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On Kalman Filtering With Faded Measurements

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Abstract— **This paper considers a sensor network where single or multiple sensors amplify and forward their measurements of a common linear dynamical system (analog uncoded transmission) to a remote fusion centre via noisy fading wireless channels. We show that the expected error covariance (with respect to the fading process) of the time-varying Kalman filter is bounded and converges to a steady state value, based on some general earlier results on asymptotic stability of Kalman filters with random parameters. More importantly, we provide explicit expressions for sequences which can be used as upper bounds on the expected error covariance, for specific instances of fading distributions and scalar measurements (per sensor). Numerical results illustrate the effectiveness of these bounds.**

I. Introduction

Due to a recent steady growth of activity in sensor networks with a large number of nodes monitoring an environment/object in various applications, multi-sensor based estimation of random processes under limited resources and communication constraints have led to new challenging filtering problems. In particular, estimation of dynamical systems based on multiple sensors under these constraints is known to be a potentially hard problem. Motivated by the asymptotic optimality of analog forwarding based communication [1] for a Gaussian sensor network, we focus on a similar amplify and forward strategy based sensor network estimating a linear dynamical system. In this scenario, the sensors simply amplify and forward their measurements of the linear system to a remote fusion centre via fading channels, where they are received in noise. Assuming perfect phase synchronization, the fading channel power gains are modelled as positive random ergodic processes with continuous distributions with an independent and identically distributed nature from one transmission time to the next. The optimal state estimation filter at the fusion centre is still a (time-varying) Kalman filter. Using some rather general asymptotic stability results for linear systems with ergodic parameters from [2], [3], we show that the expected (with respect to the fading process) error covariance matrix of the Kalman filter remains bounded and converges to a steady state matrix from arbitrary positive semidefinite matrices as initial conditions. This result is in contrast with recent results in [4] which show that the expected error covariance matrix for unstable systems in a situation where measurements can be lost with a non-zero probability can become unbounded if this loss probability exceeds a certain threshold. While this observation may not be surprising from the results and discussions in [3] and [5], we believe that this observation needs to be made (some partial results for the scalar case for Rayleigh fading were reported in [6]) in a general sense. In addition, for special cases of vector state and scalar measurements, and scalar state and measurements (for both single sensor and multiple sensor scenarios), and specific fading distributions, we provide explicit bounding matrix (or scalar) sequences that overbounds the expected error covariance matrix and also converges to a steady state value. These bounds provide a simple way to compute realistic (and often quite tight) bounds on the expected error covariance, and can be quite useful in situations when one wants to minimize the expected error covariance for such sensor network based estimation problems to optimally allocate resources across multiple sensors. When an exact recursive expression for the average error covariance is not available, one can minimize its upper bound instead for which we provide exact recursive formulas. Problems of this nature can be solved by dynamic programming techniques and will be addressed elsewhere.

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II. System Model

We consider a discrete-time linear time invariant system that represents a phenomenon of interest (for example, the trajectory of a moving object) given by

$$x_{k+1} = Ax_k + w_k \tag{1}$$

where $x_k \in \mathbb{R}^n,\ w_k \in \mathbb{R}^n, A \in \mathbb{R}^{n\times n}$ and w_k follows a Gaussian distribution with zero mean and variance $\Sigma_w \geq 0$. Note here that a matrix $V \geq 0$ implies that V is a positive semidefinite matrix. Similarly, $V > 0$ implies V is a positive definite matrix. We also assume that the initial distribution of x_0 is Gaussian with mean zero and covariance matrix $P_0 \geq 0$. Note that in this work, we allow A to be an unstable matrix. In fact the results presented in this paper are interesting only when A is unstable, just as in [4].

This system is observed by a sensor or a number of sensors which yield discrete-time measurements of the state of the system. These measurements are then sent over a wireless medium to a central processing unit called the Fusion Centre (FC). We assume that the sensors use analog forwarding [1] to send the measurements to the FC, i.e, they simply amplify and forward their measurements to the FC. Due to the randomly time-varying nature of the wireless medium,

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the FC then receives faded versions of all the measurements in additive noise either separately (orthogonal access) or as a sum of all received measurements in noise (non-orthogonal access). We consider the single sensor and multiple sensor case separately.

Single sensor case

In this case, the linear time invariant system (1) is observed by a single sensor which produces a discrete-time measurement y_k which is given by

$$y_k = Cx_k + v_k \tag{2}$$

where $y_k \in \mathbb{R}^m,\ v_k \in \mathbb{R}^m,\ C \in \mathbb{R}^{m\times n}$. We assume that v_k is Gaussian distributed with zero mean and variance $\Sigma_v > 0$. Denoting the i-th element of the measurement and measurement noise vectors as y_k^i and v_k^i respectively where $i = 1, 2, \dots, m$, we assume that the sensor transmitter amplifies the component y_k^i by a factor α_k^i and sends it to the FC over a fading channel with channel gain $h_{k,i}$. We assume that the channel undergoes slow fading such that the phase of the complex channel can be estimated and compensated for at the receiver, so that essentially $h_{k,i}$ represents the real-valued envelope of the complex channel gain. We also assume that the channel gain remains constant over the time interval to send the i-th component, $i = 1, 2, \dots, m$ but one can have $h_{k,i} \neq h_{k,j},\ i \neq j,\ i,j \in \{1, 2, \dots, m\}$. This assumption is valid when each measurement interval is much larger than the coherence time of the fading channel, which is likely to be the case in low bandwidth sensor network applications. Denoting $h_k = (h_{k,1}\ h_{k,2} \dots h_{k,m})$, we also assume that h_k is independently and identically distributed according to a continuous fading distribution $f(h)$ such that $P(h_{k,i} > 0) = 1,\ \forall k, i$.

The FC receives a scaled version of each component of the measurement vector added with measurement noise in additive noise at the FC, which represents the channel noise in the communication channel between the sensor and the FC. We assume that all the measurement components are sent separately to the FC via orthogonal channels within the measurement time interval. The received signal vector at the FC then can be written as

$$z_k = H_k C x_k + H_k v_k + n_k \tag{3}$$

where $H_k = diag(\alpha_k^1 h_{k,1}\ \alpha_k^2 h_{k,2} \dots \alpha_k^m h_{k,m})$ and $n_k = (n_k^1\ n_k^2 \dots n_k^m)'$ represents the channel noise vector. For simplicity, we assume that n_k^i, n_k^j are mutually independent for $i \neq j$ and n_k^i is Gaussian distributed with zero mean and variance σ_i^2. Thus, n_k is Gaussian distributed with zero mean and variance $\Sigma_n = diag(\sigma_1^2\ \sigma_2^2 \dots \sigma_m^2)$. We also assume that n_k is white.

For simplicity, in this paper we will also assume that $\alpha_k^i = 1$ for all $i = 1, 2, \dots, m$ and for all $k = 1, 2, \dots$. As a result, we now have $H_k = diag(h_{k,1}\ h_{k,2} \dots h_{k,m})$. The overall state-space model for this system can now be written as

$$x_{k+1} = Ax_k + w_k,\ z_k = H_k C x_k + \bar{v}_k \tag{4}$$

where $\bar{v}_k = H_k v_k + n_k$. Since H_k is a diagonal matrix, it is easy to see that $\bar{v}_k$ is Gaussian distributed with zero mean and time-varying covariance matrix $R_k = H_k \Sigma_v H_k + \Sigma_n$.

