Localness of Certain Banach Modules

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Communicated by the Editors

1. Let X be a compact subset of the complex plane $\mathbf{C}$, and let $\mathcal{R}(X)$ denote the space of C^∞ functions on $\mathbf{C}$ which agree on a neighbourhood of X with rational functions. For $0 \leqq m \in \mathbf{Z}$, let $\overline{\mathcal{P}}_m$ denote the space of conjugate-analytic polynomials of degree at most m. Then for $\beta > 0$, $[\mathcal{R}]_{\beta,X}$ (the closure in Lip (β, X) of $\mathcal{R}(X)$) is a Banach algebra, and $[\mathcal{R}\ \overline{\mathcal{P}}_m]_\beta$ is a finitely generated $[\mathcal{R}]_\beta$-module. These modules arise naturally in the study of rational approximation in high Lipschitz norms [7]. The purpose of this note is to show that $[\mathcal{R}\overline{\mathcal{P}}_m]_\beta$ is *locally determined* in a certain sense, for $\beta \notin \mathbf{Z}$. Precisely speaking, the result is as follows:

Theorem. *Let* $0 < \beta \notin \mathbf{Z}$, $0 \leqq m \in Z$, *and let* X *be a compact subset of* $\mathbf{C}$. *Suppose* $f : X \to \mathbf{C}$ *and every point* $a \in X$ *has a closed neighbourhood* U *such that*

$$f \in [\mathcal{R}(X \cap U)\overline{\mathcal{P}}_m]_{\beta, X\cap U} .$$

Then

$$f \in [\mathcal{R}(X)\overline{\mathcal{P}}_m]_{\beta,X} .$$

The first result of this kind was Bishop's theorem [3, p. 51] that *the uniform closure on* X *of* $\mathcal{R}(X)$ *is local, for* X *compact in* $\mathbf{C}$. Kallin [4] showed that this fails for sets in higher dimensional $\mathbf{C}^n$. Weinstock [9] proved a localness theorem for approximation by solutions of elliptic partial differential equations. He worked in the space of germs of C^k functions, where k is less than the order of the operator. These germ spaces are unsuitable for approximation at orders at and beyond the order of the operator. The localness result of the present paper is the first which involves approximation beyond the order of the operator. We conjecture that Weinstock's results can be extended to all nonintegral orders if one formulates them in terms of Lip β spaces instead of germ spaces.

The question whether $[\mathcal{R}\ \overline{\mathcal{P}}_m]_\beta$ is local for *integral* $\beta > 0$ remains open. There is some evidence in favor of localness: the necessary and sufficient condition for

$$[\mathcal{R}]_1 = [\mathcal{E}]_1$$

which is given in [5], is a local condition.

Indiana University Mathematics Journal, ©, Vol. 24, No. 12 (1975)

Our theorem raises a natural question, namely: *give a local description of*

$$[\mathcal{R}(X)\overline{\mathcal{P}}_m]_\beta$$

for $0 < \beta \notin \mathbf{Z}$. Some work on this question has been done [6, 7], but there are more questions than answers.

I am grateful to T. Gamelin and J. Garnett for their helpful comments.

2. There are several equivalent ways of thinking of the space Lip (β, X) [7, §4; 8, Chapter VI]. For our present purpose it is simplest to think of it as the space of all functions on X which have extensions in Lip $(\beta, \mathbf{C})$. The norm of an element $f \in$ Lip (β, X) is $\|f\|_{\beta,X} = \inf\{\|g\|_{\beta,\mathbf{C}} : g \in \text{Lip}(\beta, \mathbf{C}), g = f \text{ on } X\}$. Here $\|g\|_{\beta,\mathbf{C}}$ means

$$\sum_{0 \leq i+j \leq p} \left\|\frac{\partial^{i+j} g}{\partial x^i\, \partial y^j}\right\|_u + \sum_{i+j=p} \left\|\frac{\partial^p g}{\partial x^i\, \partial y^j}\right\|_\alpha,$$

where $\beta = p + \alpha$, $p \in \mathbf{Z}$, $0 < \alpha \leqq 1$, $\|\cdot\|_u$ is the uniform norm, and

$$\|h\|_\alpha = \sup\left\{\frac{|h(u) - h(v)|}{|u - v|^\alpha} : u, v \in \mathbf{C}, u \neq v\right\}.$$

Lip (β, X) is *not* the same as the space of germs on X of functions in Lip $(\beta, \mathbf{C})$.

