International Journal of Infrared and Millimeter Waves, Vol. 23, No. 5, May 2002 (© 2002)

"PHASE CENTERS" OF FAR INFRARED MULTI-MODED HORN ANTENNAS

E. Gleeson,[1] J. A. Murphy,[1] and B. Maffei[2]

[1]*National University of Ireland*
Maynooth, Co. Kildare, Ireland
[2]*University of Wales*
Cardiff, CF24 3YB
United Kingdom

Received February 27, 2002

Abstract

Multi-moded horn antennas can be used as high efficiency feeds for bolometers when diffraction limited resolution is not required. For example, such horns are proposed for the PLANCK Surveyor, a satellite telescope due to be launched in 2007 to make definite measurements of the Cosmic Microwave Background. In a previous paper we described an accurate approach involving electromagnetic modelling using a rigorous mode matching technique to obtain both the horn aperture fields and the corresponding far field radiation patterns. In this paper we extend this description to determine the "phase center" of such horns when used on large reflecting telescopes. The "phase center" is ill defined as the individual spatially coherent fields making up the far field pattern all appear to come from different phase centers. The best average phase center location is therefore redefined in terms of the virtual beam waist position behind the horn aperture at which the focus of the telescope should be located in order to optimise angular resolution and on-axis gain for the beam on the sky. A number of alternative techniques to locating the phase center are discussed in detail in this paper.

Key Words: Multi-moded Horn Antennas, Phase center, Virtual Waist, Modal analysis.

0195-9271/02/0500-0711/0

1. Introduction

For a single moded system the phase center of a horn antenna is the point along its axis which is the best fit center of curvature for the phase front of the far field radiation pattern. It is convenient to think of a horn antenna as a radiator rather than a receiver because the radiation characteristics are the same for both cases (theorem of reciprocity). The phase center should coincide with the focus of the telescope to optimise the gain of the system as a whole. However, the phase center of a multi-moded horn is an ill-defined concept since the field is only partially spatially coherent [1]. Each independent spatially coherent component field making up the beam can have a separate center of phase curvature. An average phase center position can be computed by optimising the predicted radiation patterns on the sky when the horn is placed close to the telescope focus. However, a more useful re-designation of the phase center would be the virtual waist position for the composite beam behind the horn aperture [2,3].

To locate the virtual waist we propagate the horn aperture fields backwards into virtual space behind the horn aperture plane and determine the position at which the beam appears narrowest (see Fig. 1). The individual spatially coherent component fields must be propagated separately and combined in quadrature to determine the intensity pattern. Since the phase front of the horn aperture is convex, the effective phase center will be located inside the horn. Furthermore, as the phase error term $s = a^2 / 2\lambda L$ increases (where a is the radius of the horn aperture and L is the slant length of the horn) the phase center moves towards the back of the horn.

In an imaging telescope configuration, the far field radiation pattern of the horn is the field that effectively illuminates the telescope aperture. On the other hand the beam on the sky produced by the horn (taking the magnification of the telescope into account) is a diffraction limited image of the virtual fields at the telescope focus assuming that the focus lies behind the horn aperture. These virtual fields can be estimated using near field diffraction techniques if the spatially coherent component fields at

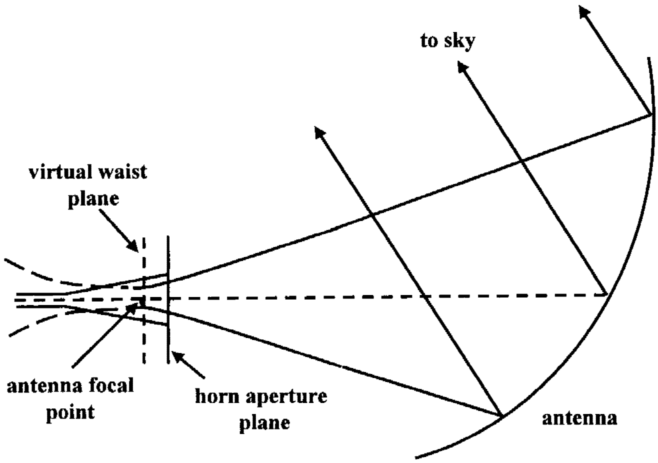

Figure 1. Set-up of a horn aperture with respect to an antenna illustrating the horn virtual waist and antenna focal point.

the horn aperture are available. The effects of telescope truncation can often be neglected in determining the position of the best phase center. For example in CMB experiments (in which multi-moded horns are used e.g. PLANCK [4]) the beam edge tapers at the telescope are often very low (typically –30dB) and to a good approximation the telescope can be assumed to be of infinite extent. In these cases the field on the sky will be an image of the horn field at the telescope focus without any significant broadening because of the low level of spatial filtering at the telescope aperture. The effects of aberrations may be important however for pixels located near the edge of the field of view.

Because of the extra degrees of freedom resulting from having more than one independent coherent field propagating, the relationship between the on-axis gain, the resolution (full width half maximum) and the form of the beam on the sky is more complex than for a single mode horn. Determining the "phase center" is therefore a process of positioning the horn aperture with respect to the telescope focus to optimise whichever beam characteristic [3] is of interest. For most applications it is either the gain or the resolution that has to be optimised. The form of the beam, however, may also be of importance.

