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QUADRATIC PRINCIPAL INDECOMPOSABLE MODULES AND STRONGLY REAL ELEMENTS OF FINITE GROUPS

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Abstract. Let P be a principal indecomposable module of a finite group G in characteristic 2 and let φ be the Brauer character of the corresponding simple G-module. We show that P affords a non-degenerate G-invariant quadratic form if and only if there are involutions $s, t \in G$ such that st has odd order and $\varphi(st)/2$ is not an algebraic integer.

We then show that the number of isomorphism classes of quadratic principal indecomposable G-modules is equal to the number of strongly real conjugacy classes of odd order elements of G.

1. Introduction

Let G be a finite group and let p be a prime. Let $\mathbb{A}$ be the ring of algebraic integers in $\mathbb{C}$ and let $\mathfrak{M}$ be a maximal ideal of $\mathbb{A}$ containing p. Set R as the localisation of $\mathbb{A}$ at $\mathfrak{M}$. So R is a complete discrete valuation ring with unique maximal ideal $J = R\mathfrak{M}$, $F := \mathrm{Frac}(R)$ is the field of algebraic numbers and $k := R/J$ is the algebraic closure of $\mathrm{GF}(p)$. Then (F, R, k) is a so-called p-modular system for G. Note that F and k are splitting fields for all subgroups of G. We say that an RG-module has quadratic type if it affords a nondegenerate G-invariant quadratic form.

Let P be a principal indecomposable RG-module. The principal indecomposable character of P is the ordinary character of the FG-module $P \otimes_R F$. Now P has a unique maximal submodule $\mathrm{Soc}(P)$ containing $J(R)G$. The irreducible Brauer character corresponding to P is the Brauer character of the kG-module $P/\mathrm{Soc}(P)$.

Recall that $g \in G$ is p-regular if its order is prime to p and p-singular if its order is divisible by p. Also an element of G is real if it is conjugate to its inverse, strongly real if it is inverted by an involution and weakly real if it is real but not strongly real. In particular the elements which square to the identity are strongly real.

Our main result is to give an efficient way of determining in characteristic 2 if a self-dual principal indecomposable module has quadratic type:

Theorem 1. *Let $p = 2$ and let P be a self-dual principal indecomposable RG-module with principal indecomposable character Φ and corresponding irreducible Brauer character φ. Then the following are equivalent:*

(i) *P has quadratic type.*
(ii) *$\varphi(g) \notin 2R$, for some strongly real 2-regular $g \in G$.*

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(iii) $\frac{\Phi(g)}{|C_G(g)|} \in 2R$, *for all weakly real* 2*-regular* $g \in G$.

As a consequence, we obtain:

Corollary 2. *If* $p = 2$ *then the number of quadratic type principal indecomposable* RG*-modules equals the number of strongly real* 2*-regular conjugacy classes of* G.

Equivalently the number of non-quadratic type self-dual principal indecomposable RG-modules equals the number of weakly real 2-regular conjugacy classes of G.

For p odd, each principal indecomposable RG-module affords a non-degenerate G-invariant quadratic form or symplectic bilinear form. We do not know how to determine the number of each using the p-regular conjugacy classes of G.

2. The type of principal indecomposable modules

Prior work has classified the type of principal indecomposable kG-modules using character theory or ring theory, as we outline in this section. Let (F, R, k) be as in the introduction. Then the group of units $\mathrm{U}(R)$ of R consists of all p'-roots of unity in $\mathbb{C}$. So the projection $R \to k$ induces an isomorphism $\mathrm{U}(R) \cong k^\times$ of multiplicative groups (which of course depends on the choice $\mathfrak{M}$ of maximal ideal).

Let M be an RG-module (kG-module) which is finitely generated and free as R-module. We say that M has quadratic or symplectic type if M affords a non-degenerate G-invariant R-valued (k-valued) quadratic form or symplectic bilinear form, respectively. It is known that when $p \neq 2$ each indecomposable kG-module is either of quadratic type or of symplectic type. When $p = 2$, each quadratic type kG-module is of symplectic type, but not conversely.

The ring multiplication in RG makes it into a module over itself, called the regular RG-module. The direct summands of this module are called principal indecomposable RG-modules. We say that a principal indecomposable RG-module P is trivial if $P/\operatorname{Rad}(P)$ is the trivial kG-module. As was shown by R. Brauer, the number of principal indecomposable RG-modules equals the number of p-regular (elements of order prime to p) conjugacy classes of G and the number of self-dual principal indecomposable RG-modules equals the number of real p-regular conjugacy classes of G.

Each self-dual irreducible FG-module is orthogonal or symplectic as it affords a non-degenerate G-invariant F-valued symmetric bilinear form or symplectic bilinear form, respectively. We say that the corresponding irreducible character of G has orthogonal or symplectic type, respectively. The type can be detected by computing the Frobenius-Schur (F-S) indicator of the character; symmetric type irreducible characters have F-S indicator $+1$ and symplectic type irreducible characters have F-S indicator -1.

Suppose that P is a self-dual principal indecomposable RG-module and let Φ be the principal indecomposable character of P. Then Φ is real-valued. Suppose first that p is odd. Then by [Wi76] and [Th84] some real-valued irreducible character of G occurs with odd multiplicity in Φ. Moreover P has quadratic or symplectic type, as this character has orthogonal or symplectic type, respectively.

Suppose then that $p = 2$. Then by [GW93], some orthogonal irreducible character of G occurs with odd multiplicity in Φ. As a consequence, P may have quadratic type but it cannot have symplectic type. If some symplectic irreducible character of G occurs with odd multiplicity in Φ, then P cannot have quadratic type. However P may be of non-quadratic type without the occurrence of such a character.