Assumption 2.1: We make the standard assumption that the pair (A, Σ_w) is stabilizable and the pair (A, C) is detectable.

For technical reasons, in order to use some results from [3] later, we will also make the assumption:

Assumption 2.2: (i) A is invertible, and (ii) $\max(0, \log ||H_0 C||)$ is integrable. (iii) $\{H_k\}$ is stationary and ergodic.

For discrete time systems which are obtained by discretizing a continuous time system, Assumption 2.2 (i) will be satisfied. Assumption 2.2 (ii) is also satisfied by common fading distributions such as Rayleigh and Nakagami. Assumption (iii) is automatically satisfied since we assumed $\{H_k\}$ to be i.i.d., though the results of [3] can also hold in more general cases where the channel has memory.

In what follows, we will also consider several special cases of the above general model (4). In particular, we will consider the following scalar state/scalar measurement model:

$$x_{k+1} = ax_k + w_k,\ z_k = h_k x_k + h_k v_k + n_k \tag{5}$$

where x_k, z_k, w_k, v_k, n_k are all scalar random processes (with $w_k \sim N(0, \sigma_w^2),\ v_k \sim N(0, \sigma_v^2),\ n_k \sim N(0, \sigma_n^2)$). We have taken $c = 1$ without loss of any generality, as both sides can be scaled by c to obtain the model described above. Just as before, we assume that $\{h_k\}$ is a sequence of i.i.d random variables distributed according to a continuous fading distribution $f(h)$ such that $P(h_k > 0) = 1,\ \forall k$.

In addition, we will consider a vector state/scalar measurement model for the single-sensor case given by

$$x_{k+1} = Ax_k + w_k,\ z_k = h_k \bar{c} x_k + h_k v_k + n_k \tag{6}$$

where $\bar{c} \in \mathbb{R}^{1\times n}$ is given by $\bar{c} = (\bar{c}_1, \bar{c}_2, \dots, \bar{c}_n)$ and v_k, n_k are scalars, where x_k, w_k follow the same model as in (4), and $n_k, \{h_k\}$ are described by the same models as in (5).

Multisensor case

In the multisensor case, we assume that the dynamical process (1) is observed by M sensors each producing a scalar measurement. While the general results on convergence and bounds derived in the next section for the single sensor case can be extended to the multisensor case, for simplicity, we stick to a scalar state and scalar measurement (per sensor) model. The sensors can then communicate their measurements to the FC via either a multi-access channel [1] (where all sensors transmit simultaneously without the time/frequency division multiplexing) or via orthogonal channels [7]. In the multi-access case, we assume that the phase shift in each channel is compensated by distributed transmit beamforming so that the measurements from all sensors add up coherently at the FC. Mathematically, the signal model for the multisensor multi-access case is given by

$$x_{k+1} = ax_k + w_k,\ z_k = \sum_{i=1}^{M} h_{k,i}(c_i x_k + v_k^i) + n_k. \quad (7)$$

We assume that $v_k^i \sim N(0, \sigma_{vi}^2)$, $n_k \sim N(0, \sigma_n^2)$ and v_k^i, v_k^j are mutually independent for $i \neq j,\ i,j \in \{1,2,\ldots,M\}$. Similarly, $\{h_{k,i}\},\{h_{k,j}\}$ are statistically independent for $i \neq j,\ i,j \in \{1,2,\ldots,M\}$. Note that $h_{k,i}$ may not be identically distributed for all i. Due to the distributed transmit beamforming assumption, note that $h_{k,i}, \forall i$ denotes the non-negative channel fading amplitude. We also assume here that $c_i > 0$ for all sensors without loss of generality. Note that if c_l for sensor l is negative, one can choose the amplification factor for this sensor as -1 instead of 1. This assumption is there to ensure that the distributed transmit beamforming scheme works effectively.

In the orthogonal access case, the FC simply receives a vector consisting of the individual *faded* measurements from all the sensors. We can write the signal model as

$$x_{k+1} = ax_k + w_k,\ z_k^i = h_{k,i} c_i x_k + h_{k,i} v_k^i + n_k^i, i = 1,\ldots,M \quad (8)$$

where z_k^i denotes the received signal (at the FC) from the i-th sensor and n_k^i is the channel noise for the i-th sensor's channel. In this case, the FC observation consists of the vector $(z_k^1, z_k^2, \ldots, z_k^M)'$ and the other modelling assumptions regarding $\{h_{k,i}\}, v_k^i$ remain the same as in (7). Note that unlike (7) however, there is no need to assume $c_i > 0, \forall i$ in this case.

III. Convergence results and bounds on the expected error covariance matrix

In this section, we present some convergence and boundedness results on the average (over the channel fading distribution) error covariance matrix for the optimal one step ahead predictor for the vector state vector measurement system (4). Later we will specialise these results for specific fading distributions. Using the knowledge of these distributions and inequalities involving some special functions, we derive more specific bounds for these cases.

We assume that the FC has full knowledge of the system matrices and noise covariances including the time-varying channel fading matrices H_k. The above state-space model (4) is a linear time-varying system and the optimal predictor (filter) for this system is a time-varying Kalman predictor (filter), that can be constructed at the FC. Denote $\mathcal{Z}_k = (z_1, z_2, \ldots, z_k)$ and $\mathcal{H}_k = (H_1, H_2, \ldots, H_k)$, and define the one step ahead optimal predictor and its error covariance as

$$\hat{x}_{k+1|k} = E[x_{k+1}|\mathcal{Z}_k, \mathcal{H}_k]$$
$$P_{k+1|k} = E[(x_{k+1} - \hat{x}_{k+1|k})(x_{k+1} - \hat{x}_{k+1|k})'|\mathcal{Z}_k, \mathcal{H}_k] \quad (9)$$

where $'$ denotes the transpose operation. In the following, we will use the shorthand notation P_{k+1} for $P_{k+1|k}$. Using the time-varying Kalman filtering equations, one can easily derive that the prediction error covariance matrix P_k satisfies the following discrete-time time-varying Riccati equation:

$$\begin{aligned} P_{k+1} =& AP_kA' + \Sigma_w \\ &- AP_kC'H_k(H_kCP_kC'H_k + R_k)^{-1}H_kCP_kA' \end{aligned} \quad (10)$$

with $R_k = H_k\Sigma_v H_k + \Sigma_n$. It is straightforward to show that the above equation can be rewritten as

$$P_{k+1} = A[P_k - P_kC'(L_k^{-1} + CP_kC' + \Sigma_v)^{-1}CP_k]A' + \Sigma_w \quad (11)$$

where $L_k = H_k\Sigma_n^{-1}H_k$. It should be obvious that $E_{\mathcal{H}_k}[P_{k+1}] = E_{\mathcal{H}_{k-1}}[G(P_k)]$ where the expectation is taken with respect to the channel realization history $\mathcal{H}_k = \{H_1, H_2, \ldots, H_k\}$, and

$$\begin{aligned} &G(P_k) \\ &= E_{H_k}[A[P_k - P_kC'(L_k^{-1} + CP_kC' + \Sigma_v)^{-1}CP_k]A' + \Sigma_w|P_k] \\ &= A[P_k - P_kC'E_{H_k}[(L_k^{-1} + CP_kC' + \Sigma_v)^{-1}]CP_k]A' + \Sigma_w \end{aligned} \quad (12)$$

where the second line in the above equation follows due to the fact that P_k is adapted to $\mathcal{H}_{k-1}$ and $\{H_k\}$ is a sequence of i.i.d random matrices. Below, for notational simplicity, we will drop the subscript from the expectation operator whenever the random process over which the expectation is taken is obvious from the context. Define the space of $n \times n$ positive semidefinite matrices as $\mathcal{S}_n$. Then we have the following property for $G : \mathcal{S}_n \to \mathcal{S}_n$, whose proof can be found in Appendix A.