In this paper the phase center of a multi-moded horn antenna is computed using two alternative approaches. The first approach involves decomposing the horn aperture field into Associated Laguerre Gaussian Beam modes and propagating these backwards to locate the virtual waist for the beam. As the modes propagate their width, radii of curvature and phase slippage evolve. An equivalent method involves using a Fresnel transformation of the field rather than a Gaussian Beam decomposition to locate the waist. However, this has the disadvantage that the transform becomes unstable for small propagation distances behind the aperture. Assuming then the waist fields are imaged onto the sky, in both cases the phase center is fixed by the point at which the beam characteristic of interest is optimised. The second approach is most appropriate for cases with non-negligible edge taper at the telescope and involves coupling the farfield of the horn to the telescope and optimising the system by adjusting the distance between the aperture and telescope, which effectively changes the phase curvature of the beam at the telescope aperture. The radiation pattern on the sky is then obtained by summing in quadrature the Fourier transforms of the far fields of the independent spatial modes of the horn with the appropriate phase error across the telescope aperture. Again the phase center position corresponds to the distance behind the horn aperture where the focus of the telescope should lie to optimise the beam characteristic of interest.

A mode matching technique can be used to compute the radiation pattern at the aperture of the horn [1,5]. In this technique a corrugated conical horn structure is regarded as a sequence of cylindrical waveguide segments with the radius stepping between the top and bottom of the corrugation slots. A smooth walled profile on the other hand is approximated by a series of cylindrical monotonically increasing radii giving a stair like profile. The natural modes of propagation for each segment are the *TE* and *TM* modes of a uniform cylindrical waveguide. There is a sudden change in the guide radius at the interface between two segments and the power carried by the individual modes is scattered between the backward propagating modes in the first guide segment and the forward propagating modes in the second guide segment. The mode matching technique is based on matching the total transverse field in the two guides at the junction so that the total complex power is conserved with the usual boundary conditions applying to the fields at the conducting walls. Track is also kept of the evanescent modes in the guide as these can clearly propagate as far as the next step in the profile.

Because of the cylindrical symmetry of the junction discontinuity for the case of regular conical horns, the scattering matrices $S^{(n)}$ for each azimuthal order are computed separately for all those values of n for which modes can propagate [1]. This is because scattering only occurs between modes with the same azimuthal number as they have the same z-component dependence on $\cos(n\phi)/\sin(n\phi)$. Each column, j, of the total transmission S_{21} matrix corresponds to the scattered mode coefficients for a particular mode, ψ_{nj}, at the input to the system [5]. Thus each independent spatially coherent component in the guide $\bar{\boldsymbol{e}}^i$ can in general be written as a total sum which is of the form $\bar{\boldsymbol{e}}^i = \sum_j A_j^i \boldsymbol{e}_j^G$ where $\boldsymbol{e}_j^G$ are the TE and TM modes in the guide segment and the A_j^i are derived from the scatter matrices $S_{21}{}^{(n)}$ and the modal expansion coefficients of the input independent coherent fields.

This paper is organised as follows. In section 2 we consider the quasi-optical determination of the virtual waist position as an approach for determining the phase center using Laguerre modes and an equivalent method involving a Fresnel transformation. In section 3 we discuss the coupling of the field from the horn antenna to a telescope and how the diffraction limited image of the horn virtual waist is produced on the sky. In section 4 we conclude with a discussion of the results of our study.

2. Determination of the horn virtual waist fields

Because of cylindrical symmetry any spatially coherent field, $E(r,\phi,z)$, at the aperture of a conical horn antenna (smooth walled or corrugated) can be expressed conveniently as a linear sum of Associated Laguerre Gaussian Beam modes [6] i.e.

$$E(r,\phi,z)=\sum_n C_n \Psi_n(r,\phi,z). \qquad (1)$$

These scalar Laguerre modes have the following mathematical form for a beam travelling in the positive z direction given by

$$\begin{pmatrix} \Psi_m^{\alpha,\cos}(r,\phi,z) \\ \Psi_m^{\alpha,\sin}(r,\phi,z) \end{pmatrix} = \psi_m^{\alpha}(r,\phi)\exp(-ikz+i(2m+\alpha+1)\varphi_0(z))\begin{pmatrix} \cos\alpha\phi \\ \sin\alpha\phi \end{pmatrix} \qquad (2)$$

where $$\psi_m^{\alpha}(r,\phi)=\sqrt{\frac{2(2-\delta_{0\alpha})m!}{\pi W^2(m+\alpha)!}}\exp\left(\frac{-r^2}{W^2}\right)\left(\frac{2r^2}{W^2}\right)^{\frac{\alpha}{2}}\times L_m^{\alpha}(2r^2/W^2)\exp(-ik(r^2/2R))$$

and α is an integer representing the degree of the Laguerre polynomial. W is called the beam width parameter and is a measure of the scale size of the beam, R is the phase front radius of curvature and φ_0 is the phase slippage of the fundamental mode with respect to a plane wave between the waist and the plane of interest (see fig.2). A mode travelling in the negative z direction is obtained by setting z as $-z$ where $z=0$ at the aperture. The mode is normalised in the sense that the generalised power $\int\left|\Psi_m^{\alpha}\right|^2 rdrd\phi$ is unity over any transverse plane.

