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ON STEADY-STATE PROPERTIES OF CERTAIN MAX-PLUS PRODUCTS

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Abstract: The asymptotic properties of inhomogeneous products in the max-plus algebra context have been investigated. In particular, for products involving matrices with the same unique critical circuit, we have obtained some sufficiency conditions under which the rank of the final product matrix is less than or equal to the length of the critical circuit of the matrices in the product. For a product comprising of matrices with the same unique critical circuit of length 1, the asymptotic rank is 1.

Key Words: Max-plus algebra, asymptotic properties, discrete-event systems

1 Introduction

In this paper we study the asymptotic behaviour of inhomogeneous products of matrices in the max-plus context. Max-plus algebra allows the manipulation of matrices using the operations of maximisation and addition, and is one of many possible methods of analysing problems in the area of discrete event dynamical systems (DEDS). In particular, the question we would like answered is whether a sufficiently long product may be written as a sum of a small number of dyads[1]? As such, the 'fundamental' equation under investigation is of the form:

$$\Gamma(k) = A_k \otimes A_{k-1} \otimes A_{k-2} \otimes \ldots \otimes A_1 \qquad (1.1)$$

where each matrix on the right can be assumed to be chosen in some arbitrary manner from a (normally) finite set. The problem is an extension of one for which there are existing results in [1], where analogous results for homogeneous products have been derived.

It has been shown that for nonnegative matrices conforming to conventional laws of algebra, the results for homogeneous products – in effect the Perron-Frobenius theory – have an analogous extension to inhomogeneous products [6]. Also, many of the ideas pertinent to nonnegative matrices obeying conventional laws of linear algebra, starting for example with the concepts of irreducibility and aperiodicity, apply to max-plus algebra; the concepts of the Perron-Frobenius eigenvalue and eigenvector of a positive matrix have their parallels too and both concepts are helpful for studying powers of matrices in the max-plus context. Furthermore, under certain conditions, a limiting rank property exists for for homogeneous products (all A_i are identical) [1], which leads us to expect that a similar result for (1.1) should lead to a product matrix expressible as a single dyad, or a sum of a small number of dyads.

What are some possible applications of our results? Let us mention one. The long term behaviour of a system described by max-plus algebra ideas is often encapsulated in the power of a matrix A^k, where k becomes large. It is known that for a wide class of matrices, there exists a K such that for all $k \geq K$, the following relation holds

$$A^k = \lambda^k y \otimes z^T \qquad (1.2)$$

for some vectors y, z. Here z^T denotes matrix transpose, and in conventional algebra terms, (1.2) says $(A^k)_{ij} = k\lambda + y_i + z_j$. From (1.2) it follows the long term evolution of a system given some prescribed initial conditions can be characterised as

$$x(k) = A^k x(0) = \lambda^k y \otimes (z^T \otimes x(0)) \qquad (1.3)$$

for all $k \geq K$. This indicates that the initial condition $x(0)$ is forgotten after K time instants (apart from a probably inessential scaling, $z^T \otimes x(0)$). Roughly speaking, this would indicate for a manufacturing line or a train network defined by max-plus equations that there exists an insensitivity to certain initial parameter settings after a time K (this is potentially valuable). Further, if at some intermediate operating time, the values of certain variables are for some reason perturbed, the system after a further finite time reverts to a steady state form of operation, forgetting the perturbations.

Now in a practical situation, the entries of the A matrix are subject at least to small fluctuations, and on occasions to big fluctuations. Consequently, instead of having the system behaviour determined by A^k, it may be determined by $\Gamma(k) = A_k \otimes A_{k-1} \otimes \ldots \otimes A_1$, where the A_i have entries very close to A (in the case of small fluctuations, which will be typical), or the A_i may have entries substantially different from A. The question that naturally arises is, will the property of forgetting of initial conditions continue to hold in the inhomogeneous case? One is then seeking a result like

$$\begin{aligned} x(k) &= A_k \otimes A_{k-1} \otimes \ldots \otimes A_1 \otimes x(0) \\ &= y(k) \otimes (z^T \otimes x(0)) \end{aligned} \qquad (1.4)$$

The vector $y(k)$ may be of the form $\lambda(k) \otimes \bar{y}(k)$ in which $\bar{y}(k)$ is bounded for all k, and $\lambda(k)$ is a scalar governing the general growth of entries of $x(k)$.