Lemma 3.1: The matrix-valued function $G(X)$ is a concave non-decreasing function of $X \in \mathcal{S}_n$.

Now given (A, C) is a detectable pair, it is easy to show that so is (A, H_kC) for H_k invertible, i.e. $h_{k,i} > 0, \forall i$. Then (A, H_kC) is a detectable pair almost surely, and intuitively we would expect P_k to be bounded almost surely for all k. For A invertible, a rigorous argument can be given by using results from [3], also see [8]. Now, we present a convergence result for $E[P_k]$ as $k \to \infty$.

Lemma 3.2: Staring with any $P_0 \in \mathcal{S}_n$, $E[P_k]$ converges to a bounded matrix $\Gamma^* \in \mathcal{S}_n$, where P_k satisfies the discrete-time Riccati equations (11).

Proof: We can easily verify that the notions of "weakly stabilizable and weakly detectable almost surely" introduced in [3] are satisfied. Then by Theorem 5.1 of [3], we know that there exists a unique stationary process $\{\bar{P}_k\}$, with $E[\bar{P}_k]$ constant $\forall k$. That $E[\bar{P}_k] \equiv \Gamma^*$ is bounded follows from equation (9) of [3], by setting e.g. $n = 0$ to give a bound on $\bar{P}_0$. Furthermore, Theorem 5.3 of [3] shows that $\{P_k\}$ starting from any initial condition P_0 is exponentially convergent to the stationary process $\{\bar{P}_k\}$. Hence $E[P_k]$ starting from any P_0 will also converge to $E[\bar{P}_k] = \Gamma^*$ as $k \to \infty$. ■

In general, analytically evaluating $E[P_k]$ is difficult. Furthermore, even though $E[P_k]$ will be bounded, it is not clear how one can obtain explicit upper bounds for arbitrary fading distributions. We now provide a result on a sequence of deterministic positive semidefinite matrices that overbounds

$E[P_k]$, $\forall k$, and also converges to a limit as $k \to \infty$. Note that we can write

$$E[P_{k+1}] = E_{H_k}[E[AP_kA' - AP_kC'(L_k^{-1}+CP_kC'+\Sigma_v)^{-1}CP_kA'+\Sigma_w|H_k]].$$

Denote $V_k = E[P_k]$. By concavity

$$V_{k+1} \leq E_{H_k}[AV_kA' - AV_kC'(L_k^{-1} + CV_kC' + \Sigma_v)^{-1}CV_kA' + \Sigma_w].$$

Then we have:

Theorem 3.3: For the state space model (4), let $\{Z_k\}$ be defined by

$$Z_{k+1} = E_{H_k}[AZ_kA' - AZ_kC'(L_k^{-1} + CZ_kC' + \Sigma_v)^{-1}CZ_kA' + \Sigma_w] \quad (13)$$

where $Z_0 = E[P_0]$, $L_k = H_k\Sigma_n^{-1}H_k$, and the components of the diagonal matrix H_k are identically and independently distributed with continuous distributions. Then $E[P_k] = V_k \leq Z_k$, and $Z_k \to Z^*$ as $k \to \infty$. $E[P_k]$ starting from $E[P_0] = V_0$ converges to a limiting value Γ^* such that $\Gamma^* \leq Z^*$.

The proof can be found in [9]. The bounding matrix sequence $\{Z_k\}$ mentioned above is still difficult to compute in general due to the difficulty of explicitly evaluating the expectation (with respect to the fading gain matrix H_k) of the nonlinear term in the right hand side of equation (13) above. In the next few sections we show how one can calculate (where evaluating the above expectation is possible) precise bounds for particular cases of systems (e.g, scalar systems, systems with vector states and scalar measurements and in case of multisensor systems - scalar state and scalar measurements) along with given fading distribution(s) for the channel(s) connecting the sensor(s) to the FC.

A. Single sensor: scalar state and measurement

In this section, we consider the scalar state and measurement model for a single sensor, given by (5). Specializing the error-covariance notation P_k to p_k for the scalar case, it is straightforward to show that p_k satisfies the following scalar discrete-time Riccati equation

$$p_{k+1} = \sigma_w^2 + \frac{a^2 p_k(h_k^2\sigma_v^2 + \sigma_n^2)}{h_k^2(p_k + \sigma_v^2) + \sigma_n^2}. \quad (14)$$

Denoting h_k^2 by r, where the time index k has been removed as $\{h_k\}$ is a sequence of i.i.d. random variables, we have

$$\gamma_{k+1} \leq \sigma_w^2 + a^2 E_r\left[\frac{\gamma_k(r\sigma_v^2 + \sigma_n^2)}{r(\gamma_k + \sigma_v^2) + \sigma_n^2}\right] \quad (15)$$

where $\gamma_k = E[p_k]$. In order to establish explicit upper bounds on $E[p_k]$ as $k \to \infty$, we consider two specific fading distributions, namely Rayleigh fading and Nakagami fading.

Rayleigh fading: In this case r is exponentially distributed with mean $\frac{1}{\lambda}$ such that $r \sim \lambda\exp(-\lambda r)$. It can then be easily shown that

$$E_r\left[\frac{\gamma_k(r\sigma_v^2 + \sigma_n^2)}{r(\gamma_k + \sigma_v^2) + \sigma_n^2}\right] = \frac{\gamma_k\sigma_v^2}{\gamma_k + \sigma_v^2} E_r\left[\frac{r + \frac{\sigma_n^2}{\sigma_v^2}}{r + \frac{\sigma_n^2}{\gamma_k + \sigma_v^2}}\right] = \frac{\gamma_k}{\gamma_k + \sigma_v^2}\left[\sigma_v^2 + \frac{\lambda\sigma_n^2\gamma_k}{\gamma_k + \sigma_v^2}e^{m_k}E_1(m_k)\right]$$

where $m_k = \frac{\lambda\sigma_n^2}{\gamma_k + \sigma_v^2}$ and $e^x E_1(x) = \int_0^\infty \frac{e^{-u}}{u+x}du$, with $E_1(x)$ being the exponential integral $E_1(x) = \int_x^\infty \frac{e^{-t}}{t}dt$. Hence

$$\gamma_{k+1} \leq \sigma_w^2 + \frac{a^2\gamma_k}{\gamma_k + \sigma_v^2}\left[\sigma_v^2 + \frac{\lambda\sigma_n^2\gamma_k}{\gamma_k + \sigma_v^2}e^{m_k}E_1(m_k)\right].$$