For single mode conical corrugated horn antennas the beam width parameter at the aperture, W_a, is often set to $0.6435a$ where a is the width of the horn aperture and the radius of curvature of the beam at this position is set to the slant length, L, of the horn. This optimises the power in the fundamental hybrid HE_{11} horn mode [7]. The position of the waist, Δ, for the beam mode set (which is not necessarily precisely equivalent to the phase center for the horn field, of course) is given by the following relationship $\Delta = L/(1+(\lambda L/\pi W_a^2)^2)$. The beam width parameter at the waist, W_0 is given by $W_0 = W_a/(1+(\pi W_a^2/\lambda L)^2)^{1/2}$. The radius of curvature,

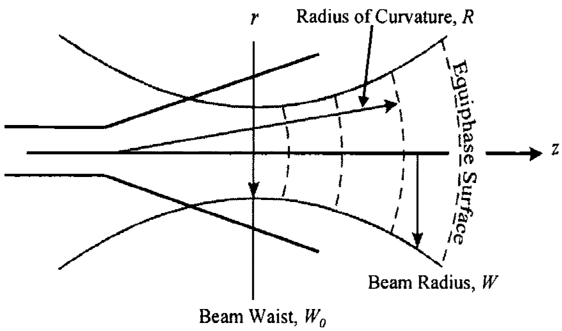

Figure 2. Laguerre Gaussian beam propagation, illustrating beam radius *W*, waist W_0, radius of curvature *R*, and equi-phase surfaces as a function of *z*, for any mode.

R, the width, *W*, and phase slippage, φ_0, of the beam a distance *z* from the aperture are described using the usual relationships [7].

For a particular choice of *R* and *W* in fact, an infinite set of free space modes, $\Psi_m^\alpha(r,\phi,z)$, exists which can be used to describe one of the components of any complex scalar field, $E(r,\phi,z)$,

$$E(r,\phi,z)=\sum_{\alpha,m}^{\infty} A_{m\alpha}\Psi_m^{\alpha\cos}(r,\phi,z)+\sum_{\alpha,m}^{\infty} B_{m\alpha}\Psi_m^{\alpha\sin}(r,\phi,z) \quad (3)$$

where the $A_{m\alpha}$ and $B_{m\alpha}$ terms (modal coefficients) are given by

$$A_{m\alpha} = \int_r\int_\phi E(r,\phi,z)\left[\Psi_m^{\alpha\cos}(r,\phi,z)\right]^* r\,dr\,d\phi \quad (4)$$

$$B_{m\alpha} = \int_r\int_\phi E(r,\phi,z)\left[\Psi_m^{\alpha\sin}(r,\phi,z)\right]^* r\,dr\,d\phi \quad (5)$$

The transverse electric field of cylindrical horns and waveguides can be written as a linear sum of *TE* and *TM* modes where the cylindrical components of the two orthogonal sets of *TE* and *TM* modes are given respectively by

$$\begin{pmatrix} \mathrm{e}_{nl}^{TE,\cos} \\ \mathrm{e}_{nl}^{TE,\sin} \end{pmatrix} = \sqrt{\frac{(2-\delta_{n0})}{\pi a^2(1-(n/q_{nl})^2)J_n^2(q_{nl})}} \left[\frac{nJ_n(q_{nl}r/a)}{q_{nl}r/a} \begin{pmatrix} \cos n\phi \\ -\sin n\phi \end{pmatrix} \hat{\mathrm{r}} - J_n'(q_{nl}r/a) \begin{pmatrix} \sin n\phi \\ \cos n\phi \end{pmatrix} \hat{\boldsymbol{\varphi}} \right] \tag{6}$$

$$\begin{pmatrix} \mathrm{e}_{nl}^{TM,\cos} \\ \mathrm{e}_{nl}^{TM,\sin} \end{pmatrix} = \sqrt{\frac{(2-\delta_{n0})}{\pi a^2 J_{n+1}^2(p_{nl})}} \left[J_n'(p_{nl}r/a) \begin{pmatrix} \cos n\phi \\ \sin n\phi \end{pmatrix} \hat{\mathrm{r}} + \frac{nJ_n(p_{nl}r/a)}{p_{nl}r/a} \begin{pmatrix} -\sin n\phi \\ \cos n\phi \end{pmatrix} \hat{\boldsymbol{\varphi}} \right] \tag{7}$$

in which q_{nl} represents the *lth* root of $J'_n(z)$, p_{nl} represents the *lth* root of $J_n(z)$ and a is the radius of the horn aperture. Here the constants of proportionality were chosen for convenience to normalise the generalised power $\iint_{Aperture} \left| e_{nl}^{TE/TM} \right|^2 r dr d\phi$ to unity.