The paper is organised as follows. Section 2 outlines the max-plus algebra; Section 3 discusses some sufficient conditions for inhomogeneous products involving matrices with length-1 critical circuit. Section 4 offers some concluding remarks.

2 Background

We will now recall some established results in the max-plus algebra, see [1, 2] for a more complete overview; in addition,

[1]This is essentially a requirement of the product to have a certain rank, when adopting the notion of Schein rank in the max-plus context.

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we will state some relevant graph theoretic concepts, adapted from analysis of nonnegative matrices [4].

2.1Max-plus Algebra

Definition 2.1 (Max-plus Algebra) *The max-plus algebra* $(\mathbb{R}_{max}, \oplus, \otimes)$ *is defined as follows:*

1. $\mathbb{R}_{max} \stackrel{\text{def}}{=} \mathbb{R} \cup \{-\infty\}$, *where* $\mathbb{R}$ *is the set of real numbers,*
2. $x \oplus y \stackrel{\text{def}}{=} \max(x, y)$, *and*
3. $x \otimes y \stackrel{\text{def}}{=} x + y$.

In particular, $\oplus$ is commutative over $\mathbb{R}_{max}$, and $\otimes$ is distributive over $\oplus$. Furthermore $\epsilon \stackrel{\text{def}}{=} -\infty$ and $e \stackrel{\text{def}}{=} 0$, so that $x \oplus \epsilon = x$ and $x \otimes e = x \; \forall x \in \mathbb{R}$.

For $A = \{a_{ij}\}$ and $B = \{b_{ij}\}$, the following are defined:

Definition 2.2 (Scalar multiplication)

$$(c \otimes A)_{ij} = c \otimes a_{ij} = c + a_{ij} \tag{2.1}$$

Definition 2.3 (Matrix sum)

$$(A \oplus B)_{ij} = a_{ij} \oplus b_{ij} = \max(a_{ij}, b_{ij}) \tag{2.2}$$

Definition 2.4 (Matrix product)

$$\begin{aligned}(A \otimes B)_{ij} &= (a_{i1} \otimes b_{1j}) \oplus \ldots \oplus (a_{in} \otimes b_{nj}) \\ &= \bigoplus_k a_{ik} \otimes b_{kj} = \max_{k=1,2,\ldots,n} (a_{ik} + b_{kj}) \end{aligned} \tag{2.3}$$

Definition 2.5 ((Schein) Rank of a matrix) *Schein rank of an* $n \times p$ *matrix* Z *is the smallest integer* k *such that* $Z = B \otimes C$, *where* B *and* C *are* $n \times k$ *and* $k \times p$ *respectively, i.e.* $Z = U_1 \otimes V_1^T \oplus U_2 \otimes V_2^T \oplus \ldots \oplus U_k \otimes V_k^T$, *where* U_i *and* V_i *denote column vectors and the superscript* T *denotes transpose in the normal sense. Various alternatives to the definition of rank can be found in [2]; the definition adopted here is the most convenient for our paper.*

2.2Graph Concepts

Definition 2.6 (Directed graph) *A directed graph (or digraph for short)* $\mathcal{G}$ *is defined as a pair* $(\mathcal{V}, \mathcal{E})$, *where* $\mathcal{V}$ *is a set of elements called nodes, numbered from 1 to n, and* $\mathcal{E}$ *is the set of directed arcs joining any node pair. An arc joining nodes* i *and* j *is denoted as* $i \rightarrow j$.

Definition 2.7 (Path) *A path is defined as a sequence of nodes* $(i_1, i_2, \ldots, i_p)$ *such that there is an arc from node* i_{j-1} *to node* i_j *for* $j = 2, 3, \ldots, p$. *We will denote a path either in full as* $i_1 \rightarrow i_2 \rightarrow i_3 \rightarrow \cdots \rightarrow i_p$, *or* $i_1 \rightarrow\rightarrow i_p$, *indicating only the terminating nodes for short.*

Definition 2.8 (Circuit) *A circuit is a path for which the initial and terminating nodes are identical.*

Definition 2.9 (Precedence graph of a matrix) *The precedence graph* $\mathcal{G}(A)$ *corresponding to an* $n \times n$ *matrix* A *is a weighted digraph with n nodes. Each arc in* $\mathcal{G}(A)$ *takes the value of* a_{ij} $(\neq \epsilon)$.