Reduction mod J induces a bijection between the principal indecomposable RG-modules and the principal indecomposable kG-modules. Under this bijection a principal indecomposable RG-module has quadratic type if and only if the corresponding principal indecomposable kG-module has quadratic type. So for convenience we can and do work with principal indecomposable kG-modules.

Write $1_G = e_1 + \cdots + e_m$ where the e_i are pairwise orthogonal primitive idempotents in kG (i.e. $e_ie_j = \delta_{ij}e_i$, for $1 \leq i, j \leq m$ and $m \geq 1$ is as large as possible). Then each kGe_i is a principal indecomposable kG-module, and all principal indecomposable kG-modules have this form. Now the map $g \to g^{-1}$, for $g \in G$, extends to an involutary k-algebra anti-automorphism ^{o} of kG, called the contragredient map. Each e_i^o is a primitive idempotent in kG and kGe_i^o is isomorphic to the dual module $(kGe_i)^* := \mathrm{Hom}(kGe_i, k)$ of kGe_i.

We must distinguish between p odd and $p = 2$:

Lemma 3 (Landrock-Manz)**.** *Let p be an odd prime and let e be a primitive idempotent in kG. Then kGe has quadratic type if and only if there is a primitive idempotent $f \in kG$, with $kGf = kGe$, such that $f = f^o$.*

In contrast, if $p = 2$, Gow and Willems showed that there is a primitive idempotent $f \in kG$, with $kGf = kGe$, such that $f = f^o$ if and only if kGe is the projective cover of the trivial kG-module. Moreover, they proved the following analogue of Lemma 3:

Lemma 4 (Gow-Willems)**.** *Let $p = 2$ and let e be a primitive idempotent in kG. Then kGe has quadratic type if and only if there is an involution $t \in G$ and a primitive idempotent $f \in kG$, with $kGf = kGe$, such that $t^{-1}ft = f^o$.*

Outline Proof. We may assume that kGe is not the projective cover of the trivial kG-module. Then it is known that each G-invariant symmetric bilinear form on kGe is symplectic and is the polarization of a G-invariant quadratic form on kGe (see [GW93, Proposition 2.2]).

Let B_1 be the symmetric bilinear form on kG with respect to which the elements of G form an orthonormal basis of kG. Then B_1 is non-degenerate and G-invariant. Its adjoint is the contragredient map. Next recall that $\mathrm{End}_{kG}(kG)$ can be identified with the opposite ring kG^{op}. Here $x \in kG^{op}$ defines the kG-homomorphism $y \to yx$, for all $y \in kG$. So if we define $B_x(y, z) := B_1(yx, z)$, for all $y, z \in kG$, then B_x is a G-invariant bilinear form on kG and $\{B_x \mid x \in kG\}$ give all G-invariant bilinear forms on kG. Moreover, B_x is non-degenerate, symmetric or symplectic as x is a unit in kG, $x = x^o$ or $x = x^o$ and $x_1 = 0$, respectively.

Next $(kGe)^* \cong kGe^o$. So the space of G-invariant bilinear forms on kGe can be identified with $ekGe^o$. As a consequence, each G-invariant bilinear form on kGe is the restriction from kG to kGe of a G-invariant bilinear form B_{exe^o}, for some unique $exe^o \in ekGe^o$. Now suppose that B_{exe^o} is a non-degenerate symplectic bilinear form on kGe. Write $exe^o = \sum_{g\in G} x_g g$. Then $x_1 = 0_k$ and $x_g = x_{g^{-1}}$, for all $g \in G$. As kGe is indecomposable, $B_{x_g(g+g^{-1})}$ is degenerate on kGe. So there is an involution $t \in G$ such that $B_{x_t t}$ restricts to a non-degenerate (symplectic) bilinear form on kGe. Thus $x_t \neq 0_k$ and B_t restricts to a non-degenerate bilinear form on kGe. Let $f \in kG^{op}$ be the projection onto kGe, with kernel the complement of kGe with respect to B_t. Now B_t has adjoint $x \mapsto tx^o t$ on kG^{op}. So f is a primitive idempotent in kG with $kGe = kGf$ such that $tf^o t = f$.

Conversely, given an involution $t \in G$ and an idempotent $f \in kG$ as in the statement, it is easy to see that B_t restricts to a non-degenerate symplectic bilinear form on kGe. □

If kGe has quadratic type and t is as in the conclusion of the Lemma, we say that t inverts e or kGe, or the projective indecomposable character of kGe.

Fong's Lemma states that when $\mathrm{char}(k) = 2$ every non-trivial self-dual irreducible kG-module affords a non-degenerate G-invariant symplectic form. It is clear that this form is unique up to a non-zero scalar. In [M16, Proposition 5.8] the second author showed:

Lemma 5. *Suppose that $p = 2$ and that M is a non-trivial self-dual irreducible kG-module, with Fong form B. Then the projective cover of M has quadratic type if and only if $B(tm, m) \neq 0$ for some $m \in M$ and involution $t \in G$.*

Outline Proof. The annihilator of M in kG is a maximal 2-sided ideal $\mathrm{Ann}(M)$ of kG with $kG/\mathrm{Ann}(M) \cong \mathrm{End}_k(M)$. Suppose first that the principal indecomposable module of M is orthogonal. Then by Lemma 4 there is an involution $t \in G$ and a primitive idempotent $f \in kG$, such that $kGf/\mathrm{Rad}(kGf) \cong M$ and $t^{-1}ft = f^o$. Set $x^{ot} := tx^o t$, for all $x \in kG$. Then ^{ot} is an involutary k-algebra anti-automorphism of kG. Now $\mathrm{Ann}(M)$ is invariant under ^{ot}. So ^{ot} induces an involutary k-algebra anti-automorphism on $\mathrm{End}_k(M)$. It is readily established that $g^{ot} = g^{-t}$, for all $g \in G$ in $\mathrm{End}_k(M)$.