The aperture field of the horn obtained using a mode matching technique [1] expressed in terms of *TE* and *TM* modes is converted to Associated Laguerre Gaussian Beam modes. Waveguide modes with $\cos n\phi$ or $\sin n\phi$ angular dependence clearly only couple to Laguerre modes with the same $n\phi$ angular dependence. Each wave guide mode can be re-expressed as:

$$e_{nl}^{TE} = \sqrt{\frac{1}{2\pi a^2\left(1-(n/q_{nl})^2\right)J_n(q_{nl})^2}} \left[J_{n-1}(q_{nl}r/a)\hat{e}_{n-1} + J_{n+1}(q_{nl}r/a)\hat{e}_{n+1} \right] \tag{8}$$

$$e_{nl}^{TM} = \sqrt{\frac{1}{2\pi a^2 J_{n+1}(p_{nl})^2}} \left[J_{n-1}(p_{nl}r/a)\hat{e}_{n-1} - J_{n+1}(p_{nl}r/a)\hat{e}_{n+1} \right] \tag{9}$$

where the pair of orthogonal unit vectors $\hat{e}_{n-1}$ and $\hat{e}_{n+1}$ expressed in terms of Cartesian unit vectors are given by $\hat{e}_{n-1} = \cos(n-1)\phi\,\hat{\imath} - \sin(n-1)\phi\,\hat{\jmath}$ and $\hat{e}_{n+1} = \cos(n+1)\phi\,\hat{\imath} + \sin(n+1)\phi\,\hat{\jmath}$. This then simplifies the analysis involved in expressing the wave guide modes as a linear sum of Associated Laguerre modes. Clearly the waveguide modes couple to normalised linear combinations of two free space Laguerre modes of the form

$$\boldsymbol{\Psi}_m^{\alpha}(r,\phi)=\sqrt{\frac{2(2-\delta_{0\alpha})m!}{\pi W^2(m+\alpha)!}}\exp\left(\frac{-r^2}{W^2}\right)\left(\frac{2r^2}{W^2}\right)^{\frac{\alpha}{2}}L_m^{\alpha}\left(\frac{2r^2}{W^2}\right)\exp\left(-jk\frac{r^2}{2R}\right)\hat{\boldsymbol{e}}_{\alpha} \tag{10}$$

where $\alpha = n \pm 1$. We can therefore for example write

$$\boldsymbol{e}_{nl}^{TE}=\sum_{m}^{M}\left(T_{m,l}^{(n)}\boldsymbol{\Psi}_m^{n-1}+T_{m+M,l}^{(n)}\boldsymbol{\Psi}_m^{n+1}\right) \tag{11}$$

$$\boldsymbol{e}_{nl}^{TM}=\sum_{m}^{M}\left(T_{m,l+L}^{(n)}\boldsymbol{\Psi}_m^{n-1}+T_{m+M,l+L}^{(n)}\boldsymbol{\Psi}_m^{n+1}\right) \tag{12}$$

where the $T_{i,j}^{(n)}$ can be regarded as the elements of a transformation matrix $T^{(n)}$ for waveguide modes of azimuthal order n and the $T_{i,j}^{(n)}$ are given by

$$T_{m,l}^{(n)}=\int_r\int_\phi \boldsymbol{\Psi}_m^{n-1^*}.\boldsymbol{e}_{nl}^{TE}\, rdrd\phi \qquad T_{m+M,l}^{(n)}=\int_r\int_\phi \boldsymbol{\Psi}_m^{n+1^*}.\boldsymbol{e}_{nl}^{TE}\, rdrd\phi$$

$$T_{m,l+L}^{(n)}=\int_r\int_\phi \boldsymbol{\Psi}_m^{n-1^*}.\boldsymbol{e}_{nl}^{TM}\, rdrd\phi \qquad T_{m+M,l+L}^{(n)}=\int_r\int_\phi \boldsymbol{\Psi}_m^{n+1^*}.\boldsymbol{e}_{nl}^{TM}\, rdrd\phi \tag{13}$$

M is the number of Laguerre modes used in the analysis for a given value of *n* and *2L* is the total number of *TE* plus *TM* modes of azimuthal order *n* used in the expansion of the horn aperture fields. Thus, at the horn aperture the overall field for each independent spatially coherent field component in the guide, $\bar{\boldsymbol{e}}^i$, can be re-expressed as follows

$$\bar{\boldsymbol{e}}^i=\sum_j^{2L}A_j^i\boldsymbol{e}_j^G=\sum_j^{2L}\left(\sum_m^{M}T_{m,j}^{(n)}A_j^i\boldsymbol{\Psi}_m^{n-1}+T_{m+M,j}^{(n)}A_j^i\boldsymbol{\Psi}_m^{n+1}\right) \tag{14}$$

where *j* corresponds to either *l* or *l+L* depending on whether the mode is *TE* or *TM*, respectively.

Once the Laguerre Gaussian expansion coefficients are determined the component modes can be propagated and re-summed to determine the fields at another plane either in front of the horn (forward direction) or in the virtual half space behind the horn (backward direction). The Laguerre mode definition used in this summation includes the phase slippage as in equation (2). The total beam is made up of an incoherent sum of component independent spatially coherent fields, $\bar{e}^i$ and thus can be computed by adding in quadrature the propagated modal sums given in equation (14). The position behind the aperture where the beam is narrowest corresponds to the virtual waist region of the horn and can be regarded as the phase center. One advantage of this approach is that once the $T^{(n)}$ matrix is computed it is horn independent as long as the modal beam parameter W is scaled with the horn aperture size. An alternative less general approach would be to express the independent spatially coherent $\bar{e}^i$ fields directly as Laguerre Gaussian Beam mode expansions. The Laguerre Gaussian approach is completely equivalent to applying a Fresnel transformation to such fields as will now be discussed.

