of teaching itself.

Transformation

At the heart of the second member of the KQ is Shulman's observation that the knowledge base for teaching is distinguished by " … the capacity of a teacher to *transform* the content knowledge he or she possesses into forms that are pedagogically powerful" (Shulman, 1987, p. 15, emphasis added). This dimension picks out behaviour that is directed towards a student (or a group of students), and which follows from deliberation and judgement informed by foundation knowledge. The choice and *use of examples* has emerged as a rich vein for reflection and critique, and one of the most prevalent codes observed in practice (Rowland, 2008).

Connection

The next dimension concerns the *coherence* of the planning or teaching displayed across an episode, lesson or series of lessons. Our conception of connection includes the *sequencing* of topics of instruction within and between lessons, including the ordering of tasks and exercises. To a significant extent, these reflect deliberations and choices entailing not only knowledge of structural connections within mathematics, but also awareness of the relative cognitive demands of different topics and tasks, and the implementation of strategies to remove (or lessen) obstacles to learning.

Contingency

Our final dimension concerns the teacher's response to classroom events that were not anticipated in the planning. This dimension of the KQ is about the ability to 'think on one's feet': it is about *contingent action.* Whilst the teacher's intended actions can be planned, the students' responses cannot. The teachers' response to students' unexpected contributions is one of the most low-inference codes of the KQ.

Many moments or episodes within a session can be understood in terms of two or more of the four units; for example, a **contingent** response to a student's suggestion might helpfully **connect** with ideas considered earlier. Furthermore, the application of content knowledge in the classroom always rests on **foundational** knowledge.

The KQ is a lens through which the observer 'sees' classroom mathematics instruction. It offers a four-dimensional framework against which mathematics lessons can be discussed, with a focus on their subject-matter content, and the teacher's related knowledge and beliefs.

This framework has been used in different contexts and for different purposes. For instance, Rowland (2012) used the KQ to examine situations in which mathematical knowledge surfaces in primary and secondary mathematics. He concludes that elementary mathematics teaching poses challenges which are qualitatively different from those confronting secondary mathematics teachers. However, the mathematics knowledge primary mathematics teachers must possess is neither less profound nor easier to acquire than that of secondary teachers. Turner and Rowland (2011) describe a project in which the framework was used to guide pre-service teachers in a process of personal reflection on their teaching. The participants found that the KQ helped them to focus more effectively on the mathematical content of their lessons and its enhancement. The authors reported that this enhanced focus on mathematical content knowledge had a positive influence on its further development. There was also evidence that the KQ helped the participants to develop a more learner-centred view of teaching and one in which conceptual understanding rather than procedural fluency was emphasised. Other recent studies using the KQ have focussed on contingent moments in the classroom (e.g. Rowland & Zazkis, 2013).

METHODOLOGY

The accounts which form the data for this study were written using the Discipline of Noticing (Mason, 2002). This advocates that practitioners write 'brief-but-vivid' accounts of incidents that they have noticed in their practice. Mason (2002) defines a brief-but-vivid account as

> one which readers readily find relates to their experience. Brevity is obtained by omitting details which divert attention away from the main issue. The aim is to locate a phenomenon, so the less particular the description, the easier this is, without becoming so general as to be of no value….Thus description is as factual as possible. (p.57)

He advises that these accounts should also avoid justification of incidents or actions, and should therefore be 'accounts of' rather than 'accounting for' a particular situation. The first three authors of this paper had written brief-but-vivid accounts of their teaching over a two-year period. These focused on notable incidents that occurred while they were teaching, but are not reflections or descriptions of a whole lecture. For more details, see O'Shea, Breen and Meehan (2017).

For this paper each of the three lecturers chose one of their modules; only the accounts relating to that module which contained references to mathematical knowledge were analysed (20 accounts for Lecturer 1, 29 for Lecturer 2, and 38 for Lecturer 3). Lecturer 1 chose a one-semester Introduction to Analysis module for 27 second-year students (this module was delivered separately to 7 Pure Maths students and 20 Science students), Lecturer 2 also chose a one-semester Introduction to Analysis module for a group of 75 second-year students, while Lecturer 3 chose a year-long Differential Calculus module for a group of 49 first-year students. All three lecturers aimed to foster dialogue in their classrooms, perhaps because of their interest in educational research and the relatively small class sizes in these modules.

When coding the data we compared our accounts with the descriptions of each of the 21 codes associated to the KQ framework, with reference to the examples available at www.knowledgequartet.org. We began the coding process by first coding a small set of accounts together. Then each lecturer coded her own set of accounts and passed on her analysis to the other two lecturers in turn. They coded the accounts independently before comparing their analysis with that of the original instructor. All three discussed any discrepancies and agreed on the final coding.

During the coding process, we felt that the names of a few of the codes did not fully reflect the terminology used in teaching mathematics at the university level. We interpreted the code *Teacher Demonstration (to explain a procedure)* to also encompass teacher demonstration to explain a proof. We chose to use the code *Choice of Example* (CE) to include particular instances of an abstract concept or a general procedure; and, as the rehearsal of a procedure or 'exercise' (Rowland, 2008), and also for non-routine tasks. We also applied the code *Responding to Students' Ideas* (RSI) from the Contingency Dimension to encompass instances where the lecturer had to respond to a *lack* of students' ideas.