Lip (β, X) itself is local, in the sense that if $f : X \to \mathbf{C}$ is such that every point has a closed neighbourhood U with $f \in$ Lip $(\beta, X \cap U)$, then it follows that $f \in$ Lip (β, X). For, let $U_1, U_2, \cdots, U_n$ be a covering of X by closed balls such that $f \in$ Lip $(\beta, X \cap U_i)$ for $i = 1, 2, \cdots, n$. For each i there exists a function $f_i \in$ Lip $(\beta, \mathbf{C})$ such that $f_i = f$ on $X \cap U_i$. Select $\varphi_1, \varphi_2, \cdots, \varphi_n \in \mathfrak{D}$ (the space of C^∞ functions on $\mathbf{C}$ with compact support) such that spt $\varphi_i \subset U_i$ and $\sum \varphi_i = 1$ on a neighbourhood of X. Let

$$g = \sum_{i=1} \varphi_i f_i .$$

Then $g \in$ Lip $(\beta, \mathbf{C})$ and it is easy to see that $g = f$ on X.

Thus, if f satisfies the hypotheses of the theorem, then $f \in$ Lip (β, X), so we might as well assume from the start that $f \in$ Lip $(\beta, \mathbf{C})$.

3. We divide the proof of the theorem into four steps.

I. Let $\mathcal{E}$ denote the space of C^∞ functions on $\mathbf{C}$, and set

$$\tilde{\mathcal{R}}_m(X) = \{h \in \mathcal{E} : (\bar{\partial})^{m+1} h = 0 \text{ on a neighbourhood of } X\}$$

where $\bar{\partial}$ denotes the operator

$$\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}.$$

Runge's Theorem [3, p. 28] carries over to Lip β norm, and states that

$$[\mathcal{R}]_\beta = [\tilde{\mathcal{R}}_0]_\beta .$$

We claim that $\tilde{\mathcal{R}}_m(X) = \tilde{\mathcal{R}}_0(X)\overline{\mathcal{P}}_m$, so that by Runge's Theorem

$$[\mathcal{R}\ \overline{\mathcal{P}}_m]_\beta = [\tilde{\mathcal{R}}_m]_\beta .$$

To see this, fix $h \in \mathfrak{D} \cap \tilde{\mathcal{R}}_m(X)$. Then $h_1 = (\bar{\partial})^m h \in \mathfrak{D} \cap \tilde{\mathcal{R}}_0(X)$. Also

$$\bar{\partial}[(\bar{\partial})^{m-1}h - \bar{z}h_1] = -\bar{z}\bar{\partial}h_1 = 0$$

on a neighbourhood of X, so $h_2 = (\bar{\partial})^{m-1}h - \bar{z}h_1 \in \tilde{\mathcal{R}}_0(X)$. Continuing, we find that h_{m+1}, defined by

$$h_{m+1} = h - \frac{\bar{z}^m h_1}{m!} - \frac{\bar{z}^{m-1}h_2}{(m-1)!} - \cdots - \bar{z}h_m$$

belongs to $\tilde{\mathcal{R}}_0(X)$, *i.e.* $h \in \tilde{\mathcal{R}}_0(X)\overline{\mathcal{P}}_m$. This suffices.

II. For $\varphi \in \mathfrak{D}$, the Cauchy transform $\hat{\varphi} \in \mathcal{E}$ is defined by

$$\hat{\varphi}(z) = \frac{1}{\pi}\int \frac{\varphi(\zeta)}{z - \zeta}\, d\mathcal{L}^2\zeta,$$

where $\mathcal{L}^2$ is Lebesque measure on the plane. One has

$$\bar{\partial}\hat{\varphi} = \varphi = \bar{\partial}\hat{\varphi} \tag{1}$$

whenever $\varphi \in \mathfrak{D}$. A basic estimate is

$$\|\hat{\varphi}\|_{\beta+1,\mathbf{C}} \leqq K(\beta, d)\, \|\varphi\|_{\beta,\mathbf{C}}, \tag{2}$$

valid if $0 < \beta \notin \mathbf{Z}$, $\varphi \in \mathfrak{D}$ and diam spt $\varphi \leqq d$. Here $K(\beta, d)$ is a constant independent of φ. This estimate goes back to Bers [1, p. 9].