The Fresnel transform of each mode making up the aperture field is most conveniently expressed in the following mathematical form with the co-ordinate frames defined as in figure 3:

$$\boldsymbol{E}_{nl}^{TE/TM}(r_0,\phi_0,z)=\frac{i}{\lambda z}\int_{\phi_a=0}^{\phi_a=2\pi}\int_{r_a=0}^{r_a=a}\boldsymbol{e}_{nl}^{TE/TM}(r_a,\phi_a)\exp\left(-ik\left(z+\frac{r_a^{\ 2}+r_0^2}{2z}\right)\right)$$

$$\times\exp\left(\frac{ikr_ar_0\cos(\phi_a-\phi_0)}{z}\right)r_a dr_a d\phi_a \quad (15)$$

In this case $\boldsymbol{e}_{nl}^{TE/TM}$ can be written as in equations (8) and (9) in terms of their Cartesian $\hat{\imath}$ and $\hat{\jmath}$ components which contain products with azimuthal ϕ terms of the form $\cos(n\pm1)\phi_a$ and $\sin(n\pm1)\phi_a$. On integrating analytically over ϕ_a [8] these terms are converted to terms of the form $J_{n\pm1}(kr_ar_0/z)2\pi i^{n\pm1}\cos(n\pm1)\phi_0/\sin(n\pm1)\phi_0$ reducing the Fresnel integration to a combination of terms involving functional variation in r_0

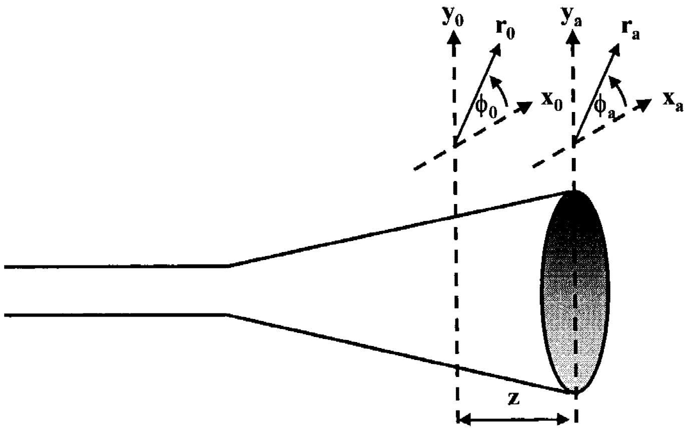

Figure 3. Fourier transform of the aperture field a distance z behind the aperture.

of the form

$$I_{n\pm1}(r_0)\sim\int_{r=0}^{r=a}J_{n\pm1}(\chi\, r_a/a)J_{n\pm1}(kr_ar_0/z)\exp\left(-ikr_a^{\,2}/2z\right)r_a dr_a \qquad (16)$$

where $\chi = p_{nl}$ or q_{nl}. The fields can clearly be reconstructed in the region behind the horn and thus the virtual waist phase center located as before. Analytically the process is identical to the procedure described for Gaussian Beam Mode analysis. However the numerical approximations involved in the two procedures are different. In fact Fresnel transforms have the disadvantage that we cannot let $z\to 0$ (for small values of z the transform becomes unstable computationally) and therefore we cannot probe the field at the aperture.

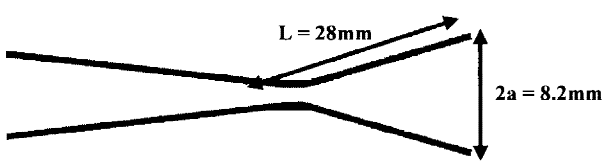

Figure 4. Conical corrugated back to back horn antenna.

As an example we take one possible conical corrugated horn antenna model being proposed for the High Frequency Instrument [9] on the PLANCK Surveyor (see fig. 4). The example discussed is based on such a horn with a design frequency of 545GHz, aperture diameter 8.2mm and slant length 28mm.

Figure 5 - 6 show contour plots of the virtual total field intensity pattern in the vicinity of the horn virtual waist computed using Associated Laguerre Gaussian Beam modes and for comparison also using Fresnel diffraction. There is a clear tendency for the Fresnel approach to begin to break down as $z \to 0$ (fields close to the horn aperture). The virtual waist phase center can be chosen as either the point where the on-axis intensity is highest or where the 3dB width is narrowest. By placing the telescope focal point at the waist an image of that plane is formed on the sky if no significant truncation of the beam occurs at the telescope. Since the horn is fed by a black body cavity both polarisations of each mode propagate and therefore all beam patterns are axially symmetric.

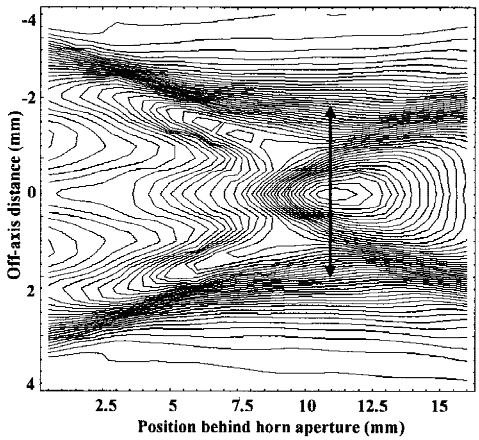

Figure 5. Beam intensity pattern in the vicinity of the horn virtual waist using Associated Laguerre Gaussian Beam modes (highest contour level is 65.26 (arbitrary units), contour spacing is 1.31). The vertical line indicates the position of the minimum FWHM.