Definition 2.10 (Weight and Length of Paths/Circuits) *The weight* $w(\rho)$ *of a path* $\rho = i_1 \rightarrow i_2 \rightarrow \cdots \rightarrow i_{l-1} \rightarrow i_l$ *is the sum of the weights of the individual arcs. The length* $l(\rho)$ *of the same path is equal to the number of arcs in the path. The average weight of a path is its weight divided by its length:* $w(\rho)/l(\rho) = (a_{i_l i_{l-1}} + a_{i_{l-1} i_{l-2}} + \ldots + a_{i_3 i_2} + a_{i_2 i_1})/(l-1)$. *The circuit mean is the average weight of a circuit.*

Definition 2.11 (Critical circuit) *Any circuit of maximum average weight is called a critical circuit.*

Definition 2.12 (Transition graph of a matrix) *A transition graph* $\mathcal{T}(A)$ *associated with A depicts the node-to-node transitions,* (a_{ij} *being the weight of the directed arc* $j \rightarrow i$), *as shown in Fig. 1.*

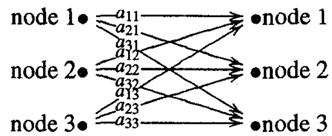

Figure 1: The transition graph of A, a (3×3) matrix.

Remark 2.1 In view of Definitions 2.4, 2.9 and 2.12, the matrix product $C = A \otimes B$ may be visualised as the concatenation of two transition graphs in the order shown in Fig. 2.

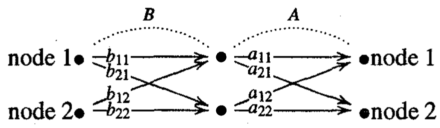

Figure 2: Multiplication of two (2×2) matrices in max-plus algebra.

Definition 2.13 (Geometrical Equivalence) *We speak of geometrical equivalence between* $\mathcal{G}(A)$ *and* $\mathcal{G}(B)$, *when the two graphs are identical in the number of nodes, the critical circuit(s), and also in the distribution of* ϵ*-elements.*

Definition 2.14 (Irreducibility, Strongly connectedness) *A square matrix A is irreducible if no permutation matrix P exists such that the matrix* $\tilde{A}$ *defined as*

$$\tilde{A} = P^T \otimes A \otimes P$$

has an upper triangular block structure. In the max-plus context, upper triangular means that the elements in the lower triangular portion all have the numerical value $\epsilon = -\infty$.

The precedence graph associated with an irreducible matrix A is called strongly connected, to reflect the consequence of irreducibility that there always exists an elementary path between nodes i *and* j, $\forall i, j$. *The converse also holds.*

Definition 2.15 (Aperiodicity) *An irreducible square matrix A is aperiodic*[2] *if an* N *exists such that for all* $k \geq N$ *and for all* i, j, *it holds that* $(A^k)_{ij} \neq \epsilon$.

Lemma 2.1 *An irreducible matrix A such that* $a_{ii} \neq \epsilon$ *for at least one i, is aperiodic*[3].

Proof: From the definition of irreducibility, there exists an integer M such that $(A^M)_{jk} \neq \epsilon$ for each (j, k) pair. For a particular (j, k) pair, call the smallest of such values $M(j, k)$. Now, further suppose that node i contains a circuit of length 1, i.e. $a_{ii} \neq \epsilon$. The finiteness of $(A^{M(j,k)})_{jk}$ is equivalent to the existence of a path from node k to node j, and $M(j, k)$ then represents the minimum path length over all paths between the given nodes. However, $(A^{M(j,k)+1})_{jk}$ need not be finite in general. Now we can construct a path $k \rightarrow\rightarrow j$ by concatenating $k \rightarrow\rightarrow i$, followed by $i \rightarrow\rightarrow j$, so that $(A^{N(j,k)})_{jk} \neq \epsilon$, where $N(j, k) = M(i, k) + M(j, i)$. The existence of such a path is implied by the assumption of irreducibility.

[2] Analogous to the notion of primitivity in conventional linear algebra.

[3] This is sharper than a similar lemma in [1], which requires every diagonal entry be finite.