Next define $B_t(x, y) := B(tx, y)$, for all $x, y \in \mathrm{End}_k(M)$. Then B_t is a non-degenerate $\mathrm{C}_G(t)$-invariant symmetric bilinear form on M. Its adjoint coincides with ^{ot} on $\mathrm{End}_k(M)$, as by irreducibility of M, the image of G spans $\mathrm{End}_k(M)$. Now $f + \mathrm{Ann}(M)$ is a primitive idempotent in $\mathrm{End}_k(M)$ which is ^{ot}-invariant. It follows that B_t is non-degenerate on the 1-dimensional subspace fM of M i.e. there is $m \in fM$ such that $B(tm, m) \neq 0_k$.

Conversely, suppose that $B(tm, m) \neq 0$ for some involution $t \in G$ and some $m \in M$. Let $\hat{f}$ be orthogonal projection onto km with respect to the non-degenerate symmetric form B_t on M. By idempotent lifting (c.f. [LM, Proposition 1.4]) there is a primitive idempotent $f \in kG$ such that $f^{ot} = f$ and $f + \mathrm{Ann}(M) = \hat{f}$. Now $kGf/\mathrm{Rad}(kGf) \cong M$. So kGf is orthogonal, according to Lemma 4. □

3. Type of principal indecomposable modules of $\mathbb{R}$-elementary subgroups

Recall that a group is elementary if it has the form $C \times P$, where P is a p-group, for some prime p and C is a cyclic p'-group. Brauer's induction theorem states that every

$\mathbb{C}$-character of G is an integer combination of characters induced from linear characters of elementary subgroups of G. For the rest of this section $p = 2$.

Now an $\mathbb{R}$-elementary group has the form $C \rtimes P$, where C is cyclic of odd order and P is a 2-group such that every element of P centralizes or inverts C. The real version of Brauer's induction theorem (which is a special case of the Witt-Berman theorem) is that every $\mathbb{R}$-character of G is an integer combination of characters induced from $\mathbb{R}$-characters of $\mathbb{R}$-elementary subgroups of G. In this section we determine the type of the principal indecomposable modules of $\mathbb{R}$-elementary groups.

As usual the centralizer $\mathrm{C}_G(g)$ or extended centralizer $\mathrm{C}^*_G(g)$ of $g \in G$ is the normalizer of $\{g\}$ or $\{g, g^{-1}\}$ in G, respectively. Then $\mathrm{C}_G(g) \subseteq \mathrm{C}^*_G(g)$ and $|\mathrm{C}^*_G(g) : \mathrm{C}_G(g)| \leq 2$. Moreover g is strongly real if $g = g^{-1}$ or $\mathrm{C}^*_G(g) \setminus \mathrm{C}_G(g)$ contains an involution and weakly real if $\mathrm{C}^*_G(g)$ does not split over $\mathrm{C}_G(g)$.

Our main result here is a special case of a result on the principal indecomposable modules of solvable groups which is due to G. Navarro and the second author [MN16, Theorem 15]:

Proposition 6. *Let $g \in G$ and let $E \in \mathrm{Syl}_2(\mathrm{C}^*_G(g))$. Then a non-trivial principal indecomposable $R(\langle g\rangle \rtimes E)$-module has quadratic type if and only if g is strongly real.*

Proof. We may assume that g is real in G. Set $C := \langle g\rangle$ and $H := C \rtimes E$. Let P be a non-trivial principal indecomposable RH-module, and let Φ be the principal indecomposable character of P. Then $P \cong M{\uparrow}^H$, for some non-trivial 1-dimensional kC-module M.

Suppose first that g is strongly real. Let t be an involution in E which inverts g. Then $M{\uparrow}^{\langle g,t\rangle}$ is a self-dual irreducible $k(\langle g, t\rangle)$-module. Fong's Lemma implies that $M{\uparrow}^{\langle g,t\rangle}$ affords a symplectic geometry. Then the induced module $P \cong (M{\uparrow}^{\langle g,t\rangle}){\uparrow}^H$ affords the induced form. So P has symplectic, hence quadratic type.

Conversely suppose that P has quadratic type. Set $D = \mathrm{C}_E(g)$ and $\hat{M} := (\mathrm{Inf}^{C\times D}_{C\times D/D} M){\uparrow}^H$. Then $\hat{M}$ is a self-dual irreducible kH-module which is isomorphic to $P/\mathrm{Rad}(P)$. Let B be a Fong form on $\hat{M}$. According to Lemma 5, $B(tm, m) \neq 0_k$, for some involution $t \in H$ and some $m \in \hat{M}$. Now $\hat{M}{\downarrow}_{C\times D} = M_1 \oplus M_2$, where M_1 is an irreducible $k(C \times D)$-module and $M_2 \cong M_1^* \ncong M_1$. Write $m = m_1 + m_2$, where $m_1 \in M_1$ and $m_2 \in M_2$. Then

$$\begin{aligned} B(tm, m) &= B(tm_1, m_1) + B(tm_1, m_2) + B(tm_2, m_1) + B(tm_2, m_2) \\ &= B(tm_1, m_1) + B(tm_2, m_2), \quad \text{as } B(tm_1, m_2) = B(tm_2, m_1). \end{aligned}$$

So we may assume without loss of generality that $B(tm_1, m_1) \neq 0_k$. As $M_1 \ncong M_1^*$, B is identically zero on M_1. So $tm_1 \notin M_1$, which forces $t \in H \setminus (C \times D)$. Then $g^t = g^{-1}$. So g is strongly real. □

4. Values of principal indecomposable and Brauer characters

In this section $p = 2$. We use Proposition 6 to clarify the relationship between the strongly and weakly real 2-regular conjugacy classes of G and the quadratic and non-quadratic self-dual principal indecomposable RG-modules.