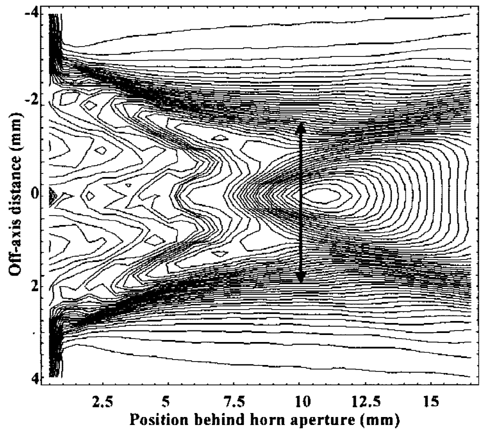

Figure 6. Beam intensity pattern in the vicinity of the horn virtual waist using direct Fresnel transformations (highest contour level is 66.0 (arbitrary units), contour spacing is 1.32). The vertical line indicates the position of the minimum FWHM.

For this horn design, the phase center in terms of the point where the on-axis intensity is highest is 11.5mm behind the aperture when Laguerre Gaussian modes are used as the basis set and 11mm behind the aperture when using smooth walled *TE* and *TM* modes. Alternatively, the phase center where the 3dB width is narrowest is 11mm behind the aperture when using Laguerre and 10mm behind the aperture when using the smooth walled modes. However because of the clear depth of field of the beams in the vicinity of the waist these discrepancies are insignificant.

3. "Phase center" by optimising the telescope beam on the sky.

In the previous discussion of the phase center it was taken that the waist (in a general sense) of the virtual field behind the horn could be assumed to be the phase center. In reality when coupled to a telescope some spatial frequency filtering of the beam will take place and this will affect the image of the horn waist on the sky with a possible shift in phase center position. An alternative approach to finding the phase center therefore involves coupling the far field of the horn to the telescope and optimising the telescope beam on the sky (see fig.7). The radiation pattern on the sky can be obtained by Fourier transformation of the far field of the horn at the telescope aperture with the relevant phase error due to the displacement, Δ, of the horn aperture from the focal plane of the telescope.

Assuming paraxiality and neglecting obliquity factors the far field at the telescope aperture of the *TE* and *TM* modes at the horn mouth is given by the Fraunhofer limit of the Fresnel transform given in equation (15)

$$\left[e_{nl}^{TE/TM}\right]^{TELESCOPE} = \frac{i}{\lambda z}\int_{\phi_a=0}^{\phi_a=2\pi}\int_{r_a=0}^{r_a=a} e_{nl}^{TE/TM}(r_a,\phi_a)\exp\left(-ik\left(z+\frac{r_T^2}{2z}\right)\right)\times \exp\left(\frac{ikr_a r_T\cos(\phi_a-\phi_T)}{z}\right) r_a dr_a d\phi_a \quad (17)$$

where in the limit as $z \to \infty$ the $kr_a^2/2z$ term is negligible.

The beam on the sky is obtained by Fourier transformation of this field with the correct phase curvature to take account of the position of the horn with respect to the telescope focus. The position for which the 3dB

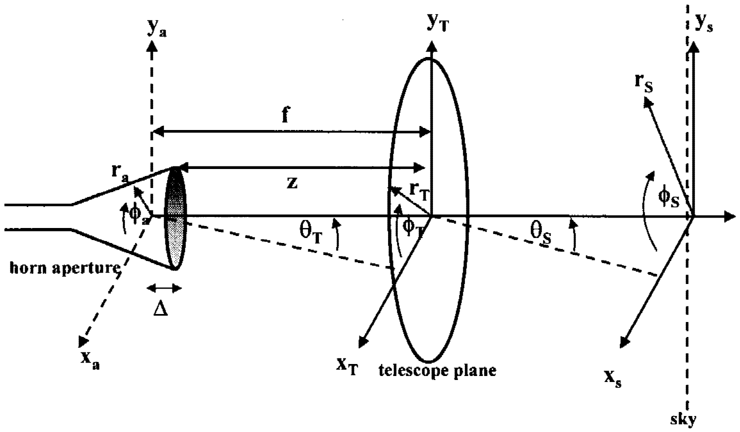

Figure 7. The far field of a horn antenna coupled to an imaging telescope and Fourier transformed onto the sky where. The diagram shows the variables used in the derivation.

level of the beam on the sky is narrowest (best angular resolution) should be close to that which optimises the on-axis gain but may not exactly coincide. In order to determine the best angular resolution the beam pattern on the sky needs to be computed as a function of the horn aperture position with respect to the telescope focus. If the truncation levels are significant for at least some of the modes, the patterns observed will not be exact images of the virtual fields in the vicinity of the telescope focal point.