RESULTS

A summary of the number and percentage of codes found in each of the four categories of the KQ for each author is given in Table 1 below. While a number of codes could be applied to some events, the one which we judged to be predominant was what was counted in this table.

KQ Dimension	Lecturer 1	Lecturer 2	Lecturer 3
Foundation	10 (20.83%)	4 (11.11%)	41 (34.74%)
Transformation	14 (29.17%)	17 (47.22%)	32 (27.12%)
Connection	7 (14.58%)	8 (22.22%)	24 (20.34%)
Contingency	17 (35.42%)	7 (19.45%)	21 (17.8%)
Total	48 (100%)	36 (100%)	118 (100%)

Table 1. Number and percentage of codes in each KQ dimension for each lecturer

On coding the accounts it became apparent that all three lecturers frequently wrote accounts about giving a task to the class or instigating a whole class discussion around a task, recording some students' responses (or lack of responses) in relation to the task, and noting what the lecturer thought or learned about student thinking. Therefore it is not surprising that the code *Choice of Example* (CE) in the Transformation Dimension was the most frequently occurring code for all three lecturers, while *Responding to Students' Ideas* in the Contingency Dimension was in each of their top three most frequently occurring codes. In many accounts, the lecturer contrasted student learning on a task with learning on the same task the previous year or with students in a different class, often noting what students found easy or difficult. The tasks were usually designed and planned by the lecturer with specific aims for student learning in mind, thus the codes *Anticipation of Complexity* (AC) in the Connection Dimension and *Awareness of Purpose* (AP) in the Foundation Dimension frequently appear for all three lecturers. Lecturer 3, who was simultaneously conducting a research project on mathematical tasks, was often explicit about the pedagogical rationale behind a given task. Consequently, another significant code for her accounts was *Theoretical Underpinning of Pedagogy* (TUP) in the Foundation Dimension.

By way of example, the following is an account from Lecturer 1, coded as RSI. She struggles to understand what the student is asking but still feels she has to respond:

> A student asked a question in the middle of a complicated proof. I didn't understand the question and asked him to ask it again. He tried but I still couldn't understand. So I explained the proof again as best I could paying attention to what I thought he had had problems with. However I realised I had made a choice. I could have continued probing until I figured out what he was asking. I decided not to do that so as not to embarrass him, but maybe I didn't really answer his question in the end.

While the majority of accounts were on incidents during lectures, some relate to preparation of tasks and lessons, or conversations with students outside of class. The following is an example of an account by Lecturer 2 coded as CE, which describes a task given to students to work on during the second lecture of the semester.

> I handed out the first Inclass Exercise of the module. It contained the following statement: There exists a university in the world, where every Analysis student achieves a final mark of

at least 90% in the module. The instructions were as follows: Write down what you would need to do to prove that Statement B is *false*. At the end of the class, a student came up to me and said that suppose there were infinitely many universities in the world, then you couldn't actually disprove the statement because you wouldn't be able to get around to all of them to check the Analysis grades. I was impressed with how he extended the statement.

Given that CE was the most frequently occurring code for all three lecturers, the following account provides another example of a task given, this time by Lecturer 1, to help students propose conjectures about the relationship between bounded and convergent sequences.

> I was talking about bounded sequences with the class today. I got them to come up with some bounded and some unbounded sequences. I tried to get the class to make conjectures by asking them to guess what the next theorem would be, or what it definitely wouldn't be. They immediately realised that there would be no theorem that said that every bounded sequence converges and then conjectured that every convergent sequence is bounded. They seemed to enjoy the process.

Next we present an account from Lecturer 3, coded as AP. She is explicit in her intentions to engage students in mathematical sense-making and on challenging students' views of mathematics as a set of rules to be learned and applied.

> Today I continued with sketching graphs of functions and asked the students to draw the graph of f(x)=1/x on its natural domain, among others. I circulated the room as they were doing this and noticed that a number of what I had considered to be the more able students were drawing the graph incorrectly (possibly confusing f(x)=1/x and g(x)=1/x^2). I have been trying to put across the idea of Calculus as a 'science' from the point of view that 'experiments/trials' can be undertaken to check 'hypotheses', results can be 'replicated' and so on, but it appears some students are disregarding this and still regard it as a collection of facts to be learnt and remembered.

Finally, we present an account from Lecturer 3. Her pedagogy is underpinned by having students take a guided-discovery approach as a classroom community (TUP).

> I tried to use a 'guided-discovery' approach to facilitate students' realization that the graph of a function and its inverse are mirror images of each other in the line y=x. However, each step of this took a lot longer than I envisaged. Moreover, I wasn't convinced at the end that the students would retain this particular piece of information longer or understand it better for having discovered it themselves as a class community.