Each kG-module M has a Brauer character φ_M; if $g \in G$ has odd order then M has a basis of eigenvectors of g. Each eigenvalue of g on M is a $2'$-roots of unity in k. So they can be lifted to $2'$-roots of unity in R, via the isomorphism $\mathrm{U}(R) \cong k^\times$. Then $\varphi_M(g)$ is defined to be the sum of these roots of unity. In particular φ_M is an $\mathbb{A}$-valued function defined on the 2-regular elements of G. As is standard, we use $\mathrm{IBr}(G)$ to denote the Brauer characters of the irreducible kG-modules.

The restriction χ^* of an ordinary character χ of G to the p-regular elements of G is a Brauer character. So $\chi^* = \sum_{\varphi\in\mathrm{IBr}(G)} d_{\chi\varphi}\varphi$, where the $d_{\chi\varphi}$ are non-negative integers, called decomposition numbers. For $\varphi \in \mathrm{IBr}(G)$ set $\Phi_\varphi := \sum_{\chi\in\mathrm{Irr}(G)} d_{\chi\varphi}\chi$. Then Φ_φ is the principal indecomposable character of G corresponding to φ. It is easy to see that Φ_φ vanishes on all p-singular elements of G.

Recall that a vertex of an indecomposable kG-module M is a subgroup V of G which is minimal subject to M being a direct summand of a module induced from V to G. J. A. Green showed that V is a p-subgroup of G, and moreover V is uniquely determined up to G-conjugacy. The next result and its corollary are due to Green.

Lemma 7. *Let M be an indecomposable kG-module, let $g \in G$ be p-regular and let $D \in \mathrm{Syl}_p(\mathrm{C}_G(g))$. Then M has a vertex V such that $\frac{\varphi_M(g)}{|D:V\cap D|}$ is an algebraic integer.*

In particular if $\varphi_M(g) \notin 2\mathbb{A}$ then D is contained in some vertex of M.

Corollary 8. *Let Φ be a principal indecomposable character of G and let $g \in G$. Then $\frac{\Phi(g)}{|\mathrm{C}_G(g)|_p}$ is an algebraic integer.*

Proof. As Φ is zero on p-singular elements, we may assume that g is p-regular. Now Φ^* is the Brauer character of a principal indecomposable kG-module and the trivial group is a vertex of this module. So $|\mathrm{C}_G(g)|_p$ divides $\Phi(g)$ in $\mathbb{A}$. □

Let Φ_1 be the trivial principal indecomposable character of G i.e. Φ_1 corresponds to the trivial 2-Brauer character φ_1 of G. In [GW93] Gow and Willems proved that $\frac{\Phi_1(1)}{|G|_2}$ is odd. We complement this result with:

Theorem 9. *If $g \in G$ is real and non-trivial then $\frac{\Phi_1(g)}{|\mathrm{C}_G(g)|_2}$ is twice an algebraic integer.*

Proof. By the second orthogonality relation $0 = \sum\limits_{\chi\in\mathrm{Irr}(G)} \chi(1)\chi(g) = \sum\limits_{\varphi\in\mathrm{IBr}(G)} \varphi(1)\Phi_\varphi(g)$. So

$$\frac{\Phi_1(g)}{|\mathrm{C}_G(g)|_2} = -\sum_{\substack{\varphi\in\mathrm{IBr}(G)\\ \varphi\neq\varphi_1}} \varphi(1)\frac{\Phi_\varphi(g)}{|\mathrm{C}_G(g)|_2}.$$

For $\varphi \neq \overline{\varphi}$ we get two equal summands. So their sum cancels mod 2. If $\varphi = \overline{\varphi}$ and $\varphi \neq \varphi_1$, Fong's Lemma implies that $\varphi(1)$ is even. The conclusion follows from these facts. □

Lemma 10. *Let g be a weakly real element of G and let Φ be a quadratic type principal indecomposable character of G. Then $\frac{\Phi(g)}{|\mathrm{C}_G(g)|_2} \in 2\mathbb{A}$.*

Proof. We may assume that g is 2-regular. Suppose for the sake of contradiction that $\frac{\Phi(g)}{|\mathrm{C}_G(g)|_2} \notin 2\mathbb{A}$. Let $E \in \mathrm{Syl}_2(\mathrm{C}_G^*(g))$ and set $H := \langle g\rangle \rtimes E$. Write $\Phi{\downarrow_H} = \sum \alpha_\Lambda \Lambda$, where Λ ranges over the principal indecomposable characters of H. Lemma 9 implies that there is a non-trivial principal indecomposable character Λ such that α_Λ is odd and $\frac{\Lambda(g)}{|\mathrm{C}_E(g)|} \notin 2\mathbb{A}$.

Let P be the projective indecomposable kG-module corresponding to Φ and let Q be the principal indecomposable kH-module corresponding to Λ. Then Q is a self-dual module which occurs with odd multiplicity α_Λ as a direct summand of $P{\downarrow_H}$. As P is orthogonal, so too is Q. So g is strongly real, according to Proposition 6. This contradiction completes the proof. □

We can refine this result, using the techniques developed in [M16]:

Lemma 11. *Let g be a real element of G and let Φ be a quadratic principal indecomposable character of G such that $\frac{\Phi(g)}{|\mathrm{C}_G(g)|_2} \notin 2\mathbb{A}$. Suppose that $t \in G$ is an involution which inverts Φ (c.f. the comment after Lemma 4). Then some conjugate of t inverts g.*