Fix $\psi \in \mathfrak{D}$ with $\psi \equiv 1$ on a neighbourhood of X, and define the continuous linear map $C : \mathcal{E} \to \mathcal{E}$ by setting

$$Cf = \widehat{\psi \cdot f}$$

for $f \in \mathcal{E}$. By (2), C extends to a continuous map $C : \text{Lip}(\beta, \mathbf{C}) \to \text{Lip}(\beta + 1, \mathbf{C})$. For $p \in \mathbf{Z}^+$ and $\varphi \in \mathfrak{D}$ consider the linear operator $T_{\varphi,p} : \mathcal{E} \to \mathcal{E}$ defined by

$$T_{\varphi,p}f = C^p[\widehat{\varphi \cdot (\bar{\partial})^{p+1}f}]$$

for $f \in \mathcal{E}$ (for instance, to calculate $T_{\varphi,1}f$, first calculate $(\bar{\partial})^2 f$, then multiply by φ, take the Cauchy transform, multiply by ψ, and finally apply the Cauchy transform again).

Claim. *If $p \in \mathbf{Z}^+$ and $\varphi \in \mathfrak{D}$, then there is a constant K depending on β, p, φ and ψ such that*

$$\|T_{\varphi,p}f\|_{\beta,\mathbf{C}} \leqq K\, \|f\|_{\beta,\mathbf{C}},$$

whenever $f \in \mathfrak{D}$.

Proof of Claim. The proof proceeds by induction on p.

In case $p = 0$, $T_{\varphi,0}$ is the classical "T_φ operator" [3, p. 29], and we have

$$T_{\varphi,0}f = \varphi f - \widehat{f\cdot\bar{\partial}\varphi},$$

as is easily seen. Hence for $0 < \beta \notin \mathbf{Z}$,

$$\begin{aligned}||T_{\varphi,0}f||_\beta &\leqq ||\varphi f||_\beta + ||\widehat{f\bar{\partial}\varphi}||_\beta \\ &\leqq K_1\,||f||_\beta + K_2\,||f\bar{\partial}\varphi||_{\beta-1}\end{aligned}$$

(by (2))

$$\leqq K_3\,||f||_\beta\,.$$

(For $\gamma \leqq 0$, $||f||_\gamma$ denotes the uniform norm.)

Now suppose the claim holds for all indices less than p, and *all* β, φ with $0 < \beta \notin \mathbf{Z}$ and $\varphi \in \mathfrak{D}$. Denoting $\varphi_i = (\bar{\partial})^i\varphi$, we have

$$(\bar{\partial})^{p+1}(\varphi f) = \varphi(\bar{\partial})^{p+1}f + \sum_{k=0}^{p}\binom{p+1}{k}\varphi_{p-k+1}(\bar{\partial})^k f,$$

hence

$$\begin{aligned}T_{\varphi,p}f = C^p[\widehat{\varphi\cdot(\bar{\partial})^{p+1}f}] &= C^p[(\bar{\partial})^{p+1}(\varphi f)] - \sum_{k=0}^{p}\binom{p+1}{k}C^p[\varphi_{p-k+1}(\bar{\partial})^k f] \\ &= T_{\psi,p-1}(\varphi f) + \sum_{k=0}^{p}\binom{p+1}{k}C^{p-k}T_{\varphi_{p-k+1},k}f,\end{aligned}$$

hence

$$\begin{aligned}||T_{\varphi,p}f||_\beta &\leqq ||T_{\psi,p-1}(\varphi f)||_\beta + \sum_{k=0}^{p}\binom{p+1}{k}\,||C^{p-k}T_{\varphi_{p-k+1},k}f||_\beta \\ &\leqq K_4\,||\varphi f||_\beta + \sum_{k=0}^{p}K_4\,||T_{\varphi_{p-k+1},k}f||_{\beta+k-p} \\ &\leqq K_5\,||f||_\beta + \sum_{k=0}^{p}K_5\,||f||_{\beta+k-p} \\ &\leqq K_6\,||f||_\beta\,.\end{aligned}$$

Thus the induction step is complete, and the claim is proved.

III. We observe that if $f \in \mathcal{E}$ and $(\bar{\partial})^{m+1}f = 0$ on a neighbourhood W of $X \cap \operatorname{spt}\varphi$, then $(\bar{\partial})^{m+1}T_{\varphi,m}f = 0$ on

$$V = [W \cup (\mathbf{C}\backslash\operatorname{spt}\varphi)] \cap [\operatorname{int}\{x : \psi(x) = 1\}],$$

which is a neighbourhood of X; for if $x \in V$, then $(\bar{\partial})^p\psi(x) = 0$ for $p = 1, 2, 3, \cdots$, hence by repeated use of (1),

$$[(\bar{\partial})^{m+1}T_{\varphi,m}f](x) = (\psi(x))^m\cdot\varphi(x)\cdot(\bar{\partial})^{m+1}f(x).$$