Using the inequality $e^x E_1(x) < \ln\left(1 + \frac{1}{x}\right)$, one can then write

$$\gamma_{k+1} \leq \sigma_w^2 + \frac{a^2\gamma_k}{\gamma_k + \sigma_v^2}\left[\sigma_v^2 + \frac{\lambda\sigma_n^2\gamma_k}{\gamma_k + \sigma_v^2}\ln\left(1 + \frac{\gamma_k + \sigma_v^2}{\lambda\sigma_n^2}\right)\right] \leq \sigma_w^2 + a^2\left[\sigma_v^2 + \lambda\sigma_n^2\ln\left(1 + \frac{\gamma_k + \sigma_v^2}{\lambda\sigma_n^2}\right)\right].$$

We will now define two new sequences $\{s_k\}, \{q_k\}$ such that

$$s_{k+1} = \sigma_w^2 + \frac{a^2 s_k}{s_k + \sigma_v^2}\left[\sigma_v^2 + \frac{\lambda\sigma_n^2 s_k}{s_k + \sigma_v^2}\exp\left(\frac{\lambda\sigma_n^2}{s_k + \sigma_v^2}\right)E_1\left(\frac{\lambda\sigma_n^2}{s_k + \sigma_v^2}\right)\right] \quad (16)$$

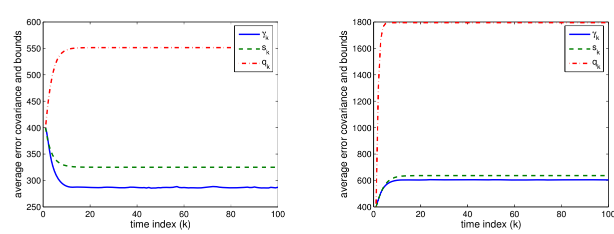
$$q_{k+1} = \sigma_w^2 + a^2\left[\sigma_v^2 + \lambda\sigma_n^2\ln\left(1 + \frac{q_k + \sigma_v^2}{\lambda\sigma_n^2}\right)\right] \quad (17)$$

with $s_0 = E[p_0], q_0 = E[p_0]$. It is obvious that $s_k \leq q_k$, $\forall k$. One can now provide bounds on $E[p_k]$ in terms of the limiting values of the above sequences as follows:

Theorem 3.4: The sequences $\{s_k\}, \{q_k\}$ defined above by (16), (17) converge to their individual limiting values s^* and q^*, respectively as $k \to \infty$. It is also true that $E[p_k] \leq s_k \leq q_k$, $\forall k$. $E[p_k]$ starting from $E[p_0]$ converges to a limiting value γ^* where $\gamma^* \leq s^* \leq q^*$.

Proof: See Appendix B. ■

The sequence $\{s_k\}$ is obviously a tighter bound than $\{q_k\}$. The reason why we use both sequences is that it is easier to prove convergence using the q_k iterations. In Figure 1, the average error covariance computed via simulations by averaging over 50000 randomly generated sample paths of length 100 is plotted against the various bounding sequences derived above.



(a) $\sigma_v = 2.0$ (b) $\sigma_v = 20.0$

Fig. 1. Average error covariance and bounds for Rayleigh fading, with $a = 1.25$, $c = 1.0$, $\sigma_w = 1.0$, $\sigma_n = 2.0$, $\lambda = 100$

Nakagami Fading: It is well known that the Nakagami-m distribution provides a very good model for land-mobile and indoor-mobile multipath propagation [10]. In this case, the channel power gain r is distributed according to the following probability distribution $p_R(r) = \frac{m^m r^{m-1}\lambda^m}{\Gamma(m)} \exp(-\lambda m r)$ where $m \in [\frac{1}{2}, \infty)$ is a parameter depicting the severity of fading, $\frac{1}{\lambda}$ is the mean channel power gain and $\Gamma(.)$ denotes the gamma function. Note that Rayleigh fading is a special case of Nakagami fading with $m = 1$. As m increases beyond 1, the severity of fading decreases. Here we consider $m = \frac{1}{2}$ which denotes fading that is more severe than Rayleigh fading. In this case, $r \sim \sqrt{\frac{\lambda}{2\pi r}} \exp(-\frac{\lambda r}{2})$.

Denoting $\gamma_k = E[p_k]$ as before, we can show from (15) that

$$\gamma_{k+1} \leq \sigma_w^2 + \frac{a^2\gamma_k}{\gamma_k + \sigma_v^2}\left(\sigma_v^2 + \frac{\sigma_n^2\gamma_k}{\gamma_k + \sigma_v^2}\frac{\lambda\sqrt{\pi}}{2u}\exp(u^2)\text{erfc}(u)\right) \quad (18)$$

where $u = \sqrt{\frac{\sigma_n^2\lambda}{2(\sigma_v^2+\gamma_k)}}$ and $\frac{\pi}{2u}\exp(u^2)\text{erfc}(u) = \int_0^\infty \frac{e^{-t^2}}{t^2+u^2}dt$. Here erfc$(x)$ denotes the complementary error function defined as $\frac{2}{\sqrt{\pi}}\int_x^\infty \exp(-t^2)dt$. Noting that $\frac{\sqrt{\pi}}{2}\exp(u^2)\text{erfc}(u) \leq \frac{1}{u+\sqrt{u^2+\frac{4}{\pi}}}$ [11], one can then easily show that

$$\gamma_{k+1} \leq \sigma_w^2 + \frac{a^2\gamma_k}{\gamma_k + \sigma_v^2}\left(\sigma_v^2 + \frac{\sigma_n\sqrt{2\lambda}\gamma_k}{\sqrt{\gamma_k + \sigma_v^2}}\frac{1}{u + \sqrt{u^2 + \frac{4}{\pi}}}\right).$$

We can now define two sequences $\{\tilde{s}_k\}, \{\tilde{q}_k\}$ as follows:

$$\tilde{s}_{k+1} = \sigma_w^2 + \frac{a^2\tilde{s}_k}{\tilde{s}_k + \sigma_v^2}\left(\sigma_v^2 + \frac{\sigma_n^2\tilde{s}_k}{\tilde{s}_k + \sigma_v^2}\frac{\lambda\sqrt{\pi}}{2\tilde{u}}\exp\left(\tilde{u}^2\right)\text{erfc}\left(\tilde{u}\right)\right) \quad (19)$$

$$\tilde{q}_{k+1} = \sigma_w^2 + a^2\left(\sigma_v^2 + \frac{\sigma_n\sqrt{2\lambda}\tilde{q}_k}{\sqrt{\frac{\sigma_n^2\lambda}{2}} + \sqrt{\frac{\sigma_n^2\lambda}{2} + \frac{4}{\pi}(\tilde{q}_k + \sigma_v^2)}}\right) \quad (20)$$

with $\tilde{s}_0 = E[p_0], \tilde{q}_0 = E[p_0]$, and $\tilde{u} = \sqrt{\frac{\sigma_n^2\lambda}{2(\sigma_v^2+\tilde{s}_k)}}$. We then similarly have the following theorem which states:

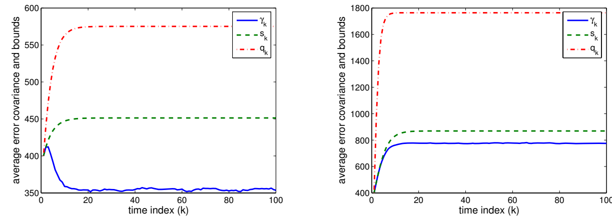
Theorem 3.5: For the case of Nakagami$(\frac{1}{2})$ fading, the sequences $\{\tilde{s}_k\}, \{\tilde{q}_k\}$ defined above by (19), (20) converge to their individual limiting values $\tilde{s}^*$ and $\tilde{q}^*$, respectively as $k \to \infty$. It is also true that $E[p_k] \leq \tilde{s}_k \leq \tilde{q}_k$, $\forall k$. $E[p_k]$ starting from $E[p_0]$ converges to a limiting value $\tilde{\gamma}^*$ where $\tilde{\gamma}^* \leq \tilde{s}^* \leq \tilde{q}^*$.