Sketch Proof. By Lemma 4 there is a primitive idempotent f in kG such that kGf is the principal indecomposable kG-module corresponding to Φ and B_t restricts to a non-degenerate symplectic form on kGf. Let $H = \langle g\rangle \rtimes E$ and Q be as in Lemma 10; so Q is a projective indecomposable kH-module which occurs with odd multiplicity in $kGf{\downarrow_H}$. Then Q is a non-degenerate component of the symplectic module $(kGf, B_t){\downarrow_H}$. Now $(kG, B_t) = (k\langle t\rangle, B_t){\uparrow^G}$. So Q is a non-degenerate component of

$$(1) \qquad (k\langle t\rangle, B_t){\uparrow^G}{\downarrow_H} = \perp_{\langle t\rangle aH\subseteq G} (k\langle t^a\rangle, B_{t^a}){\downarrow_{\langle t^a\rangle\cap H}}{\uparrow^H}.$$

Let $a \in G$. First suppose that $\langle t^a\rangle \subseteq H$. Then the right hand side has a summand

$$(2) \qquad (k\langle t^a\rangle, B_{t^a}){\uparrow^H} \cong (kH, B_{t^a}).$$

The other possibility is that $t^a \notin H$. Then $\langle t^a\rangle \cap H = 1$ and $(k\langle t^a\rangle, B_{t^a}){\downarrow_1} \cong (k^2, \hat{B}_1)$ is a symplectic plane. Let $1 = e_1 + \cdots + e_u$ be a decomposition of 1 as a sum of pairwise orthogonal primitive idempotents in kH. For each i, the bilinear form B_1 on kG defines a perfect G-pairing $kGe_i \times kGe_i^o \to k$. We set $(kGe_i \oplus kGe_i^o, \hat{B}_1)$ as the corresponding symplectic *paired* module. It is important to note that this module has no proper non-degenerate component. Then the summand on the right hand side of (1) has the orthogonal decomposition

$$(3) \qquad (k^2, \hat{B}_1){\uparrow^H} = \perp_{i=1}^{u} (kHe_i \oplus kHe_i^o, \hat{B}_1).$$

Now Q is an indecomposable non-degenerate submodule of an orthogonal sum of modules of the form (2) and (3). As the modules in (3) have no such submodules, we deduce that there is $a \in G$ such that $s := t^a \in H$ and Q is a non-degenerate submodule of (kH, B_s). Equivalently there is a primitive idempotent $e \in kH^{op}$ such that $kHe \cong Q$ and $e^{os} = e$. Now $k(C \times D)e$ is a projective indecomposable $k(C \times D)$-module which is not self-dual (as it is not the projective cover of the trivial $k(C \times D)$-module). So $s \notin (C \times D)$. We conclude that $g^s = g^{-1}$. □

Complementing Lemma 10, we have:

Lemma 12. *Let g be a strongly real 2-regular element of G and let P be a non-quadratic self-dual principal indecomposable RG-module with associated $\varphi \in \mathrm{IBr}(G)$. Then $\varphi(g) \in 2\mathbb{A}$.*

Proof. We note that $\varphi(1)$ is even by Fong's Lemma. So we may assume that $g \neq 1$. As before let $E \in \mathrm{Syl}_2(\mathrm{C}^*_G(g))$ and set $H := \langle g \rangle \rtimes E$.

Suppose for the sake of contradiction that $\varphi(g) \notin 2\mathbb{A}$. Write $\varphi{\downarrow_H} = \sum \beta_\lambda \lambda$, where λ ranges over the irreducible Brauer characters of H. So by hypothesis there is $\lambda \in \mathrm{IBr}(H)$ such that β_λ is odd and $\lambda(g) \notin 2\mathbb{A}$.

Let Q be the projective indecomposable kH-module corresponding to λ. As g is strongly real, it follows from Proposition 6 that Q has quadratic type. Now by Frobenius-Nakayama reciprocity P occurs with odd multiplicity β_λ in $Q{\uparrow^G}$. As $P \cong P^*$, we conclude that P has quadratic type. This contradiction completes the proof. □

Let M be the irreducible kG-module whose Brauer character is φ. In [M16] the second author assigned a symplectic vertex T to M; T is a minimal subgroup of G such that M is an orthogonal direct summand of a symplectic kT-module induced up to G. Just as with Green vertices, T is uniquely determined up to G-conjugacy. In view of Lemma 11 we hazard the following, which would complement Lemma 11 and strengthen Lemma 12:

Conjecture 13. *Let M be the irreducible kG-module, let φ be the Brauer character of M and let g be a real 2-regular element of G such that $\varphi(g) \notin 2\mathbb{A}$. Then M has an extended defect group which contains an extended defect group of g but which is not contained in $\mathrm{C}_G(g)$.*

5. Type of principal indecomposable RG-modules

Recall that R is the localisation of the ring of algebraic integers at a maximal ideal containing 2. In particular R has a unique maximal ideal J. Suppose that G has ℓ 2-regular conjugacy classes, r real 2-regular conjugacy classes and s strongly real 2-regular conjugacy classes. So G has ℓ irreducible 2-Brauer characters and r real irreducible 2-Brauer characters. Suppose also that G has σ quadratic type principal indecomposable RG-modules.

List the irreducible 2-Brauer characters of G as $\varphi_1, \dots, \varphi_\ell$ and let $g_1, \dots, g_\ell$ be a set of representatives for the 2-regular conjugacy classes of G. Set Φ_k as the principal indecomposable character of G corresponding to φ_k, for $k = 1, \dots, \ell$.

The second orthogonality relation (see Theorem 9) give equations in R:

$$\sum_{k=1}^{\ell} \frac{\Phi_k(g_i^{-1})}{|\,\mathrm{C}_G(g_i)|} \varphi_k(g_j) = \delta_{ij}, \quad \text{for } 1 \leq i, j \leq \ell.$$

Suppose that g_i and g_j are real in G. Then $\frac{\Phi_k(g_i^{-1})}{|\,\mathrm{C}_G(g_i)|}\varphi_k(g_j) = \frac{\overline{\Phi}_k(g_i^{-1})}{|\,\mathrm{C}_G(g_i)|}\overline{\varphi}_k(g_j)$. So the contribution of the non-real φ_k to the above displayed sum is zero, modulo J.