Now, as $a_{ii} \neq \epsilon$, the path lengths $N(j,k)+1, N(j,k)+2$, etc. for paths joining nodes k to j, can be obtained by concatenating a path $k \to\to i$ of length $M(i,k)$ and cycling through the length-1 $i \to i$ circuit an appropriate number of times, and concatenating a path $i \to\to j$ of length $M(j,i)$. This implies that there will always be a path from node k to node j of any length $n \geq N(j,k)$. Consequently, $(A^n)_{jk} \neq \epsilon$ for all $n \geq N$ for all j, k, where $N = \max_{j,k}(N(j,k))$. ∎

We now state an extension to the concept of aperiodicity involving product of different matrices.

Lemma 2.2 *Let $\mathcal{X} = \{X_1, X_2, \ldots, X_N\}$ be a set of geometrically equivalent irreducible $(n \times n)$ matrices, all have at least one commonly located finite diagonal element, in the $(1,1)$ position, say. Consider a product of matrices chosen from this set in some arbitrary order. For a sufficiently large number of terms in this product (the number being independent of the particular factors of the product), the resulting product matrix will consist exclusively of finite entries.*

Proof: Similar to that of Lemma 2.1, since the argument relies on the existence of paths, i.e. finite path weight. ∎

3 Single Critical Circuit of Length 1

In this section we shall make the following assumptions:

Assumption 3.1 *The matrices A_i, $i \in \{1, 2, \ldots, k\}$, are chosen from a set $\mathcal{X} = \{X_1, X_2, \ldots, X_N\}$ of geometrically equivalent irreducible matrices, each with a unique critical circuit of length 1, at node 1.*

Assumption 3.2 [4] *The matrices A_i have a critical circuit weight of zero, for all i.*

Each constituent matrix of the product in (1.1) has the same unique critical circuit of length 1. It follows that any homogeneous product, i.e. power, of such a matrix has the same property. This motivates us to study a condition under which an inhomogeneous product of the type (1.1) also has the same critical circuit property as its constituents[5]. Note that while a power of a matrix with a unique critical circuit of length 1 also has such a critical circuit, it is not necessarily that the case that the power is geometrically equivalent to the original matrix, e.g. powers of an aperiodic matrix. By analogy, we are not seeking condition for all inhomogeneous products of a set of matrices to be geometrically equivalent to the original matrices.

The set of matrices under consideration in this section will now be characterised accordingly.

Lemma 3.1 *Consider a set of $(n \times n)$ matrices $\mathcal{X} = \{X_1, X_2, \ldots, X_N\}$ obeying assumptions 3.1 and 3.2. Let $A_{\text{sup}} = X_1 \oplus X_2 \oplus \ldots \oplus X_N$. If A_{sup} has a unique critical circuit of length 1, it is necessarily of average weight 0 at node 1, and any product of terms chosen from $\mathcal{X}$ is guaranteed to have the same unique length-1 critical circuit at node 1 as A_{sup}. Conversely, if any product of terms chosen from $\mathcal{X}$ has a unique length-1 critical circuit at node 1, A_{sup} must have this property.*

[4] If this is not the case, a given matrix can be 'normalised' by subtracting (in the conventional sense) the maximum circuit weight from it.

[5] As we shall see, this will ensure the rank 1 property for sufficiently long inhomogeneous products

Proof: Let us first observe why, if A_{sup} has a unique critical circuit of length 1, it is necessarily of weight 0 at node 1. As each $X_i \in \mathcal{X}$ has this property, every diagonal entry of every X_i (except for the $(1,1)$ entry) is negative, the definition of A_{sup} ensures that $(A_{\text{sup}})_{11} = 0$, and $(A_{\text{sup}})_{ii} < 0$ for $i > 1$. Hence if A_{sup} is known to have a unique length-1 critical circuit, it is necessarily of weight 0 at node 1.

To complete the proof, we note that by the definition of A_{sup}, it is clear that for all choices of $A_k, \ldots, A_1$ selected from $\mathcal{X}$

$$\begin{aligned}\left[A_{\text{sup}}^k\right]_{mn} &\geq \left[A_k \otimes A_{k-1} \otimes \ldots \otimes A_1\right]_{mn} \\ &= \max_{A_k, A_{k-1}, \ldots, A_1} \left[A_k \otimes A_{k-1} \otimes \ldots \otimes A_1\right]_{mn}\end{aligned}$$