The proof is similar to that of Theorem 3.4 and is omitted.

Figure 2 shows the simulated average error covariance for Nakagami$(\frac{1}{2})$ fading for the same set of parameter values as in Figure 1. The interpretation of the graphs is similar to that of Figure 1. Note that as expected, the average error covariance performance and the corresponding bounds are generally worse than those for Rayleigh fading.

B. Single sensor: vector state and scalar measurement model and Rayleigh fading

Considering now the vector state scalar measurement model (6), it is easy to show that the corresponding time-



(a) $\sigma_v = 2.0$ (b) $\sigma_v = 20.0$

Fig. 2. Average error covariance and bounds for Nakagami$(\frac{1}{2})$ fading, with $a = 1.25$, $c = 1.0$, $\sigma_w = 1.0$, $\sigma_n = 2.0$, $\lambda = 100$

varying discrete-time Riccati equation is given by

$$P_{k+1} = \Sigma_w + A\left[P_k - \frac{P_k\bar{c}'\bar{c}P_k}{\bar{c}P_k\bar{c}' + \sigma_v^2}\frac{r}{r + \beta_k}\right]A' \quad (21)$$

where $r = h_k^2$ and $\beta_k = \frac{\sigma_n^2}{\bar{c}P_k\bar{c}'+\sigma_v^2}$. Note that $\bar{c}P_k\bar{c}' + \sigma_v^2$ is clearly a scalar quantity. Taking expectations on both sides and using the concavity of the right hand side in the above equation (Lemma 3.1), one can then write (by using $\bar{V}_k = E[P_k]$)

$$\bar{V}_{k+1} \leq \Sigma_w + A\bar{V}_kA' - \frac{A\bar{V}_k\bar{c}'\bar{c}\bar{V}_kA'}{\bar{c}\bar{V}_k\bar{c}' + \sigma_v^2}E_r\left[1 - \frac{\bar{\beta}_k}{r + \bar{\beta}_k}\right]$$

where $\bar{\beta}_k = \frac{\sigma_n^2}{\bar{c}\bar{V}_k\bar{c}'+\sigma_v^2}$. In the case of the channel being Rayleigh faded with mean $\frac{1}{\lambda}$, we can then derive the following upper bounding sequence of positive semidefinite matrices $\{\bar{Z}_k\}$:

$$\begin{aligned}\bar{Z}_{k+1} =& \Sigma_w + A\bar{Z}_kA' \\ &- \frac{A\bar{Z}_k\bar{c}'\bar{c}\bar{Z}_kA'}{\bar{c}\bar{Z}_k\bar{c}' + \sigma_v^2}\left[1 - \bar{m}_k\exp(\bar{m}_k)E_1(\bar{m}_k)\right]\end{aligned}$$

with $\bar{Z}_0 = V_0$ and $\bar{m}_k = \frac{\lambda\sigma_n^2}{\bar{c}\bar{Z}_k\bar{c}'+\sigma_v^2}$. Following Theorem 3.3, we can now conclude that $E[P_k] = \bar{V}_k \leq \bar{Z}_k$, and $\bar{Z}_k \to \bar{Z}^*$ as $k \to \infty$. $E[P_k]$ starting from $E[P_0] = \bar{V}_0$ converges to a limiting value $\bar{\Gamma}^*$ such that $\bar{\Gamma}^* \leq \bar{Z}^*$. Note that similar results can be obtained for other types of continuous fading disributions, however explicit bounding sequences for these fading distributions are not provided here to avoid repetition.

C. Multiple sensors

In this section, we consider the case of multiple sensors where each sensor observes a scalar state process and makes a scalar measurement. Both the multi-access (7) and the orthogonal access (8) schemes will be considered. Although these results can be extended to the vector state and scalar measurement (per sensor) case in a similar manner to the previous section, we do not include such results to maintain simplicity. We only consider the case of Rayleigh fading.

1) Multi-access case: Recall the signal model (7). The error covariance p_k satisfies the recursion

$$p_{k+1} = \sigma_w^2 + \frac{a^2p_k(\sum_{i=1}^M h_{k,i}^2\sigma_i^2 + \sigma_n^2)}{(\sum_{i=1}^M h_{k,i}c_i)^2p_k + \sum_{i=1}^M h_{k,i}^2\sigma_i^2 + \sigma_n^2}.$$

Denoting $\gamma_k = E[p_k]$ and letting $r_i = h_i^2$ (dropping the time subscript k), it follows similar to before that

$$\gamma_{k+1} \leq \sigma_w^2 + E_{\mathbf{r}}\left[\frac{a^2\gamma_k(\sum_{i=1}^M r_i\sigma_i^2 + \sigma_n^2)}{(\sum_{i=1}^M \sqrt{r_i}c_i)^2 p_k + \sum_{i=1}^M r_i\sigma_i^2 + \sigma_n^2}\right]$$
$$\leq \sigma_w^2 + E_{\mathbf{r}}\left[\frac{a^2\gamma_k(\sum_{i=1}^M r_i\sigma_i^2 + \sigma_n^2)}{\sum_{i=1}^M r_i c_i^2\gamma_k + \sum_{i=1}^M r_i\sigma_i^2 + \sigma_n^2}\right]$$

where $\mathbf{r} = (r_1, \ldots, r_M)$, and the second inequality follows since the c_i's are assumed to be positive. Since we are dealing with Rayleigh fading, $r_i \sim \lambda_i \exp(\lambda_i r_i)$.

