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Maximum Likelihood Estimation of Time-series with Markov Regime

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Abstract

In this paper, we consider the estimation of various Markov-modulated time-series. We obtain maximum likelihood estimates of the time-series parameters including the Markov chain transition probabilities and the time-series coefficients using the EM (Expectation Maximization) algorithm. Also the recursive EM algorithm is used to obtain on-line parameter estimates. Simulation studies show that both algorithms yield satisfactory results.

1 Introduction

Signal Model: Let s_k denote a N_s-state irreducible Markov chain with states $\{1, 2, \dots, N_s\}$ with transition probability matrix $\Pi = (\pi_{mn})$, $\pi_{mn} = P(s_{k+1} = n | s_k = m)$ and initial state probability $\pi = (\pi_m)$, $\pi_m = P(s_1 = m)$. Define the Markov-modulated polynomials as follows:

$$\begin{aligned} A(z^{-1}, s_k) &= 1 + \sum_{i=1}^{p} a_i(s_k) z^{-i} \\ B(z^{-1}, s_k) &= 1 + \sum_{i=1}^{q} b_i(s_k) z^{-i} \\ C(z^{-1}, s_k) &= 1 + \sum_{i=1}^{r} c_i(s_k) z^{-i} \end{aligned} \quad (1)$$

where z^{-1} denotes the delay operator and k denotes discrete-time. Let $A(m) \triangleq (a_1(m) \ \dots a_p(m))'$, $B(m) \triangleq (b_1(m) \ \dots b_q(m))'$, $C(m) \triangleq (c_1(m) \ \dots c_r(m))'$. In this paper, we consider estimation of any one of the following second-order stationary Markov-modulated time-series models:

$$\textbf{ARX}: \quad A(z^{-1}, s_k) y_k = B(z^{-1}, s_k) u_k + w_k \quad (2)$$

$$\textbf{MAX}: \quad y_k = B(z^{-1}, s_k) u_k + C(z^{-1}, s_k) w_k \quad (3)$$

$$\textbf{ARMA}: \quad A(z^{-1}) y_k = C(z^{-1}, s_k) w_k \quad (4)$$

where u_k, y_k are the measured input and output at time k, $w_k \sim$ white $N(0, \sigma^2)$ is independent of s_k and ϕ is the parameter vector consisting of polynomial coefficients and Markov chain parameters (e.g., $\phi = (A(m), B(m), \Pi, \sigma^2)$ for (2)). We assume u_k to be persistently exciting [4]. We also assume that $A(z^{-1}, s_k)$, $B(z^{-1}, s_k)$ and $C(z^{-1}, s_k)$ are coprime to each other for each m, $m \in \{1, 2, \dots, N_s\}$.

Notations: $Y_k = (y_1, \dots, y_k)^T$, $U_k = (u_1 \dots u_k)^T$, $Z_k = (Y_k, U_k)$ denotes the observed "incomplete" data. $S_k = (s_1 \dots s_k)^T$, $Y_t^k = (y_t \dots y_k)^T$ and $U_t^k = (u_t \dots u_k)^T$ where superscript T denotes transpose.

Estimation Objectives: We use the Expectation Maximization (EM) algorithm [7] to obtain maximum likelihood (ML) estimates of ϕ, given Y_T, U_T (when appropriate) in Sec. 2. Also based on the recursive EM algorithm [2], an on-line estimation scheme is presented in Sec. 3.

In [5], the EM algorithm and a recursive EM algorithm are used to estimate Markov-modulated AR processes which is a special case of our model (2) with $B = 0$. The three models we consider in this paper can be regarded as an extension of the work in [5]. Applications of such estimation algorithms can be found in [6], [5] and in the references therein.

Remark 1: Models (2), (3) or (4) are special cases of the Markov-modulated ARMAX model

$$A(z^{-1}, s_k) y_k = B(z^{-1}, s_k) u_k + C(z^{-1}, s_k) w_k \quad (5)$$

However, unlike (2), (3) and (4), ML estimation of (5) is computationally prohibitive since it requires computing probability density functions over all N_s^T realisations of a N_s state T point Markov chain. For similar reasons, we forbid $A(z^{-1})$ in (4) to be Markov-modulated.

Remark 2: Deriving stationarity criteria for Markov-modulated time-series is a difficult problem. For example, two switching, separately second order AR stationary processes can result in an unstable system – whereas two individually unstable AR processes can be stabilized when allowed to switch according to a Markov regime. For sufficient conditions on the second-order stationarity of Markov-modulated time- series, see [5].