IV. Now suppose $f \in \operatorname{Lip}(\beta, \mathbf{C})$ and

$$f \in [\mathcal{R}(X \cap U_i)\overline{\mathcal{P}}_m]_{\beta, X\cap U_i} \tag{3}$$

for each element U_i of a finite collection $\{U_1, U_2, \cdots, U_q\}$ of closed balls whose interiors cover X. Modify f off X (if need be) so as to ensure that spt f is compact. Extend $\{U_i\}_1^q$ to a finite covering $\{U_i\}_1^Q$ of spt f by closed balls such that the interiors also cover spt f and $U_i \cap X = \varnothing$ for $q + 1 \leqq i \leqq Q$. Choose functions $\varphi_1, \varphi_2, \cdots, \varphi_Q \in \mathfrak{D}$ such that spt $\varphi_i \subset U_i$ and $\sum \varphi_i \equiv 1$ on a neighbourhood of spt f. Consider the function

$$\begin{aligned} f_1 &= f - \sum_{i=1}^{Q} T_{\varphi i, m} f \\ &= f - C^m[\widehat{(\bar\partial)^{m+1} f}] \\ &= f - C^m(\bar\partial)^m f. \end{aligned}$$

We have $(\bar\partial)^{m+1} f_1 = (\bar\partial)^{m+1} f - \psi^m(\bar\partial)^{m+1} f = 0$ on a neighbourhood of X, so $f_1 \in \tilde{\mathcal{R}}_m(X)$. Also, since $(\bar\partial)^{m+1} T_{\varphi i, m} f = \psi^m \varphi_i (\bar\partial)^{m+1} f$, it follows that

$$(\bar\partial)^{m+1} \sum_{i=q+1}^{Q} T_{\varphi i, m} f = 0$$

on a neighbourhood of X, so that

$$\sum_{q+1}^{Q} T_{\varphi i, m} f \in \tilde{\mathcal{R}}_m(X).$$

Fix i, $1 \leqq i \leqq q$. Then by (3) we may choose a sequence $\{f_N\} \in \tilde{\mathcal{R}}_m(X \cap U_i)$ such that $\|f - f_N\|_{\beta, X\cap U_i} \downarrow 0$. By modifying f_N off $X \cap U_i$ we may assume that $\|f - f_N\|_{\beta, \mathbf{C}} \downarrow 0$. By part II, it follows that

$$\|T_{\varphi i, m} f - T_{\varphi i, m} f_N\|_{\beta, \mathbf{C}} \to 0.$$

By part III, $T_{\varphi i, m} f_N \in \tilde{\mathcal{R}}_m(X)$, hence $T_{\varphi i, m} f \in [\tilde{\mathcal{R}}_m(X)]_{\beta, X}$. Thus

$$f = f_1 + \sum_{1}^{q} T_{\varphi i, m} f + \sum_{q+1}^{Q} T_{\varphi i, m} f$$

belongs to $[\tilde{\mathcal{R}}_m(X)]_{\beta, X}$. This concludes the proof.

It is easy to see that the proof of our theorem can be adapted to prove the following extension of Bishop's theorem: *the uniform closure of* $\mathcal{R}(X)\overline{\mathcal{P}}_m$ *is local whenever* X *is compact in* $\mathbf{C}$ *and* $0 \leqq m \in \mathbf{Z}$. In place of the inequality (2) one uses the estimate

$$\|\hat\varphi\|_{u, \mathbf{C}} \leqq K(d)\, \|\varphi\|_{u, \mathbf{C}},$$

which holds for all $\varphi \in \mathfrak{D}$ and $d > 0$ with diam spt $\varphi \leqq d$.

4. We close with a simple, but useful, application of the operator C introduced in §3, part II.

If $X \subset \mathbf{C}$ is compact and $0 < n \in \mathbf{Z}$, we denote the closure of $\mathcal{E}$ in Lip (n, X) by $D^n(X)$.

Theorem. *Let $0 < n \in \mathbf{Z}$, $0 \leqq j \in \mathbf{Z}$, and let $X \subset \mathbf{C}$ be compact. Suppose $\mathcal{R}(X)\overline{\mathcal{P}}_j$ is not uniformly dense in $C(X)$. Then*

$$[\mathcal{R}\overline{\mathcal{P}}_{j+n}]_n \neq D^n(X).$$

Proof. Suppose $\mathcal{R}(X)\overline{\mathcal{P}}_j$ is not uniformly dense in $C(X)$. Let U be the largest open subset of $\mathbf{C}$ such that $U \cap X$ has zero area, and let $Y = X \backslash U$. Then every nonempty relatively open subset of Y has positive area. Since $\mathcal{R}(X)\overline{\mathcal{P}}_j$ is not dense in $C(X)$, there exists a finite complex Borel regular measure μ with support in X such that $\mu \perp \mathcal{R}(X)\overline{\mathcal{P}}_j$. We claim that μ must actually be supported in Y. To see this, fix $\varphi \in \mathfrak{D}$ with $Y \cap \text{spt } \varphi = \emptyset$ and form the measure

$$\nu = \varphi \cdot \mu + (\bar{\partial}\varphi) \cdot \hat{\mu} \cdot \mathcal{L}^2.$$