Assume without loss of generality that the sensors are ordered such that

$$\frac{c_1^2}{\sigma_1^2} \geq \frac{c_2^2}{\sigma_2^2} \geq \cdots \geq \frac{c_M^2}{\sigma_M^2}. \quad (22)$$

We have

$$E_{\mathbf{r}}\left[\frac{\sum_{i=1}^M r_i\sigma_i^2 + \sigma_n^2}{\sum_{i=1}^M r_i c_i^2\gamma_k + \sum_{i=1}^M r_i\sigma_i^2 + \sigma_n^2}\right]$$
$$= \frac{\sigma_1^2}{c_1^2\gamma_k + \sigma_1^2} E_{r_2,\ldots,r_M}\left[1 + (\mathcal{A}_1 - \mathcal{B}_1)e^{\mathcal{B}_1}E_1(\mathcal{B}_1)\right]$$

where $\mathcal{A}_1 = \frac{\lambda_1}{\sigma_1^2}\left(\sum_{i=2}^M r_i\sigma_i^2 + \sigma_n^2\right)$, and $\mathcal{B}_1 = \frac{\lambda_1}{c_1^2\gamma_k + \sigma_1^2}\left(\sum_{i=2}^M r_i(c_i^2\gamma_k + \sigma_i^2) + \sigma_n^2\right)$. By assumption (22), $\mathcal{A}_1 - \mathcal{B}_1 \geq 0$. Using the inequality $e^x E_1(x) < \ln(1 + \frac{1}{x})$ would result in very complicated expressions which are difficult to work with for $M > 2$. For a simpler expression, we will instead use the looser inequality $e^x E_1(x) < \frac{1}{x}$. Then

$$E_{\mathbf{r}}\left[\frac{\sum_{i=1}^M r_i\sigma_i^2 + \sigma_n^2}{\sum_{i=1}^M r_i c_i^2\gamma_k + \sum_{i=1}^M r_i\sigma_i^2 + \sigma_n^2}\right]$$
$$\leq \frac{\sigma_1^2}{c_1^2\gamma_k + \sigma_1^2} E_{r_2,\ldots,r_M}\left[1 + \frac{\mathcal{A}_1 - \mathcal{B}_1}{\mathcal{B}_1}\right]$$
$$= E_{r_2,\ldots,r_M}\left[\frac{\sum_{i=2}^M r_i\sigma_i^2 + \sigma_n^2}{\sum_{2=1}^M r_i(c_i^2\gamma_k + \sigma_i^2) + \sigma_n^2}\right]$$
$$= \frac{\sigma_2^2}{c_2^2\gamma_k + \sigma_2^2} E_{r_3,\ldots,r_M}\left[1 + (\mathcal{A}_2 - \mathcal{B}_2)e^{\mathcal{B}_2}E_1(\mathcal{B}_2)\right]$$

where $\mathcal{A}_2 = \frac{\lambda_2}{\sigma_2^2}\left(\sum_{i=3}^M r_i\sigma_i^2 + \sigma_n^2\right)$, and $\mathcal{B}_2 = \frac{\lambda_2}{c_2^2\gamma_k + \sigma_2^2}\left(\sum_{i=3}^M r_i(c_i^2\gamma_k + \sigma_i^2) + \sigma_n^2\right)$. By assumption (22), we also have $\mathcal{A}_2 - \mathcal{B}_2 \geq 0$. Continuing this process, we eventually arrive at

$$E_{\mathbf{r}}\left[\frac{\sum_{i=1}^M r_i\sigma_i^2 + \sigma_n^2}{\sum_{i=1}^M r_i c_i^2\gamma_k + \sum_{i=1}^M r_i\sigma_i^2 + \sigma_n^2}\right]$$
$$\leq \frac{\sigma_M^2}{c_M^2\gamma_k + \sigma_M^2}\left[1 + (\mathcal{A}_M - \mathcal{B}_M)e^{\mathcal{B}_M}E_1(\mathcal{B}_M)\right]$$

where $\mathcal{A}_M = \frac{\lambda_M}{\sigma_M^2}\sigma_n^2, \mathcal{B}_M = \frac{\lambda_M}{c_M^2\gamma_k + \sigma_M^2}\sigma_n^2$. We can define the sequence

$$s_{k+1} = \sigma_w^2 + \frac{a^2 s_k\sigma_M^2}{c_M^2 s_k + \sigma_M^2}\left[1 + (\tilde{\mathcal{A}}_M - \tilde{\mathcal{B}}_M)e^{\tilde{\mathcal{B}}_M}E_1(\tilde{\mathcal{B}}_M)\right]$$

with $\tilde{\mathcal{A}}_M = \frac{\lambda_M}{\sigma_M^2}\sigma_n^2, \tilde{\mathcal{B}}_M = \frac{\lambda_M}{c_M^2 s_k + \sigma_M^2}\sigma_n^2$, and convergence properties of this sequence can be proved similar to Theorem 3.4. Recalling assumption (22), we thus see that we are bounded by the result assuming just the "worst" sensor in terms of the sensor SNR c_i^2/σ_i^2.

An alternative bound:
Consider the following inequality for $x_i \geq 0$,

$$\frac{1}{\sum_{i=1}^M x_i} \leq \frac{1}{M^2}\sum_{i=1}^M \frac{1}{x_i}, \quad (23)$$

which is a consequence of the well-known result that the arithmetic mean is greater than or equal to the harmonic mean. We will use this inequality to derive an alternative bound. A more attractive feature of this bound is that the parameters for all of the sensors will appear in the expression obtained. Applying the inequality (23), we have

$$E_{\mathbf{r}}\left[\frac{\sum\limits_{i=1}^M r_i\sigma_i^2 + \sigma_n^2}{\sum\limits_{i=1}^M r_i(c_i^2\gamma_k + \sigma_i^2) + \sigma_n^2}\right] = E_{\mathbf{r}}\left[\frac{\sum\limits_{i=1}^M r_i\sigma_i^2 + \sigma_n^2}{\sum\limits_{i=1}^M (r_i(c_i^2\gamma_k + \sigma_i^2) + \frac{\sigma_n^2}{M})}\right]$$
$$\leq \frac{1}{M^2}E_{\mathbf{r}}\left[\sum_{i=1}^M \frac{\sum_{j=1}^M r_j\sigma_j^2 + \sigma_n^2}{r_i(c_i^2\gamma_k + \sigma_i^2) + \frac{\sigma_n^2}{M}}\right].$$

We can evaluate

$$E_{\mathbf{r}}\left[\frac{\sum_{j=1}^M r_j\sigma_j^2 + \sigma_n^2}{r_i(c_i^2\gamma_k + \sigma_i^2) + \frac{\sigma_n^2}{M}}\right] = E_{r_i}\left[\frac{r_i\sigma_i^2 + \sum_{j\neq i}\frac{\sigma_j^2}{\lambda_j} + \sigma_n^2}{r_i(c_i^2\gamma_k + \sigma_i^2) + \frac{\sigma_n^2}{M}}\right]$$
$$= \frac{\sigma_i^2}{c_i^2\gamma_k + \sigma_i^2}[1 + (\mathcal{C}_i - \mathcal{D}_i)e^{\mathcal{D}_i}E_1(\mathcal{D}_i)]$$

with $\mathcal{C}_i = \frac{\lambda_i}{\sigma_i^2}\left(\sum_{j\neq i}\frac{\sigma_j^2}{\lambda_j} + \sigma_n^2\right), \mathcal{D}_i = \frac{\lambda_i\sigma_n^2}{M(c_i^2\gamma_k + \sigma_i^2)}$. Hence an alternative bounding sequence is

$$t_{k+1} = \sigma_w^2 + a^2 t_k\frac{1}{M^2}\sum_{i=1}^M \frac{\sigma_i^2}{c_i^2 t_k + \sigma_i^2}\left[1 + (\tilde{\mathcal{C}}_i - \tilde{\mathcal{D}}_i)e^{\tilde{\mathcal{D}}_i}E_1(\tilde{\mathcal{D}}_i)\right]$$

where $\tilde{\mathcal{C}}_i = \frac{\lambda_i}{\sigma_i^2}\left(\sum_{j\neq i}\frac{\sigma_j^2}{\lambda_j} + \sigma_n^2\right), \tilde{\mathcal{D}}_i = \frac{\lambda_i\sigma_n^2}{M(c_i^2 t_k + \sigma_i^2)}$.