We may choose our notation so that $\varphi_1, \ldots, \varphi_r$ are the real irreducible 2-Brauer characters of G and $g_1, \ldots, g_r$ are in the real 2-regular conjugacy classes of G. Define the $r \times r$-matrices:

$$A = \left[\frac{\Phi_j(g_i^{-1})}{|\,\mathrm{C}_G(g_i)|}\right], \quad B = [\varphi_i(g_j)]\,.$$

They involve only the real principal indecomposable characters, the real irreducible Brauer characters and the real 2-regular elements of G. By the work above $AB \equiv I \pmod{J}$.

We further refine our notation so that $\varphi_1, \ldots, \varphi_\sigma$ are the Brauer characters of the quadratic principal indecomposable RG-modules and $g_1, \ldots, g_s$ are in the strongly real 2-regular conjugacy classes of G. Thus $\varphi_{\sigma+1}, \ldots, \varphi_r$ are the Brauer characters of the non-quadratic principal indecomposable RG-modules and $g_{s+1}, \ldots, g_r$ are in the weakly real 2-regular conjugacy classes of G. The matrices A and B have corresponding block forms:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.$$

So A_{11} is the $s \times \sigma$ submatrix of A with rows and columns indexed by the strongly real classes and quadratic type principal indecomposable characters. Likewise B_{11} is the $\sigma \times s$ submatrix of B with rows and columns indexed by Brauer characters of quadratic principal indecomposable RG-modules and strongly real classes, respectively.

Now each term in A_{21} has the form $\frac{\Phi_k(g_i^{-1})}{|\,\mathrm{C}_G(g_i)|}$ where g_i is weakly real and Φ_k has quadratic type. Likewise each term in B_{21} has the form $\varphi_k(g_j)$ where φ_k is the Brauer character of a non-quadratic principal indecomposable RG-module and g_j is strongly real. So according to Lemmas 10 and 12, all these terms belong to J. Thus

$$A_{21} \equiv 0 \pmod{J} \quad \text{and} \quad B_{21} \equiv 0 \pmod{J}.$$

It follows from this that $A_{11}B_{11} \equiv I \pmod{J}$ and $A_{22}B_{22} \equiv I \pmod{J}$. In particular A_{11} and A_{22} have full row rank (mod J). So $s \leq \sigma$ and $r - s \leq r - \sigma$. We conclude that $s = \sigma$. This proves our main theorem, which we restate here for the convenience of the reader:

Theorem 14. *The number of strongly real* 2*-regular conjugacy classes of* G *equals the number of quadratic type principal indecomposable* RG*-modules and the number of weakly real* 2*-regular conjugacy classes of* G *equals the number of non-quadratic type self-dual principal indecomposable* RG*-modules.*

The analysis above translates into the following criteria for strong and weak reality, which proves Theorem 1:

Proposition 15. *Let* $p = 2$ *and let* P *be a self-dual principal indecomposable* RG*-module with principal indecomposable character* Φ *and corresponding irreducible Brauer character* φ*. Then:*

(i) *P has quadratic type if and only if $\varphi(g) \notin 2R$, for some strongly real 2-regular $g \in G$.*
(ii) *P has non-quadratic type if and only if $\frac{\Phi(g)}{|C_G(g)|} \notin 2R$, for some weakly real $g \in G$.*

6. Strong and weak projective indecomposable modules

In this section we give examples of strong and weak principal indecomposable characters, many of which involve quasisimple finite groups. In addition to [Atlas], we use the notation and decomposition matrices provided by the Modular Atlas homepage [MOC].

Let $g \in G$ be 2-regular. If g is strongly real, $tgt = g^{-1}$, for some involution $t \in G$. Then $s := gt$ is an involution which is conjugate to t in the dihedral group $\langle g, t\rangle$ and $g = st$. Conversely, if $g = uv$ where $u, v \in G$ are involutions, then v inverts g and so g is strongly real. So by the class multiplication formula, g is strongly real if and only if

$$\sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(t)^2 \chi(g)}{\chi(1)} \neq 0,$$

for some involution $t \in G$. We note that the character table of G determines the prime divisors of the orders of the elements of G. In particular the character table determines the 2-regular conjugacy classes of G. However the character table of a group does not generally determine which conjugacy classes of 2-elements are involutions e.g. D_8 and Q_8 have the 'same' character tables.

Example 16. $G = 2.A_5$. *All three non-trivial 2-regular classes of G are weakly real, for example because G has a unique involution. So all three non-trivial 2-principal indecomposable characters Φ_2, Φ_3 and Φ_4 of $2.A_5$ are of non-quadratic type. This also follows from the fact that Φ_2 contains the symplectic irreducible characters χ_6 and χ_9 with odd multiplicity 1, Φ_3 contains the symplectic irreducible characters χ_7 and χ_9 with odd multiplicity 1 and Φ_4 contains the symplectic irreducible character χ_8 with odd multiplicity 1.*

Example 17. $G = \mathrm{Sp}(4,5)$, *or* $2.\mathrm{S}_4(5)$ *in* [Atlas] *notation. Then G has 60 irreducible $\mathbb{C}$-characters, 26 of which are faithful. All non-faithful characters are orthogonal and all faithful characters are symplectic. Now G has fifteen 2-regular classes, all of whom are real. By examination of the 2-decomposition matrix, each of the 2-principal indecomposable characters Φ_i, for $i \in \{2, 3, 4, 6, 8, 9, 10, 11, 12, 13, 14\}$ contain at least one symplectic irreducible constituent with odd multiplicity. So these twelve principal indecomposable characters are non-quadratic. By computing the square of the involution class $2B$, we see that $1A$, $3A$ and $5D$ are the strongly real 2-regular classes of G. Now $\varphi_5 = \chi_6^* - \chi_1^*$ and $\chi_6(5D) = 0$. So $\varphi_5(5D) = -1$ is odd. Similarly $\varphi_7 = \chi_{11}^*$ and $\chi_{11}(3A) = 5$ is odd. So Φ_1, Φ_5 and Φ_7 are the quadratic principal indecomposable characters of G.*

