CHANGE DETECTION IN MARKOV-MODULATED TIME SERIES

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ABSTRACT

In this paper, we address the problem of online change detection of Markov-modulated time series models. For simplicity, we look at Auto-regressive time-series models the parameters of which are modulated by a finite-state homogeneous Markov chain. We propose a Cumulative Sum based statistical test to detect abrupt changes is such processes. Computation of average run length functions, in particular, mean delay in detection and mean time between false alarms are particularly difficult to obtain in closed form for such processes. Although there are ways to approximate such computation, we do not address those issues in this paper. Simulation studies illustrate the detection capability of our proposed test.

1. INTRODUCTION

Detection of abrupt changes in signals and systems is a topic of continued interest and has applications in fault detection in navigational systems, onset detection in seismic signal processing, segmentation of speech signals etc [1]. Such abrupt changes are characterized by small changes of system parameters that can potentially have devastating effects on the behaviour of the system if accumulated over time. An asymptotic local approach to early detection of "slight" changes has been presented in [2] [3]. Most of the signals treated in [1] are time-series models (linear or nonlinear) and algorithms are presented that develop on-line methods for detecting changes of parameters of such models. Recently, in [4], an algorithm for detection of parameter changes in a hidden Markov model using the log-likelihood function has been given. In the case of a hidden Markov model, a CUSUM-procedure is developed since although the observations are not independent before and after the change (generally speaking), one can easily calculate the log-likelihood function provided the parameters before and after the change are known.

In our paper, we develop a similar method for detection of changes in Markov-modulated time series models, where the parameters of the time-series are modulated by a Markov chain, in the sense that the time-series parameters are constant over segments with abrupt changes from segment to segment. Such so-called "segmentation" models are used in econometrics, seismology, geology and image analysis. Algorithms for estimation of such time-series are given

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in [5]. Markov-modulated time-series models can be also viewed as *random coefficient* time-series which are used to model the stochastic stability of short run market equilibrium under variations of supply [6]. It is thus important that we develop an on-line method for detecting changes in the parameter sets and the underlying hidden Markov model modulating the time-series. We present a brief analysis of our method in the next subsection under the assumption that we know the parameter sets before and after the change.

As a remark for situations where the parameters after the change might not be known, we would like to add that one can consider extending the Generalized Likelihood Ratio (GLR) tests for such Markov modulated processes.

2. SIGNAL MODEL AND ON-LINE CHANGE DETECTION ALGORITHM

For simplicity, we take a Markov-modulated autoregressive (AR) process as the basis of our analysis which is given by

$$y_k = a_1(s_k)y_{k-1} + a_2(s_k)y_{k-2} + \ldots + a_m(s_k)y_{k-m} + v_k \tag{1}$$

To make the analysis simple, we also assume that $y_k, v_k \in \mathbb{R}$, s_k is a homogeneous first-order Markov chain belonging to a finite-discrete state space. v_k is a Normally distributed noise process with a density $N(0, \sigma^2)$. The transition probability of the Markov chain is given by $P = (p_{ij})$ where $p_{ij} = P(s_{k+1} = i \mid s_k = j)$, $i, j \in \{1, 2, \ldots, N\}$. The initial probability distribution of s_0 is given by π such that $P(s_0 = i) = \pi(i)$. Hence, the complete parameter space of the Markov-modulated AR process can be specified by $\lambda = (P, \pi, a, \sigma)$ where $a = (a_1\ a_2\ \ldots\ a_m)$ where the dependence of a_i on s_k has been suppressed. We assume that λ can belong to two distinct parameter quadruple λ_H, λ_K.

Remark 2.1 Note that deriving stationarity criteria for Markov-modulated time-series is a difficult problem in the sense that two switching separately second order AR stationary processes can result in an unstable system whereas two individually unstable AR processes can be stabilized when allowed to switch according to a Markov regime. For the sake of our analysis, we assume that our individual AR processes are stable and so is the switched process.

Note from [4] that a sequential CUSUM-like procedure in a manner similar to Page's recursive test can be written as a recursion in the test statistic T_k, $k \in \mathbb{N}$ in the following manner:

$$\begin{aligned} T_k &= \max\{0, T_{k-1} + g(k)\} \\ g(k) &= \log\left(\frac{f_K(y_k \mid y_{k-1}, \ldots, y_0)}{f_H(y_k \mid y_{k-1}, \ldots, y_0)}\right) \end{aligned} \tag{2}$$

where $T_0 = 0$ and f_K, f_H denote the density function when $\lambda = \lambda_K$ or $\lambda = \lambda_H$ respectively.

Obviously $g(k)$ calculates the difference between the log-likelihood functions according to parameter quadruple λ_K and λ_H. Next, we present a formula of calculating this log-likelihood function which can be easily derived.

Note that $\lambda_K = (P^K, \pi^K, a^K, \sigma^K)$ and similarly λ_H can be expressed. Define $B_k^l = diag(b^l(y_k, 1), \ldots, b^l(y_k, N))$ where $b^l(y_k, i) = \exp(-\frac{(y_k - a^{l\prime} Y_{k-1}^{k-m})^2}{2\sigma^{l2}})$ denotes the probability density function of observing y_k given that the state of the Markov chain is i under the parameter quadruple λ^l, $l = K, H$ where $Y_{k-1}^{k-m} \triangleq (y_{k-1}, \ldots, y_{k-m})'$.