2) Orthogonal access case: Recalling the orthogonal access model (8), it can be shown using the matrix inversion lemma that the error covariance satisfies

$$p_{k+1} = \sigma_w^2 + \frac{a^2 p_k}{1 + p_k\sum_{i=1}^M \frac{h_{k,i}^2 c_i^2}{h_{k,i}^2\sigma_i^2 + \sigma_n^2}}.$$

Letting $\gamma_k = E[p_k], r_i = h_i^2$, we have

$$\gamma_{k+1} \leq \sigma_w^2 + E_{\mathbf{r}}\left[\frac{a^2\gamma_k}{1 + \gamma_k\sum_{i=1}^M \frac{r_i c_i^2}{r_i\sigma_i^2 + \sigma_n^2}}\right].$$

We can compute

$$E_{\mathbf{r}}\left[\frac{1}{1 + \gamma_k\sum_{i=1}^M \frac{r_i c_i^2}{r_i\sigma_i^2 + \sigma_n^2}}\right]$$
$$= \frac{\sigma_1^2 E_{r_2,\ldots,r_M}\left[1 + (\mathcal{A}_1 - \mathcal{B}_1)e^{\mathcal{B}_1}E_1(\mathcal{B}_1)\right]}{\sigma_1^2(1 + \gamma_k\sum_{i=2}^M \frac{r_i c_i^2}{r_i\sigma_i^2 + \sigma_n^2}) + c_1^2\gamma_k}$$

where

$$\mathcal{A}_1 = \frac{\lambda_1 \sigma_n^2}{\sigma_1^2}, \mathcal{B}_1 = \frac{\lambda_1 \sigma_n^2 (1 + \gamma_k \sum_{i=2}^{M} \frac{r_i c_i^2}{r_i \sigma_i^2 + \sigma_n^2})}{\sigma_1^2 (1 + \gamma_k \sum_{i=2}^{M} \frac{r_i c_i^2}{r_i \sigma_i^2 + \sigma_n^2}) + c_1^2 \gamma_k}.$$

We note that here $\mathcal{A}_1 - \mathcal{B}_1$ is always positive, unlike the multi-access case where we needed the extra assumption (22). Again using the inequality $e^x E_1(x) < \frac{1}{x}$, we obtain

$$\begin{aligned} & E_{\mathbf{r}} \left[\frac{1}{1 + \gamma_k \sum_{i=1}^{M} \frac{r_i c_i^2}{r_i \sigma_i^2 + \sigma_n^2}} \right] \\ & \leq E_{r_2, \ldots, r_M} \left[\frac{1}{1 + \gamma_k \sum_{i=2}^{M} \frac{r_i c_i^2}{r_i \sigma_i^2 + \sigma_n^2}} \right] \leq \ldots \\ & \leq \frac{\sigma_M^2}{c_M^2 \gamma_k + \sigma_M^2} [1 + (\mathcal{A}_M - \mathcal{B}_M) e^{\mathcal{B}_M} E_1(\mathcal{B}_M)] \end{aligned}$$

where $\mathcal{A}_M = \frac{\lambda_M \sigma_n^2}{\sigma_M^2}, \mathcal{B}_M = \frac{\lambda_M \sigma_n^2}{\sigma_M^2 + c_M^2 \gamma_k}$. Note however that since no ordering of the sensors is assumed, we could have taken the expectations over $(r_1, \ldots, r_M)$ in any order. We can thus define an upper bounding sequence

$$s_{k+1} = \sigma_w^2 + \min_{i=1,\ldots,M} \frac{a^2 s_k \sigma_i^2}{c_i^2 s_k + \sigma_i^2} \left[1 + (\tilde{\mathcal{A}}_i - \tilde{\mathcal{B}}_i) e^{\tilde{\mathcal{B}}_i} E_1(\tilde{\mathcal{B}}_i) \right]$$

with $\tilde{\mathcal{A}}_i = \frac{\lambda_i \sigma_n^2}{\sigma_i^2}, \tilde{\mathcal{B}}_i = \frac{\lambda_i \sigma_n^2}{\sigma_i^2 + c_i^2 s_k}$. Convergence properties of the sequence can be proved similar to before.

An alternative bound:
Similar to the multi-access case, we can derive an alternative bound, again using the inequality (23). For the orthogonal case, we get:

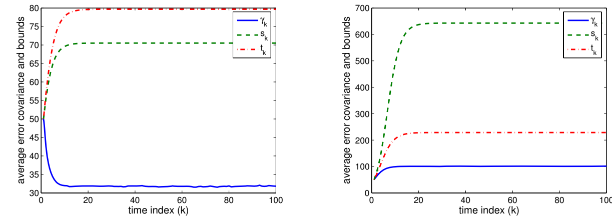
$$t_{k+1} = \sigma_w^2 + \frac{a^2 t_k}{M^2} \sum_{i=1}^{M} \frac{M \sigma_i^2 \left[1 + (\tilde{\mathcal{C}}_i - \tilde{\mathcal{D}}_i) e^{\tilde{\mathcal{D}}_i} E_1(\tilde{\mathcal{D}}_i) \right]}{M c_i^2 t_k + \sigma_i^2}$$

where $\tilde{\mathcal{C}}_i = \frac{\lambda_i \sigma_n^2}{\sigma_i^2}, \tilde{\mathcal{D}}_i = \frac{\lambda_i \sigma_n^2}{M c_i^2 t_k + \sigma_i^2}$.

3) Simulation Results: In Figure 3 we plot the simulated average error covariance for the multi-access scheme with two sensors, for different values of λ_1 and λ_2. We will plot the two general bounds s_k and t_k. We see that sometimes both s_k will be better than t_k, but sometimes the alternative bound t_k will be better. In Figure 4 we similarly plot the simulated average error covariance and two the different bounds s_k and t_k for the orthogonal access scheme with two sensors. Similar interpretations to Figure 3 apply.

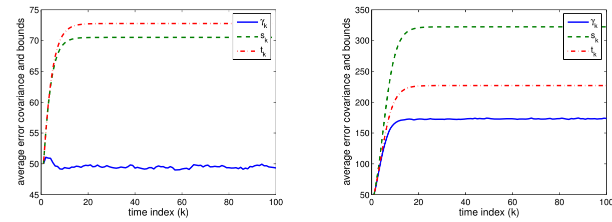
IV. Conclusion

In this paper, we considered a linear state estimation problem when measurements from single or multiple sensors are received via random fading channels at a remote fusion centre. Under some mild assumptions, we showed that the expected (with respect to the fading process) estimation error covariance at the fusion centre remains bounded and converges to a steady state value. While explicit expressions of the expected error covariance are hard to compute exactly, we provided exact deterministic bounding sequences on the average error covariance for the system models with scalar measurements (per sensor) and specific fading distributions. Numerical illustrations show that these bounds can often be quite tight.