However, the maximal weight path $\left[A_{\text{sup}}^k\right]_{mn}$ is also the sum of a set of weights of the form

$$\left[A_{\text{sup}}^k\right]_{mn} = \left[A_{\text{sup}}\right]_{mi_{k-1}} + \left[A_{\text{sup}}\right]_{i_{k-1}i_{k-2}} + \ldots + \left[A_{\text{sup}}\right]_{i_1 n}$$

which, for a particular choice of $A_k, A_{k-1}, \ldots, A_1$, leads to

$$\begin{aligned}\left[A_{\text{sup}}^k\right]_{mn} &= a(k)_{mi_{k-1}} + a(k-1)_{i_{k-1}i_{k-2}} + \ldots + a(1)_{i_1 n} \\ &\leq \max_{i_{k-1}, \ldots, i_1} \left[a(k)_{mi_{k-1}} + \ldots + a(1)_{i_1 n}\right] \\ &= \left[A_k \otimes A_{k-1} \otimes \ldots \otimes A_1\right]_{mn}\end{aligned}$$

Therefore

$$\left[A_{\text{sup}}^k\right]_{mn} = \max_{A_k, A_{k-1}, \ldots, A_1} \left[A_k \otimes A_{k-1} \otimes \ldots \otimes A_1\right]_{mn} \quad (3.1)$$

Consider an arbitrary product $\Gamma(k) = A_k \otimes A_{k-1} \otimes \ldots \otimes A_1$. Since A_{sup} has, at node 1, a unique critical circuit of length 1 and average weight 0, any powers of A_{sup} has the same property, then we have

$$\begin{aligned}0 &= (A_{\text{sup}}^k)_{11} \\ &\geq \Gamma(k)_{11} \\ &\geq a(k)_{11} + a(k-1)_{11} + \ldots + a(1)_{11} = 0\end{aligned}$$

The first inequality follows because of (3.1), and the second because the weight of the path $1 \to 1 \to \cdots \to 1$ is a lower bound on the maximum weight path $1 \to\to 1$ of length k yielding $\Gamma(k)_{11}$. It follows that $\Gamma(k)_{11} = 0$.

Suppose for some choice of the A_i there is a critical circuit $i \to i_1 \to \cdots \to i_{k-1} \to i$ for $\Gamma(k)$. Its weight is necessarily underbounded by $\Gamma(k)_{11}$. Then we would have

$$\begin{aligned}&(A_{\text{sup}}^k)_{ii_{k-1}} + (A_{\text{sup}}^k)_{i_{k-1}i_{k-2}} + \ldots + (A_{\text{sup}}^k)_{i_1 i} \\ &\qquad \geq \Gamma(k)_{ii_{k-1}} + \Gamma(k)_{i_{k-1}i_{k-2}} + \ldots + \Gamma(k)_{i_1 i} \geq 0\end{aligned}$$

However, A_{sup} and all its powers have a unique critical circuit of length 1 at node 1, with weight 0. Hence the above inequality can hold only with $i_{k-1} = i_{k-2} = \ldots = i_1 = i = 1$, and it must hold with equality. This establishes that $\Gamma(k)$ has the same unique length-1 critical circuit as A_{sup}.

For the converse, suppose (to obtain a contradiction) A_{sup} has a critical circuit consisting of $i \to i_1 \to \cdots \to i_{j-1} \to i$, where either $j > 1, i \neq 1$ or both. Since A_{sup} has a circuit $1 \rightarrow\rightarrow 1$ of weight 0, the total weight of the $i \rightarrow\rightarrow i$ circuit must be at least zero, that is, for some $X_{\alpha i}$

$$\begin{aligned} 0 &\leq (A_{\text{sup}})_{ii_{j-1}} + (A_{\text{sup}})_{i_{j-1}i_{j-2}} + \dots + (A_{\text{sup}})_{i_1 i} \\ &= (X_{\alpha_j})_{ii_{j-1}} + (X_{\alpha_{j-1}})_{i_{j-1}i_{j-2}} + \dots + (X_{\alpha_1})_{i_1 i} \\ &\leq \Gamma(j)_{ii} \end{aligned} \tag{3.2}$$

where $\Gamma(j) = X_{\alpha_j} \otimes X_{\alpha_{j-1}} \otimes \dots \otimes X_{\alpha_1}$.