Define the following forward variable $\alpha_k^l \triangleq (\alpha_k^l(1), \ldots, \alpha_k^l(N))'$ where obviously $\alpha_k^l \in \mathbb{R}^N$, such that the following recursion in α_k^l holds:

$$\begin{aligned} \alpha_k^l &= B_k^l P^l \frac{\alpha_{k-1}^l}{\sum_i \alpha_{k-1}^l(i)} \\ \alpha_0^l &= B_0^l \pi^l \end{aligned} \tag{3}$$

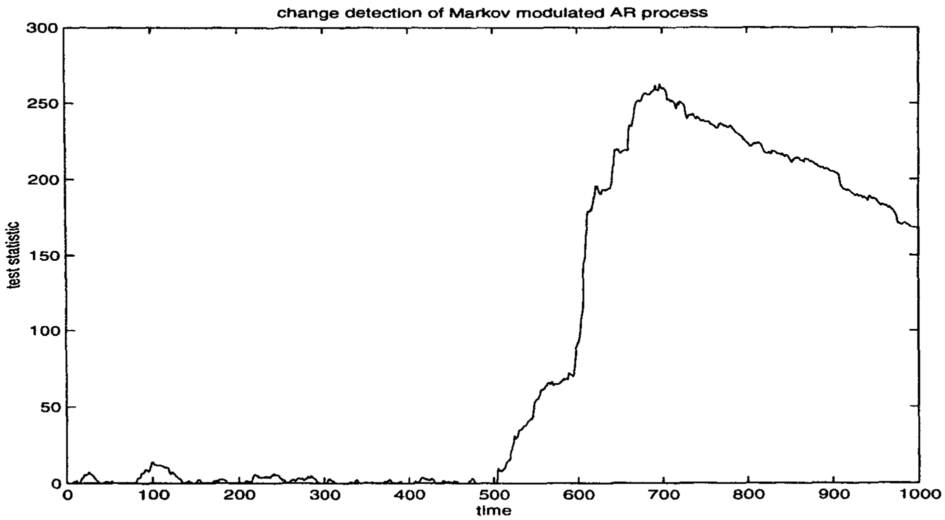

Figure 1: Plot of test statistic versus time for on-line detection of changes in Markov-modulated AR process

Remark 2.2 Note that in the right hand side of the first equation in (3), α^l_{k-1} is normalized to avoid numerical problems.

It is easy to show that according to the above recursion, $\alpha^l_k(j)$ is equal to the quantity $\frac{f_l(y_k, y_{k-1}, \dots, y_0, s_k=j|\lambda_l)}{f_l(y_{k-1}, \dots, y_0|\lambda_l)}$. Then it easily follows that $f_l(y_k \mid y_{k-1}, \dots, y_0)$, $l = K, H$ is given by

$$f_l(y_k \mid y_{k-1}, \dots, y_0) = \sum_j \alpha^l_k(j) \tag{4}$$

3. SIMULATION RESULTS

In this subsection, we present some simulation results with a second order Markov-modulated time-series modulated by a Markov chain that can take values in a 4-dimensional state space. We assume that the time-series changes from a parameter quadruple λ_H to λ_K after the first 500 points and then changes back to λ_H after the next 200 points. We choose two different transition probability matrices P^H, P^K, two very different AR parameter sets a^H, a^K and two different noise variances ${\sigma^H}^2, {\sigma^K}^2$.

Here are the details of the simulations: $s_k \in \{1,2,3,4\}$,

$$P^H = \begin{pmatrix} 0.2 & 0.4 & 0.2 & 0.7 \\ 0.4 & 0.2 & 0.3 & 0.1 \\ 0.2 & 0.3 & 0.2 & 0.1 \\ 0.2 & 0.1 & 0.3 & 0.1 \end{pmatrix} \tag{5}$$

$$P^K = \begin{pmatrix} 0.7 & 0.15 & 0.12 & 0.23 \\ 0.1 & 0.7 & 0.1 & 0.1 \\ 0.1 & 0.05 & 0.68 & 0.17 \\ 0.1 & 0.1 & 0.1 & 0.5 \end{pmatrix} \tag{6}$$

The AR parameters are given by the following matrices A^l, $l = H, K$ where $A^l_{ij} = a_j(i)$ given $\lambda = \lambda_l$. We used

$$A^H = \begin{pmatrix} 0.9 & 0.4 \\ 0.4 & 0.9 \\ -0.4 & 0.9 \\ -0.7 & 0.2 \end{pmatrix} \tag{7}$$

and

$$A^K = \begin{pmatrix} 0.2 & 0.8 \\ 0.8 & 0.2 \\ -0.5 & 0.5 \\ 0.2 & 0.6 \end{pmatrix} \tag{8}$$

Also, $\sigma^H = 1.0$, $\sigma^K = 3.0$.

The following figure shows the plot of the test statistic with time clearly showing the changes at $k = 500$ and $k = 700$.

4. CONCLUSIONS AND FUTURE WORK

We did not address the issue of computation of mean time between false alarms or mean delay in detection (more generally the average run length (ARL function)) because closed form computation of these quantities are virtually impossible when the observations are dependent. However, simulation studies can be performed to obtain empirical values of these quantities like in [4]. We believe that the behaviour of these quantities as a function of the detection threshold will be similar to those in [4].

There is also a need to obtain algorithms for change detection for more complicated Markov-modulated or semi-Markov processes or even long-range dependent processes. These issues are currently under investigation.

5. REFERENCES

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