(a) $\lambda_1 = 100, \lambda_2 = 20$ (b) $\lambda_1 = 100, \lambda_2 = 200$

Fig. 3. Average error covariance and bounds for multi-access scheme, with $a = 1.25$, $c_1 = 1.0$, $c_2 = 1.0$, $\sigma_w = 1.0$, $\sigma_1 = 1.0$, $\sigma_2 = 2.0$, $\sigma_n = 2.0$.



(a) $\lambda_1 = 100, \lambda_2 = 20$ (b) $\lambda_1 = 100, \lambda_2 = 200$

Fig. 4. Average error covariance and bounds for orthogonal scheme, with $a = 1.25$, $c_1 = 1.0$, $c_2 = 1.0$, $\sigma_w = 1.0$, $\sigma_1 = 1.0$, $\sigma_2 = 2.0$, $\sigma_n = 2.0$.

Appendix

A. Proof of Lemma 3.1

Suppose we have a positive semidefinite matrix Q_k independent of H_k such that $P_k \geq Q_k$. Note that since H_k is independent of P_k, one can write

$$\begin{aligned} G(P_k) &= E_{H_k} \left[A P_k A' - A P_k C' (L_k^{-1} + C P_k C' + \Sigma_v)^{-1} \right. \\ &\quad \left. \times C P_k A' + \Sigma_w \right] \\ &= E_{H_k} \Big[\min_{K_k} \{ (A - K_k C) P_k (A - K_k C)' + \Sigma_w \\ &\quad + K_k (\Sigma_v + L_k^{-1}) K_k' \} \Big] \\ &= E_{H_k} \big[\{ (A - K_k^* C) P_k (A - K_k^* C)' + \Sigma_w \\ &\quad + K_k^* (\Sigma_v + L_k^{-1}) K_k^{*\prime} \} \big] \\ &\geq E_{H_k} \big[\{ (A - K_k^* C) Q_k (A - K_k^* C)' + \Sigma_w \\ &\quad + K_k^* (\Sigma_v + L_k^{-1}) K_k^{*\prime} \} \big] \\ &\geq E_{H_k} \Big[\min_{K_k} \{ (A - K_k C) Q_k (A - K_k C)' + \Sigma_w \\ &\quad + K_k (\Sigma_v + L_k^{-1}) K_k' \} \Big] \\ &= G(Q_k) \end{aligned}$$

where the second line follows from the fact that the Kalman filter operates with the optimal time-varying gain K_k^*, and the fourth line follows since $P_k \geq Q_k$. This completes the proof of the non-decreasing property.

In order to prove that $G(P_k)$ is a concave function of P_k, one needs to show that

$$G(\alpha P_k^1 + (1 - \alpha) P_k^2) \geq \alpha G(P_k^1) + (1 - \alpha) G(P_k^2)$$

where P_k^1, P_k^2 are both positive semidefinite and $0 < \alpha < 1$. Suppose $\bar{P}_k = \alpha P_k^1 + (1-\alpha)P_k^2$. Then, using the fact that the Kalman filter operates with the optimal gain, we have

$$\begin{aligned}
G(\bar{P}_k) &= E_{H_k}\Big[\min_X\{(A-XC)\bar{P}_k(A-XC)' + \Sigma_w \\
&\qquad + X(\Sigma_v + L_k^{-1})X'\}\Big] \\
&= E_{H_k}\Big[\min_X\{(A-XC)(\alpha P_k^1 + (1-\alpha)P_k^2)(A-XC)' \\
&\qquad + \Sigma_w + X(\Sigma_v + L_k^{-1})X'\}\Big] \\
&= E_{H_k}\Big[\min_X\big(\alpha\{(A-XC)P_k^1(A-XC)' \\
&\qquad + \Sigma_w + X(\Sigma_v + L_k^{-1})X'\} \\
&\qquad + (1-\alpha)\{(A-XC)P_k^2(A-XC)' \\
&\qquad + \Sigma_w + X(\Sigma_v + L_k^{-1})X'\}\big)\Big] \\
&= E_{H_k}\Big[\min_X\big(\alpha f(X, P_k^1) + (1-\alpha) f(X, P_k^2)\big)\Big]
\end{aligned}$$

where $f(X, P_k) = (A-XC)P_k(A-XC)' + \Sigma_w + X(\Sigma_v + L_k^{-1})X'$. Noting that $f(X, P_k)$ is an affine function in P_k and pointwise minimum of an affine function is a concave function, it is clear that

$$\begin{aligned}
G(\bar{P}_k) &\geq E_{H_k}\Big[\alpha \min_X f(X, P_k^1) + (1-\alpha)\min_X f(X, P_k^2)\Big] \\
&= \alpha G(P_k^1) + (1-\alpha) G(P_k^2)
\end{aligned}$$

which completes the proof of concavity.

B. *Proof of Theorem 3.4*

Consider the equations (16) and (17). Note that the right hand side of (16) and (17) are increasing functions of s_k and q_k respectively, which implies that both $\{s_k\}$ and $\{q_k\}$ are monotonic sequences. Given that $\gamma_0 = E[p_0] = s_0 = q_0$, it clearly follows that $\gamma_1 \leq s_1 \leq q_1$. Using the increasing property mentioned above, one can then prove by induction that $\gamma_k \leq s_k \leq q_k,\ \forall k = 1, 2, \ldots$. We know that as a special case of Theorem 3.2, $\gamma_k = E[p_k]$ converges to a limit (say γ^*) as $k \to \infty$.

We now show that q_k also converges to a limit (denoted by q^*) as $k \to \infty$. In order to prove this, we rewrite (17) as $q_{k+1} = l(q_k)$ where $l(q_k)$ represents the right hand side of (17). It is straightforward to show that the mapping $q = l(q)$ has a unique fixed point. However, in order to prove the convergence of the recursion $q_{k+1} = l(q_k)$ we use the *standard function properties* of the function $l(q)$ [12], namely positivity, monotonicity (these two are obvious) and scalability. In order to show scalability, we have to show that $\beta l(q) > l(\beta q)$ for $\beta > 1$. This follows because

$$\ln\left(1 + \frac{\beta q_k + \sigma_v^2}{\lambda \sigma_n^2}\right) < \ln\left(1 + \frac{\beta(q_k + \sigma_v^2)}{\lambda\sigma_n^2}\right) < \beta \ln\left(1 + \frac{q_k + \sigma_v^2}{\lambda \sigma_n^2}\right)$$

where the first inequality follows since $\beta > 1$ and the second inequality follows since $\ln(1+\beta x) < \beta \ln(1+x)$ for $x > 0, \beta > 1$. Since $l(q)$ is a standard function, the recursion $q_{k+1} = l(q_k)$ will converge to the unique fixed point q^*.

Now $\{s_k\}$ is a monotonic sequence sandwiched between two convergent sequences $\{q_k\}$ and $\{\gamma_k\}$ (the limits of the two sequences are in general different). Hence $\{s_k\}$ can be bounded from both above and below, so converges to a limit s^*. Since $\gamma_k \leq s_k \leq q_k,\ \forall k = 1, 2, \ldots$, we have $\gamma^* \leq s^* \leq q^*$.

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