If $i \neq 1$ this shows that the particular $\Gamma(j)$ in question has a critical circuit other than $1 \rightarrow\rightarrow 1$. Suppose $i = 1$ and $j \neq 1$ so that $i_1 \neq 1$, define

$$\bar{\Gamma}(j) = X_{\alpha_1} \otimes X_{\alpha_j} \otimes \dots \otimes X_{\alpha_2}$$

and observe that

$$\begin{aligned} \bar{\Gamma}(j)_{i_1 i_1} &\geq (X_{\alpha_1})_{i_1 1} + (X_{\alpha_j})_{1 i_{j-1}} + \dots + (X_{\alpha_2})_{i_2 i_1} \\ &\geq 0 \end{aligned} \tag{3.3}$$

Hence there is a circuit $i_1 \rightarrow\rightarrow i_1$ for $\bar{\Gamma}(j)$ with weight at least 0, and the actual critical circuit for $\bar{\Gamma}(j)$ is not a unique length-1 circuit at node 1. The hypothesis is contradicted, so it must be the case that A_{sup} has a unique critical circuit of length 1 at node 1. ■

In the statement of the following lemma, the term circuit is used to denote a path on a transition graph associated with an inhomogeneous product, where the beginning and end nodes have the same index. A companion result to Lemma 3.1 is as follows.

Lemma 3.2 *Adopt the hypotheses as Lemma 3.1, and suppose that A_{sup} has a unique critical circuit of length 1 at node 1, with weight 0. Consider a transition graph associated with a product of terms chosen arbitrarily from $\mathcal{X}$, and let P be a path in the graph of arbitrary length, beginning and ending at node 1, with no intermediate visits to node 1. Then there exists a positive constant δ such that the average weight of P is over bounded by $-\delta$.*

Proof: Let the path P be $1 \rightarrow i_1 \rightarrow i_2 \rightarrow \dots \rightarrow i_{p-1} \rightarrow 1$. If $p = l(P) \geq n$, P necessarily contains a circuit. We can write P as the concatenation of P_1, P_c, P_2, with P_1: $1 \rightarrow\rightarrow j$, P_c: $j \rightarrow\rightarrow j$ and P_2: $j \rightarrow\rightarrow 1$, and $l(P_1) + l(P_2) < n$. Now the weights of P_1, P_c, P_2, i.e. $w(P_1), w(P_c), w(P_2)$, are overbounded by the weights of the same path computed for A_{sup}. Call these weights $w_{\text{sup}}(P_1)$, etc. Thus

$$\begin{aligned} w(P) &= w(P_1) + w(P_c) + w(P_2) \\ &\leq w_{\text{sup}}(P_1) + w_{\text{sup}}(P_c) + w_{\text{sup}}(P_2) \end{aligned} \tag{3.4}$$

Let P_{red} denote the path P_1 concatenated with P_2. Then P_{red} is a circuit from 1 to 1 of length greater than 1 but no greater than $n-1$ arcs. Since A_{sup} has a unique critical circuit, with weight 0 and length 1,

$$w_{\text{sup}}(P_{\text{red}}) < 0$$

The set of all such P_{red} is a finite set, since any P_{red} has finite length. Hence there exists $\eta_1 < 0$ such that

$$w_{\text{sup}}(P_1) + w_{\text{sup}}(P_2) \leq \eta_1 < 0$$

for all possible P_1, P_2.

Next, the unique critical circuit property of A_{sup} guarantees that for any $j \in \{2, 3, \dots, n\}$

$$\frac{w_{\text{sup}}(P_c)}{l(P_c)} \leq \eta_2 < 0$$

for some η_2. Since $l(P_1) + l(P_2) < n$, it follows easily that for some $\delta > 0$,

$$\frac{w(P)}{l(P)} \leq -\delta < 0$$

■

A related result is as follows.

Lemma 3.3 *Consider a set of $(n \times n)$ matrices $\mathcal{X}$ obeying Assumptions 3.1 and 3.2, with A_{sup} as defined in Lemma 3.1. Let P denote a path of length l_p in the transition graph associated with a product of terms chosen arbitrarily from $\mathcal{X}$, starting and ending at node i, $i \in \{2, 3, \dots, n\}$. Then any such P will have an average circuit weight of at most $-\delta < 0$, where $-\delta$ denotes either the maximum of the average circuit weight of all possible $1 \rightarrow\rightarrow 1$ circuits including nodes other than 1 but without intermediate visits to node 1, or the maximum of the $i \rightarrow i$ length-1 self-circuits, $\forall i \in \{2, 3, \dots, n\}$, depending on which is the larger.*

Proof: To obtain a contradiction, we first assume the existence of an $i \rightarrow\rightarrow i$ circuit Q, of length l_q, and having an average weight $\eta > -\delta$. We can proceed to construct a $1 \rightarrow\rightarrow 1$ circuit by the following means: concatenating an elementary $1 \rightarrow\rightarrow i$ path of length l_1 with t copies of the $i \rightarrow\rightarrow i$ circuit Q, followed by another elementary $i \rightarrow\rightarrow 1$ path of length l_2 (Fig. 3).