2 ML estimation via EM algorithm

Markov-modulated ARX estimation

The EM algorithm is an iterative procedure; each iteration involves two steps, E-step and M-step.

E Step: Following [3], the expectation of the log-likelihood function of a T-point "complete" data sequence $M_T = (Y_T, U_T, S_T)$ defined as

$$Q(\phi^{(l)}, \phi) \triangleq E\{\ln f(M_T|\phi) | Z_T, \phi^{(l)}\}$$

$$= -\frac{T}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\sum_{k=1}^{T-1}\sum_{m=1}^{N_s}\gamma_k(m)\left(A(z^{-1},m)y_k - B(z^{-1},m)u_k\right)^2 + \sum_{k=1}^{T-1}\sum_{m=1}^{N_s}\sum_{n=1}^{N_s}\xi_k(m,n)\ln\pi_{mn} + \sum_{m=1}^{N_s}\gamma_1(m)\ln\pi_m \quad (6)$$

where $\xi_k(m,n) \triangleq f(s_k = m, s_{k+1} = n|Z_T, \phi^{(l)})$ and $\gamma_k(m) \triangleq f(s_k = m|Z_T, \phi^{(l)})$. $\gamma_k(m)$ is computed via the "forward backward" procedure described in [1]. $\phi^{(l)}$ is the estimate of the parameter vector at the l-th iteration assuming the iteration procedure starts with an initial estimate $\phi^{(0)}$.

M Step: This step involves computing $\arg\max_\phi Q(\phi^{(l)}, \phi)$ to yield the estimates of π_{mn}, σ^2, $A(m)$, $B(m)$. For all the relevant details, see [6].

Markov-modulated MAX estimation

The MAX model (3) can be written in equivalent ARX form as

$$A'(z^{-1}, s_k)y_k = B'(z^{-1}, s_k)u_k + e_k \quad (7)$$

where the polynomial $A'(z^{-1}, s_k)$ is "sufficiently" long enough to ensure that e_k is almost white (see [4], pg 291 for details) and $B'(z^{-1}, s_k) = A'(z^{-1}, s_k)B(z^{-1}, s_k)$. The EM algorithm described in the previous section yields the estimates of $A'(m)$ and $B'(m)$ and hence of $B(m)$. $C(m)$ in (3) can be estimated by solving a set of *inverse Yule-Walker equations* (see pg 291, [4]). Details can be found in [6].

Markov-modulated ARMA estimation

Since A in (4) is no longer Markov-modulated, it can be estimated via a set of *Yule-Walker* equations (see pp 289, [4]). Rewriting (4) as

$$A(z^{-1})A'(z^{-1}, s_k)y_k = e_k \quad (8)$$

(where e_k and $A'(z^{-1}, s_k)$ are as defined in the previous section), estimate of $A(z^{-1})A'(z^{-1}, s_k)$ and hence $C(m)$ can be obtained via EM.

3 On-line Estimation via Recursive EM algorithm

An on-line estimation scheme can be implemented based on the recursive EM algorithm proposed in [2].

4 Simulation studies

We present simulation examples, with $N_s = 2$, $\pi_{11} = \pi_{22} = 0.9$ for on-line recursive EM algorithm. Simulation results for the off-line EM algorithm can be found in [6].

On-line estimation via recursive EM algorithm

Consider a jump time-varying 100000 point Markov-modulated MAX model with $\sigma^2 = 1$ and

$$B(1) = (0.8\ 0.3)',\ B(2) = (0.5\ 0.1)',\ C(1) = (0.5\ 0.3)',\ C(2) = (-0.4\ 0.2)' \quad t \le 20000$$

$$B(1) = (0.5\ 0.9)',\ B(2) = (-0.6\ 0.4)',\ C(1) = (0.7\ 0.5)',\ C(2) = (-0.2\ 0.5)' \quad t > 20000$$

Figure 1 shows the time evolution of the estimates when the estimation procedure starts with arbitrary initial estimates. Results for a Markov-modulated ARMA model can be found in [6].

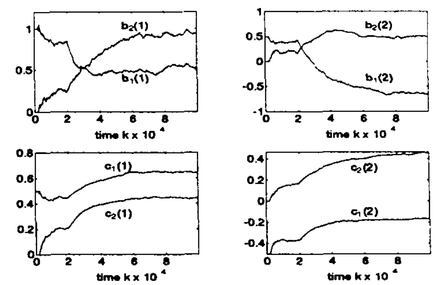

Figure 1: Time evolution of MAX parameters

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