DISCUSSION

In this paper we have used the KQ to analyse a set of accounts written as part of a professional development project that involved engaging with the Discipline of Noticing (Mason 2002). This is not the usual type of data that has been used in previous KQ studies. Typically, the researchers in those studies had access to classrooms (of either pre-service or experienced teachers), and have been able to record and analyse entire lessons. Our data is different in two key ways. Firstly we do not have recordings of entire lessons but the brief-but-vivid accounts of the instructor herself on some aspect of the class, which was memorable to her. This is a limitation because we may have chosen not to include some relevant aspects of our classes, or of our students' experience and reactions, but the accounts do shed some light on the 'attention-dependent knowledge' of the instructor (Ainley & Luntley, 2007). We did not write our accounts in order to give a representative view of our teaching, rather we concentrated on aspects which were troublesome to us. However, we do have accounts from almost every lecture in the modules considered whereas previous studies have data only from a very small number of classes with a given teacher.

In his KQ analysis of university mathematics teaching, Rowland (2009) refers to only one lecture. The analysis homes in on the foundation dimension and in particular on the beliefs of the lecturer (about mathematics and pedagogy), but does not explore the other three dimensions. Our analysis has shown that all four dimensions were present in our data. It should be noted that all three lecturers pursued an interactive approach in their classes, and perhaps the same spread of codes would not be present in an analysis of a more stereotypical university lecture.

On the other hand, the prevalence of the use of the *responding to student ideas* code for the accounts discussed here suggests that the traditional image of a lecture (in which a lecturer delivers from a pre-prepared script, rarely deviating from it, and interacts minimally with students) is not always accurate and highlighted this element of our practice for us.

In addition, given our previous focus on decision points in these accounts (O'Shea, Breen & Meehan, 2017), we may have expected the contingency dimension to be dominant but this was not the case. The KQ highlighted the importance of the other three categories in our accounts, especially the transformation dimension in the *choice of examples*. We found the framework provided a lens through which the *knowledge* brought to bear in the preparation and teaching of lessons could be viewed in a coherent and comprehensive manner.

Each of the first three authors is a mathematician and while none has any formal pedagogical training, all three conduct research in mathematics education. Many of the accounts suggested an *awareness of purpose* on the lecturers' behalf or a *theoretical underpinning to the pedagogy* used when teaching. Perhaps this is a consequence of their familiarity with the research literature. However, the fact that the instructors often contrasted student learning in the lectures for which accounts were written with that of other cohorts lends some support to the hypothesis of McAlpine and Weston (2000) that a teaching mathematician can construct knowledge of learners and pedagogy through experience and reflection.

In several accounts the three lecturers highlighted what they noticed about student thinking on a given task and reflected on this after the lecture. These reflections could be said to *inform* their knowledge about mathematics pedagogy, particularly their knowledge of content and students and knowledge of content and teaching (Ball et al., 2008), which is a component of the Foundation Dimension. Although we chose not to code these reflections since they occurred after lectures, they point to a growth of knowledge as a consequence of reflection on teaching. It seems that use of the KQ as a reflection tool could afford mathematicians (with no formal pedagogical training) an opportunity to develop pedagogical knowledge. It is interesting to compare this with Turner and Rowland's (2011) finding that the KQ afforded preservice primary teachers (typically non-specialists in mathematics) an opportunity to develop mathematical content knowledge, illustrating the usefulness of KQ to mathematics teachers of a variety of backgrounds.

Some KQ codes did not appear in our analysis. For example there were no accounts coded as displaying behaviour such as *adherence to a textbook* or *concentration on procedures*. This may be because of the nature of the mathematics taught in the given modules. We also found very few references to *use of mathematical terminology* and *overt display of subject knowledge*, which is not to say that the lecturers did not use terminology or show their subject knowledge during classes but that they did not talk about it in their accounts (perhaps because it was normal and not problematic). We used the code *identifying student errors* sparingly, even though many accounts contained instances of a lecturer noticing a problem with student understanding. In our accounts the lecturers seemed to focus more on how to respond to a student rather than being able to tell when a piece of mathematics was wrong, and so we coded these episodes using the *responding to student ideas* code. We also used this code when the lecturer was faced with a lack of student ideas, for instance when she asked a question but received no replies. It may be that this is a situation that occurs more frequently in university than in school, where the size of classes can result in unwillingness to take part in discussions. We found that the type of specialist knowledge required to teach abstract mathematics at university was accounted for in the KQ with many of the codes mentioned earlier as well as others such as *choice of representation, recognition of conceptual appropriateness* and *making connections between concepts or representations*.

Even though there are some differences in the prevalence of codes at school and university level, we believe that the KQ offers a useful lens with which to study undergraduate teaching. It has drawn our attention to the importance of different facets of lecturers' mathematical knowledge which we may otherwise have overlooked. It would be interesting to explore the relationships between the four dimensions of the Quartet, for example how the underpinning dimension of foundation knowledge influences the lecturers' choices made in the other three dimensions, and how it is in turn influenced by knowledge generated by the lecturer in a contingent moment. We used the KQ to code reflective accounts written by mathematics lecturers as they reflected on their teaching. However, we suggest it could also be used to guide the reflective process and the writing of the accounts. It would be interesting to explore whether such an approach would lead to a change in the lecturers' perspectives on teaching similar to those described by Turner and Rowland (2011).

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