The new path $\tilde{Q}$, which does not visit node 1 except at the ends, has average weight $[w(1 \rightarrow\rightarrow i) + t\eta l_q + w(i \rightarrow\rightarrow 1)]/(l_1 + tl_q + l_2)$. As $t \rightarrow \infty$, the average weight of $\tilde{Q}$ approaches that of Q, $\eta > -\delta$. This contradicts the original assumption that the maximum average weight of any $1 \rightarrow\rightarrow 1$ circuit which only includes node 1 at the end points has an average weight of less than $-\delta$. Consequently any $i \rightarrow\rightarrow i$ circuit, $i \in \{2, 3, \dots, n\}$, must have an average weight of at most $-\delta$. ■

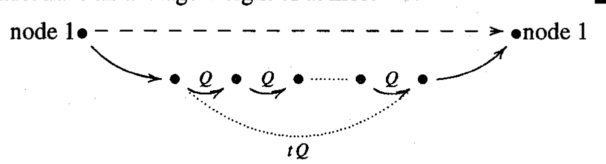

Figure 3: Forming a $1 \rightarrow\rightarrow 1$ circuit by concatenating $t \times Q$ circuits.

Corollary 3.1 *Adopt the same hypotheses as in Lemma 3.3. Consider the maximum weight path from node i to node 1 over all possible lengths with no intermediate visits to node 1, $i \in \{2, 3, \dots, n\}$. This path contains no circuits, and is also of bounded length of less than n. Similarly, the maximum weight path from node 1 to node i contains no circuits, and is of bounded length.*

Proof: Suppose the maximal weight $i \rightarrow\rightarrow 1$ path contains a circuit; then the circuit would involve nodes from $\{2, 3, \dots, n\}$. However, as shown in Lemma 3.3, the average weight of such a circuit is at most $-\delta < 0$, hence by removing this circuit, the path weight can be increased. This contradicts the assumption that the path is of maximum weight. Further, as the number of nodes is finite, absence of circuits means the path length is at most $n-1$. The same arguments apply to $1 \to\to j$ paths. Note that in the case of an $i \to\to i$ path, this upper bound is n instead of $n-1$. ∎

In view of the above results, we can present our main result guaranteeing a rank 1 product as follows.

Theorem 3.1 *Consider a max-plus product of the form:* $\Gamma(k) = A_k \otimes A_{k-1} \otimes \ldots \otimes A_1$, *where each* A_l, $l = 1, 2, \ldots, k$, *is selected from a geometrically equivalent set* $\mathcal{X} = \{X_1, X_2, \ldots, X_N\}$, *consisting of* $(n \times n)$ *irreducible matrices, obeying Assumptions 3.1 and 3.2. If*

$$A_{\sup} \overset{\text{def}}{=} X_1 \oplus X_2 \oplus X_3 \oplus \ldots \oplus X_N$$

has a unique critical circuit of length 1 at node 1, with average weight of 0, then there exists a finite K_{crit} *such that for all* $k \geq K_{crit}$, $\text{rank}(\Gamma(k)) = 1$, *i.e.* $\Gamma(k) = U \otimes V'$, *where* U *is an* $(n \times 1)$ *column vector, and* V' *is a* $(1 \times n)$ *row vector.*

Proof: Step 1: We will show that $\Gamma(k)_{ij}$ has a lower bound independent of k as $k \to \infty$, $i, j \in \{1, 2, \ldots, n\}$.

Let P_1 be any $j \to\to 1$ path, and P_3 any $1 \to\to i$ path, of length l_1 and l_3 (independent of k) respectively. Consider a path P of length k from j to i consisting of three segments: P_1, P_2, comprised of $(k - l_1 - l_2)$ repeated $1 \to 1$ transitions, and P_3. Such a path exists for all $k \geq l_1 + l_2$, and then

$$\begin{aligned} \Gamma(k)_{ij} &\geq w(P) \\ &= w(P_1) + 0 + w(P_2) \end{aligned} \qquad (3.5)$$

Step 2: We will show that, for suitably large k, a maximal weight path of the transition graph for $\Gamma(k)$ with weights summing to $\Gamma(k)_{ij}$ necessarily visits node 1.