For each waveguide mode $e_{nl}^{TE/TM}$ at the horn aperture the field on the sky is therefore given by

$$\left[e_{nl}^{TE/TM}\right]^{SKY} \propto \int_{r_a=0}^{a} \int_{r_T=0}^{D/2} \int_{\phi_a=0}^{2\pi} \int_{\phi_T=0}^{2\pi} e_{nl}^{TE/TM}(r_a, \phi_a) \exp\left(ikr_a \frac{r_T}{z} \cos(\phi_a - \phi_T) \right) \times$$
$$\exp(ikr_T \sin\theta_S \cos(\phi_T - \phi_S)) \exp(ikr_T^2 \Delta / 2f^2) r_a dr_a d\phi_a \, r_T dr_T d\phi_T \quad (18)$$

where D is the diameter of the telescope aperture and Δ is the horn displacement. The integration over ϕ_a, ϕ_T and r_a may be performed analytically as in section 2 for Fresnel integration thus reducing equation (18) to a combination of one dimensional integrals (similar to equation (16)) with respect to r_T of the form:

$$I_{n\pm1}^{SKY}(\theta_S) \sim \int_{r_T=0}^{r_T=D/2} \mathcal{J}_{n\pm1}(\chi, kar_T / f) J_{n\pm1}(kr_T \sin\theta_S) \exp(ikr_T^2 / 2R_{eff}) r_T dr_T \quad (19)$$

where R_{eff} is the effective radius of curvature of a mode because of the displacement Δ of the horn aperture from the telescope focus with $R_{eff} = f^2 / \Delta$ and where

$$\mathcal{J}_{n\pm1}(s,t) = \frac{\{s J_{n\pm1}(t) J_{n\pm1}'(s) - t J_{n\pm1}(s) J_{n\pm1}'(t)\}}{t^2 - s^2} \quad (20)$$

The total power on the sky can be computed by adding in quadrature the propagated modes given in equation 18. For simplicity the phase center of the horn can be computed by maximising the on-axis gain. In that case, only modes of azimuthal order 1 need to be included as these alone have on-axis power because of azimuthal symmetry. Therefore the on-axis gain can be computed by coupling each mode of azimuthal order 1 to an on-axis point and adding these contributions taking into account the partial coherent nature of the field.

Again in this case since paraxiality is assumed an alternative approach involving Gaussian Beam modes can be chosen. The approach is more stable especially if the horn aperture is placed close to the telescope focus and the levels of truncation are low at the telescope. The disadvantage of the modal approach in this case is that the telescope edge truncation causes mode scattering at the telescope [10] although the defocusing effect of moving the horn with respect to the telescope is readily included by its effect on the phase radius of curvature term.

Again we consider an example of the horn discussed previously. Figure 8 illustrates the intensity on the sky as a function of off-axis distance for varying displacements of the horn aperture with respect to the focus of the telescope where the level of truncation is at –30dB (PLANCK requirements) and figure 9 shows the on-axis gain at the telescope. The position at which the beam is narrowest in terms of its FWHM (resolution) is also indicated on figure 8. As can clearly be seen the FWHM varies slowly with the horn position compared to the on-axis gain which is particularly susceptible to phase slippages between component modes of the fields. In figure 10 the field on the sky is illustrated for the case of non-negligible beam edge taper at the telescope (the truncation level is –10dB). The sharp maximum in gain is lost as the higher order spatial frequencies are now significantly truncated.

For the case of negligible beam edge taper at the telescope the phase center in terms of the point where the on-axis intensity (on the sky and at the telescope) is highest is 11.5mm behind the aperture. Alternatively, the phase center where the 3dB width is narrowest is 11mm behind the aperture. The predicted phase center locations using the alternative definitions and theoretical approaches lie in the range 10mm – 11.5mm. Clearly the beam pattern (given by a vertical cut on fig. 8) on the sky also evolves in form in the region of the highest gain and resolution. This may influence the chosen positioning for the horn depending on the application. When the truncation level at the telescope is –10dB the point

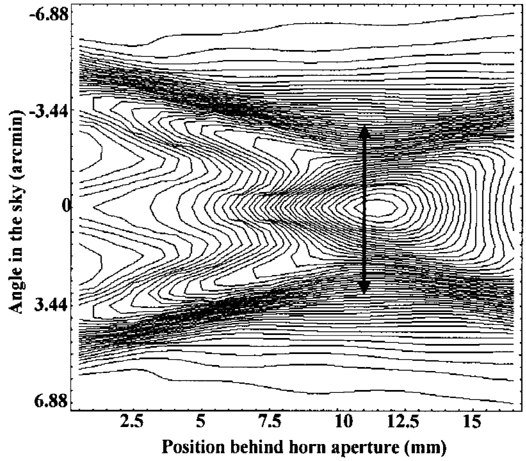

Figure 8. Beam intensity pattern on the sky for varying displacements of the horn aperture with respect to the telescope focus (highest contour level is 77.0 (arbitrary units), contour spacing is 1.54). The level of truncation at the telescope is at –30dB.

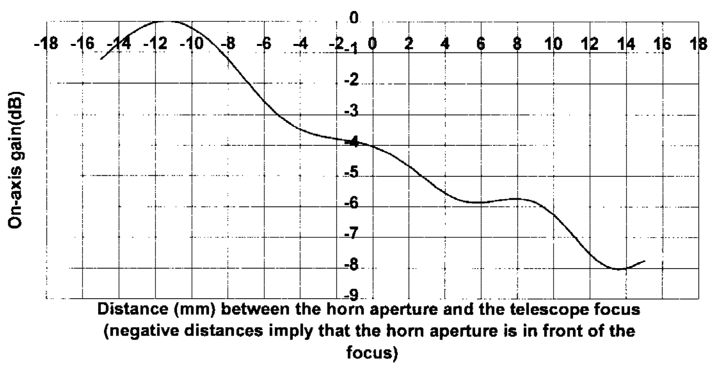

Figure 9. On-axis gain at the telescope for varying displacements of the horn aperture with respect to the telescope focus using smooth walled *TE* and *TM* modes. The level of truncation at the telescope is –30dB.