Example 18. $G = McL$ *has 24 irreducible $\mathbb{C}$-characters, of whom two, χ_{11} and χ_{13}, are symplectic and 10 are orthogonal. Also G has 11 2-regular conjugacy classes, 5 of whom are real. Examination of the square of the involution class $2A$ shows that $1A, 3B$ and $5B$ are the strongly real 2-regular classes of G. So the real 2-regular classes $3A$ and $5A$ are* *weakly real. Now* $\varphi_3 = \chi_3^*$ *and* $\chi_3(5B) = 1$ *is odd. Also* φ_{11} *is the unique real irreducible Brauer character which occurs with odd multiplicity in* χ_{15}^* *and* $\chi_{15}(3B) = 9$ *is odd. So* Φ_1, Φ_3 *and* Φ_{11} *are the quadratic indecomposable characters of* G.

Next φ_2 *and* φ_{10} *are the remaining real irreducible Brauer characters of* G. *The symplectic character* χ_{11} *occurs with odd multiplicity* 1 *in* Φ_{10}. *This confirms that* Φ_{10} *is a non-quadratic principal indecomposable character. There is a unique symplectic irreducible constituent of* Φ_2, *namely* χ_{13}, *and this occurs with even multiplicity* 2. *So we cannot use its presence to verify that* Φ_2 *is not quadratic. On the other hand* $\phi_2 = \chi_2^*$ *and the values* $\chi_2(1) = 22, \chi_2(3B) = 4$ *and* $\chi_2(5B) = 2$ *on the strongly real* 2*-regular classes are all even. This confirms that* Φ_2 *is not quadratic.*

Example 19. $G = 2.Ru$ *has* 61 *irreducible* $\mathbb{C}$*-characters,* 25 *of which are faithful. Of the faithful characters,* 7 *are symplectic and the remaining* 18 *are non-real. Of the* 36 *characters of* Ru, *two are non-real and the remaining* 34 *are orthogonal. Also* G *has* 9 *classes of elements of odd order, all of whom are real. By examining the square of the two classes of non-central involutions, we see that the weakly real* 2*-regular classes are* $15A, 29A$ *and* $29B$.

Now the symplectic character χ_{53} *has odd multiplicity* 1 *in* Φ_4. *So* Φ_4 *is not quadratic. Alternatively,* φ_4 *is the unique Brauer character which occurs with odd multiplicity in* χ_{53}^* *and* $\chi_{53}(15A) = 1$. *Next* $\varphi_6 = \chi_{49}^*$ *and* $\chi_{49}(15A) = -1$ *is odd. Similarly* $\varphi_7 = \chi_{50}^*$ *and* $\chi_{50}(15A) = -1$. *This confirms that* Φ_6 *and* Φ_7 *are not quadratic.*

We take the opportunity to correct Example 2.12 in [GW93]. This erroneously claims that a certain principal indecomposable module of a group of order 288 is weakly real. In fact the assertion, on the first line of p268, that a certain form c is B-invariant, is false.

Example 20. *Let* H *be the non-abelian group* $C_3 \rtimes C_4$ *of order* 12. *Let* σ *be the switching automorphism of* $H \times H$: $(x, y)^\sigma = (y, x)$, *for all* $x, y \in H$. *Set* $G = (H \times H)\langle\sigma\rangle$. *Now* G *is a* 2*-nilpotent group with normal Hall* $2'$*-subgroup* $N \cong C_3 \times C_3$. *Let* ω *be one of the two non-trivial linear characters of* C_3. *Then* G *has three orbits on* $\mathrm{Irr}(N)$, *with representatives* $1 \times 1, 1 \times \omega$ *and* $\omega \times \omega$, *respectively. Set* $\Phi_1 = (1 \times 1){\uparrow}^G$, $\Phi_2 = (1 \times \omega){\uparrow}^G$ *and* $\Phi_3 = (\omega \times \omega){\uparrow}^G$. *Let* Φ_i *belong to the* 2*-block* B_i *of* G. *Then* B_1, B_2 *and* B_3 *are distinct, real and nilpotent.*

The principal 2*-block* B_1 *consists of the* 8 *linear characters and* 6 *irreducible characters of degree* 2 *in* $\mathrm{Irr}(G/N)$. *Then* Φ_1 *is strongly real, as it is the trivial principal indecomposable character of* G.

The block B_2 *consists of the* 8 *irreducible characters in* $\mathrm{Irr}(G \mid 1 \times \omega)$, *each of which has degree* 4. *Four of these are orthogonal and four are symplectic. Now* $1 \times \omega$ *has stabilizer* $H \times C_6$ *and extended stablizer* $H \times H$ *in* G. *As* $H \times H$ *does not split over* $H \times C_6$, *it follows that* B_2 *is a weakly real* 2*-block of* G. *So* Φ_2 *is weakly real.*

Finally B_3 *consists of the* 5 *irreducible characters in* $\mathrm{Irr}(G \mid \omega \times \omega)$. *Four of these are orthogonal and of degree* 4. *The remaining character is symplectic and of degree* 8. *Now the stabilizer of* $\omega \times \omega$ *in* G *is* $C_6 \wr C_2$. *As* $(b^{-1}, b)\sigma$ *is an involution in* G *which inverts* $\omega \times \omega$, *the extended stablizer of* $\omega \times \omega$ *splits over its stabilizer. As a consequence*

$\mathrm{Irr}(G \mid \omega \times \omega)$ *is a strongly real* 2*-block of* G*. So* Φ_3 *is strongly real. This means that the corresponding principal indecomposable kG module has a quadratic geometry, contrary to the conclusion of* [GW93, 2.12].