Suppose that for arbitrarily large k, there is a particular $\Gamma(k)$ such that the maximal weight path P' yielding $\Gamma(k)_{ij}$ does not pass through node 1. The weight of such a path will be less than the weight $w_{\sup}(P')$ of the same path in a transition graph for $A^k_{\sup}$. Let $\bar{A}_{\sup}$ denote $A_{\sup}$ with the first column and row deleted, then the average weight $\bar{\lambda}$ of the critical circuit(s) of $\bar{A}_{\sup}$ is necessarily negative. It follows that all entries of $\bar{A}^k_{\sup}$ for large k will be of order $k\bar{\lambda}$. Now the path P' for $\Gamma(k)$ in $A^k_{\sup}$ does not visit node 1, and thus is also a path of $\bar{A}^k_{\sup}$, so that $\Gamma(k)_{ij} \leq w_{\sup}(P') \leq O(k\bar{\lambda})$ for large k. But this would imply $\Gamma(k)_{ij} \to -\infty$ as $k \to \infty$, a contradiction.

Step 3: We will show that for suitably large k, the path whose arc weights sum to $\Gamma(k)_{ij}$ starts with the maximum weight path from node j to node 1 of length $l_1^* < n$, call it P^*_{1j}, and ends with the maximum weight path from node 1 to node i of length $l_3^* < n$, call it P^*_{3i}, and contains $[k - (l_1^* + l_3^*)]$ arcs $1 \to 1$ in the middle. Note that $l_1^*, l_3^* < n$ follows by Corollary3.1.

Recall that for k suitably large, the maximal weight path P joining node j to i, i.e. $\Gamma(k)_{ij}$, necessarily passes through node 1. Let P_{1j} be the segment of P from j to the first visit to node 1, P_{3i} the last segment of P from the last visit to node 1 to node i, and P_2 the segment of P between the first and last visits to node 1. Note for the time being, we have not stated any restrictions regarding P_2. The definitions of P^*_{1j} and P^*_{3i} ensure that $w(P_{1j}) \leq w(P^*_{1j})$ and $w(P_{3i}) \leq w(P^*_{3i})$. Also, $w(P_2) \leq 0$, since the path $1 \to 1 \to \cdots \to 1$ of length $l(P_2)$ is the unique maximum weight $1 \to\to 1$ circuit. Hence

$$\Gamma(k)_{ij} \leq w(P^*_{1j}) + w(P^*_{3i})$$

On the other hand, P^* comprising of P^*_{1j}, $[k - (l_1^* + l_3^*)]$ transitions $1 \to 1$ and P^*_{3i} defines one path of length k from node j to node i, and its weight necessarily underbounds $\Gamma(k)_{ij}$, i.e.

$$w(P^*_{1j}) + w(P^*_{3i}) \leq \Gamma(k)_{ij}$$

It follows that $\Gamma(k)_{ij} = w(P^*_{1j}) + w(P^*_{3i})$, or in matrix notation

$$\Gamma(k) = \begin{bmatrix} w(P^*_{31}) \\ w(P^*_{32}) \\ \vdots \\ w(P^*_{3n}) \end{bmatrix} \otimes \begin{bmatrix} w(P^*_{11}) & \ldots & w(P^*_{1n}) \end{bmatrix} \quad (3.6)$$

∎

4 Conclusion

In this paper we have concentrated on the asymptotic properties of inhomogeneous products, building upon previous results concerning homogeneous products of matrices [1]. It is seen that the graphical approach naturally lends itself to this type of problem, and we have derived some sufficiency conditions accordingly. We are currently making progress towards tackling the problem of necessary and sufficient conditions.

As shown in Section 3, there exists a finite 'time' for which the the final product will have rank less than or equal to the length of the critical circuit. For a product consisting of matrices with the same single unique critical circuit of length 1, this final rank is 1. However, by using some probabilistic arguments, it can be shown that for product with increasing length, the results for homogeneous and inhomogeneous products of matrices with length-1 critical circuit, with suitable modifications, e.g. grouping of terms in two's, can be applied. That is, even products of matrices with the same critical circuit of length 2 will attain a rank of 1 with probability 1. Nevertheless, the finite-time result is possibly more useful in practice.

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