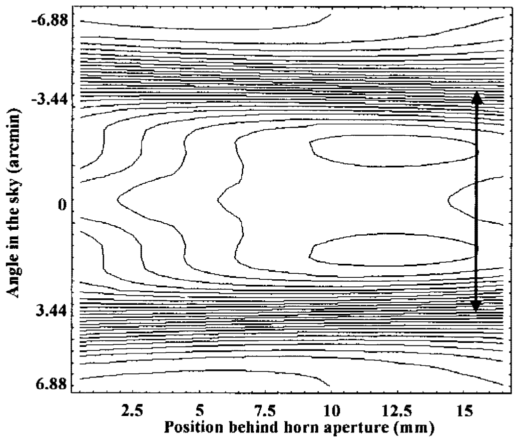

Figure 10. Beam intensity pattern on the sky for varying displacements of the horn aperture with respect to the telescope focus (highest contour level is 34.0 (arbitrary units), contour spacing is 1.36). The level of truncation at the telescope is at –10dB. The vertical line indicates the position of the minimum FWHM.

where the on-axis intensity is highest is 10mm behind the aperture and the 3dB width is narrowest at 15.5mm. However, the 3dB beam width is clearly a very slowly varying function of the position of the horn as was the case in figure 9 also.

4. Discussion and Conclusions

In this paper we discussed how useful techniques for locating the best virtual waist phase center for a multi-moded partially coherent horn antenna could be developed. The technique was based on near field diffraction of the individual spatially coherent fields at the horn aperture into the virtual propagation half-space behind the horn aperture and then summing their contributions to the intensity in quadrature. Two equivalent approaches based on either a modal or integral transformation analysis were outlined and compared for an example prototype horn for a real far IR system (HFI on PLANCK). For the case where the spatial frequency filtering of the horn beam is significant on a telescope a straightforward paraxial approach based on Fresnel transforms is outlined. The field on the sky can then be optimised for gain, resolution or form.

Clearly on real telescopes the paraxial approach may be a significant approximation, and especially for array imaging systems where aberrations may be important. Distortion of the component spatially coherent fields can result in an even more complex variation in gain, resolution and form than in the case for the paraxial perfect imaging discussed in this paper [11]. Ultimately for such cases an accurate physical optics model may be necessary for fine tuning the phase center location as computed using the straightforward approach presented in this paper.

Acknowledgements

The authors would like to acknowledge the support of Enterprise Ireland who fund this research.

References

[1] Murphy, J.A., Colgan, R., O'Sullivan, C., Maffei, B., Ade, P., "Radiation Patterns of Multi-moded Corrugated Horns for Far-IR Space Applications", *Infrared Physics and Technology*, **41**, pp. 515-528, 2001.

[2] Murphy, J.A., Padman, R., "Phase Centers of Horn Antennas Using Gaussian Beam Mode Analysis", *IEEE Transactions on Antennas and Propagation*, **38**, No.8, pp1306-1310, 1990.

[3] Wylde, R.J., Martin, D.H., "Gaussian Beam-Mode Analysis and Phase-Centers of Corrugated Feed Horns", *IEEE Transactions on Microwave Theory and Techniques*, **41**, No.10, pp1691-1699, 1993.

[4] http://astro.estec.esa.nl/Planck.html.

[5] Olver A.D., Clarricoats P.J.B., Kishk A.A. and Shafai L., *Microwave Horns and Feeds,* IEEE Press, 1994, (chapter 4).

[6] Murphy, J.A., "Aperture Efficiencies of Large Axisymmetric Reflector Antennas Fed by Conical Horns", *IEEE Transactions on Antennas and Propagation*, **36**, No.4, pp570-575, 1988.

[7] Wylde, R.J., "Millimetre Wave Gaussian Beam Mode Optics and Corrugated Feed Horns", *IEEE Proc,* **131**, pp158-262, 1984.

[8] Born M., Wolf E., *Principles of optics : Electromagnetic Theory of Propagation Interference and Diffraction of Light*, 6th edition, Pergamon Press 1987.

[9] Lamarre, J.M., Ade, P.A.R., Benoit, A., de Bernardis, P., Bock, J., Bouchet, F., Bradshaw, T., Charra, J., Church, S., Couchot. F., Delabrouille, J., Estathiou, G., Giard, M., Giraud-Heraud, Y., Gispert, R., Griffin, M., Lange, A., Murphy, J.A., Pajot, F., Puget, J.L., Ristorcelli, I., "The High Frequency Instrument on PLANCK: Design and Performance", *Astro. Letters and Communications*, **37**, pp161-170, 2000.

[10] Murphy, J.A., Withington, S., Egan,E., "Mode Conversion at Diffracting Apertures in Millimeter and Submillimeter Wave Optical Systems", *IEEE Transactions on Microwave Theory and Techniques*, **41,** No.10, pp1700-1702, 1993.

[11] Yurchenko, V., "Phase Centers of Multimoded Horns in the presence of Aberrations", in preparation for publication.