We note that the involution module of B_3 *can be constructed as follows. We have* $\mathrm{Irr}(H \mid \omega) = \{\chi_1, \chi_{-1}\}$*, where* χ_ϵ *has degree* 2 *and F-S indicator* ϵ*. Clearly* $\mathrm{C}_G(\sigma) = \Delta H \times \langle \sigma \rangle$ *in* G*. So* $\mathbb{C}_{\mathrm{C}_G(\sigma)} \uparrow^G$ *is isomorphic to* $\mathbb{C}H$*, as a module for* $\mathbb{C}H \wr C_2$*. In particular the involution module of* B_3 *has ordinary character* $\hat{\chi}_1 + \hat{\chi}_{-1}$*, where* $\hat{\chi}_\epsilon$ *is an extension of* $\chi_\epsilon \times \chi_\epsilon$ *from* $H \times H$ *to* G*. It is easy to check that both of these characters has F-S indicator* $+1$*.*

In view of the above examples, we venture the following:

Conjecture 21. $\epsilon(\Phi)$ *is even if* Φ *is a weakly real principal indecomposable character.*

7. Odd Cartan Invariants

We keep our notation φ_i, Φ_i and g_i for the irreducible 2-Brauer characters, the principal indecomposable characters and the elements of the 2-regular conjugacy classes of G, respectively. List the irreducible characters of G as $\chi_1, \ldots, \chi_k$. Then

$$\chi_i^* = \sum_{j=1}^{\ell} d_{ij}\varphi_j, \quad \Phi_i = \sum_{j=1}^{k} d_{ij}\chi_j, \quad \Phi_i = \sum_{j=1}^{\ell} c_{ij}\varphi_j,$$

where the Cartan invariants are given by $c_{ij} = \sum_{u=1}^{k} d_{ui}d_{uj}$.

Recall that $g \in G$ is said to have 2-defect zero if $\mathrm{C}_G(g)$ has odd order. In particular g has odd order. Now suppose that g is real and of 2-defect zero. Then a Sylow 2-subgroup of $\mathrm{C}_G^*(g)$ has order 2. So g is inverted by an involution, whence g is strongly real in G.

Now the $\ell \times \ell$ Cartan matrix $[c_{ij}]$ of G is a symmetric integer matrix whose invariant factors are $|\,\mathrm{C}_G(g_1)|_2, \ldots, |\,\mathrm{C}_G(g_\ell)|_2$. In particular its rank modulo 2 coincides with the number of 2-regular conjugacy classes of G which have 2-defect zero. So we get one strongly real 2-Brauer character for each real class of 2-defect zero. Our final Theorem refines this observation:

Theorem 22. *Suppose that* c_{ij} *is odd, where* φ_i *and* φ_j *are real-valued. Then there exists* w *such that* c_{iw} *is odd and* φ_w *is strongly real.*

Proof. We compute

$$\begin{aligned}\sum_{u=1}^{r}\Phi_i(g_u)\frac{\Phi_j(g_u^{-1})}{|\,\mathrm{C}_G(g_u)|} &\equiv \sum_{u=1}^{r}\sum_{v=1}^{\ell} c_{iv}\varphi_v(g_u)\frac{\Phi_j(g_u^{-1})}{|\,\mathrm{C}_G(g_u)|}, \quad \text{definition of } c_{iv}\\ &\equiv \sum_{u=1}^{r}\sum_{v=1}^{r} c_{iv}\varphi_v(g_u)\frac{\Phi_j(g_u^{-1})}{|\,\mathrm{C}_G(g_u)|}, \quad \text{as } c_{iv}\varphi_v(g_u)=c_{i\overline{v}}\overline{\varphi}_v(g_u)\\ &\equiv \sum_{v=1}^{r} c_{iv}\sum_{u=1}^{\ell}\varphi_v(g_u)\frac{\Phi_j(g_u^{-1})}{|\,\mathrm{C}_G(g_u)|}, \quad \text{as } \varphi_v(g_u)\Phi_j(g_u^{-1})=\varphi_v(g_u^{-1})\Phi_j(g_u)\\ &\equiv c_{ij} \mod J, \quad \text{by the second orthogonality relation.}\end{aligned}$$

It follows that $\Phi_i(g_u)\frac{\Phi_j(g_u^{-1})}{|\,\mathrm{C}_G(g_u)|}\notin J$, for some real g_u. But $|\,\mathrm{C}_G(g_u)|_2 \mid \Phi_i(g_u)$. So g_u must have 2-defect zero, and hence g_u is strongly real. Now $\Phi_i(g_u)=\sum_{w=1}^{\ell}c_{iw}\varphi_w(g_u)\equiv\sum_{w=1}^{r}c_{iw}\varphi_w(g_u)$. So there exists w such that $\varphi_v=\overline{\varphi}_v$, c_{iw} is odd and $\varphi_w(g_u)$ is coprime to 2. Then φ_w is strongly real, according to Lemma 12. This completes the proof.

Remarks: Let t be an involution inverting a primitive idempotent corresponding to Φ_w. Then t inverts g_u, by Lemma 11. So t is uniquely determined up to conjugacy, as g_u has defect 0. Now $c_{iw}=\sum_{x=1}^{k}d_{ix}d_{wx}$. So there exists x such that $\chi_x=\overline{\chi}_x$ and d_{ix} and d_{wx} are odd. As φ_w is strongly real, χ_x is an orthogonal irreducible character of G. ☐

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