Then [3, p. 51] $\nu \perp \mathcal{R}(X)$ and spt $\nu \subset X \cap \text{spt } \varphi$. Hence

$$\nu \perp \mathcal{R}(X \cap \text{spt } \varphi),$$

since any rational function with poles off $X \cap$ spt φ is the uniform limit on $X \cap$ spt φ of rational functions with poles off X. By the Hartogs–Rosenthal Theorem [3, p. 47], $\nu = 0$, hence $\hat{\nu} = 0$. But $\hat{\nu} = \varphi\hat{\mu}$, so we conclude that $\hat{\mu}$ vanishes off Y, hence spt μ is contained in Y [3, p. 46, (8.2)], as claimed. Thus $\mu \perp \mathcal{R}(Y)\overline{\mathcal{P}}_j$, so $\mathcal{R}(Y)\overline{\mathcal{P}}_j$ is not uniformly dense in $C(Y)$.

Let us suppose, contrary to the assertion of the theorem, that

$$[\mathcal{R}(X)\overline{\mathcal{P}}_{j+n}]_n = D^n(X).$$

We first obtain a contradiction by proving that $\mathcal{R}(Y)\overline{\mathcal{P}}_j$ is uniformly dense in $C(Y)$.

First, we claim that there is a constant $K > 0$ such that

$$|(\bar{\partial})^n g(z)| \leqq K \, ||g||_{n,Y} \tag{4}$$

whenever $g \in \mathcal{E}$ and $z \in Y$. It suffices to prove this for the points z at which Y has full area density, since the set of such points is dense in Y. Accordingly, let Y have full area density at z and for $h \in \mathbf{C}$ let $F(z, h)$ denote the set of all points of the form $z + (r + is)h$, where r and s run over all the integers between 2^{-n} and 2^n. It is well-known [10] that there exists a sequence $h_n \to 0$ such that $|h_n|^{-1}h_n \to 1$ and $F(z, h_n) \subset Y$ for $n = 1, 2, 3, \cdots$. Thus by forming difference quotients we find that each partial derivative $\partial^{i+j}g(z)/\partial x^i \partial y^j$, with $i + j = n$, is bounded by $K_1 \, ||g||_{n,Y}$, where K_1 depends only on n, and not on z or g. Taking $K = 2^n K_1$, we obtain (4).

Fix $f \in \mathfrak{D}$ and $\epsilon > 0$. Choose $h \in \mathcal{R}(X)\overline{\mathcal{P}}_{n+j}$ such that

$$||h - C^n f||_{n,X} < \epsilon/K.$$

Then, since $\bar{\partial}Cg = \psi \cdot g$ whenever $g \in \mathcal{E}$, (4) implies that

$$||(\bar{\partial})^n h - \psi^n \cdot f||_{u,Y} < \epsilon.$$

Since $(\bar{\partial})^n h \in \mathcal{R}(X)\bar{\mathcal{P}}_i \subset \mathcal{R}(Y)\bar{\mathcal{P}}_i$, and $\psi \equiv 1$ on X (and hence on Y), we conclude that f belongs to the uniform closure of $\mathcal{R}(Y)\bar{\mathcal{P}}_i$ on Y. Since $\mathfrak{D}$ is uniformly dense in $C(Y)$, we are done.

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Added in proof. The localness theorem holds for *integral* β also. The argument of step II of the proof must be changed somewhat to cover this case. The key observation is that for f and φ in $\mathfrak{D}$ and any $n \in \mathbf{Z}^+$, we have

$$\|\widehat{f\,\bar{\partial}\varphi}\|_n \leqq \|\widehat{f\,\bar{\partial}\varphi}\|_{n+1/2} \leqq K_1\, \|f\,\bar{\partial}\varphi\|_{n-1/2} \leqq K_2\, \|f\|_n\ ,$$

so that

$$\|T_{\varphi,0}f\|_n \leqq K_3\|f\|_n\ .$$

This work was partially supported by National Science Foundation grant GP-33693X.

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Date communicated: MAY 28, 1974