The tangent stars of a set,
and extensions of smooth functions.

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Abstract.

The k–th order tangent star, $\mathrm{Tan}^k(M,X)$, of a closed subset X of a C^k manifold M is defined and studied. The map $(M,X)\mapsto \mathrm{Tan}^k(M,X)$ is a covariant functor from the category of pairs to the category of stars. Given a continuous function $f : X \to \mathbf{R}$, and letting $G = \mathrm{graph} f$, we consider the star–morphism

$$\pi_* : \mathrm{Tan}^k(M\times\mathbf{R}, G) \to \mathrm{Tan}^k(M,X)$$

induced by the projection $\pi : M\times\mathbf{R}\to M$.

Theorem : The function f has a C^k extension to M if and only if π_* is a bijection.

A method for calculating $\mathrm{Tan}^k(M,X)$, and several examples, are presented, and the relations to other tangent concepts are investigated.

Running head: Tangent stars and extensions of functions.

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1. Introduction.

The purpose of this paper is to introduce the k–th order *tangent star*, denoted $\mathrm{Tan}^k X$, of an arbitrary closed set X contained in a C^k manifold, M. A star is something like a vector bundle, but there are two main differences. All the rays $S(a)$ of a star S are vector spaces, but unlike the fibres of a vector bundle, they share a *common origin*, and, more significantly, the dimension of $S(a)$ is nonconstant. The rays $\mathrm{Tan}^k(X,a)$ of $\mathrm{Tan}^k X$ are, in addition, finitely–generated modules over finite–dimensional real algebras. The formal definitions of star and tangent star are given in section 2. The relation of first–order tangent stars to the classical tangents of Denjoy, Whitney and Zariski, and the relation of higher–order tangent stars to the higher–order tangent bundles of Pohl [31] and to the paratangent spaces of Glaeser [17] are explained in section 6.

The modules $\mathrm{Tan}^1(X,a),\ldots,\mathrm{Tan}^k(X,a)$ carry a lot of information about the germ of the set X at the point a, and are invariant under C^k diffeomorphisms. These invariants (including associated integral invariants) have considerable discriminating power, and may prove useful in connection with singularities and critical points. However, our immediate motivation for introducing them is the study of smooth extensions. We wanted to sharpen **Whitney's** extension theorem.

Whitney's theorem is well–known, has been frequently used [1, 3, 4, 8, 11, 12, 14, 15, 17, 20, 21, 23, 24, 26, 34, 36, 37, 38, 41], and has been developed in various directions [2, 5, 6, 7, 10, 15, 19, 22, 25, 27, 28, 33, 35]. It is usually described as giving conditions on a function $f : X \to \mathbf{R}$ (where $X \subset \mathbf{R}^d$ is an arbitrary closed set), necessary and sufficient for the existence of a C^k function $\bar{f} : \mathbb{R}^d \to \mathbf{R}$, such that $\bar{f}|X = f$. This description is a bit misleading. The theorem actually says [39] that such an $\bar{f}$ exists if and only if some other functions f_i (corresponding to each d–term multi–index i of order $|i| \le k$), all mapping X to $\mathbf{R}$ continuously, and satisfying a technical condition, exist. The technical condition says, essentially, that the jet (f_i) satisfies Taylor's theorem on X, in a locally uniform way. The trouble about the theorem is that the functions f_i are not, in general, uniquely determined By f. The theorem is more properly described as an extension theorem for C^k *jets*, rather than functions. In some potential applications, f is given, but suitable f_i are not to hand. This well–known problem (cf. [10]) has sometimes been overcome in an ad–hoc fashion [9, 13, 18, 32]. The objective of our investigation was to find a result which would meet the following criteria:

> Given a closed set $X \subset \mathbb{R}^d$ and a function $f : X \to \mathbf{R}$, we should be able to determine whether or not f has a C^k extension by examining only quantities which are *uniquely* and *explicitly* calculable from the values of f on X. Furthermore, we should be able to write down a *formula* or *algorithm* for an extension, if there is one.

Granted, there is an element of vagueness about these criteria, but on any interpretation they call for substantial sharpening of Whitney's result.

In this paper, we present a result which meets the first criterion: it tells us whether or not there exists a C^k extension, using only explicitly–calculable quantities.

Given X and f, we may insist that f be continuous, and then its graph is a closed subset of $\mathbf{R}^{d+1}$. The projection induces a natural map π_* from $\mathrm{Tan}^k(\mathrm{graph} f)$ onto $\mathrm{Tan}^k X$, in a way described below (cf. Section 3). We prove that f **has a $\mathbf{C}^k$ extension to $\mathbb{R}^d$ if and only if the map π_* is bijective**. The proof of sufficiency given in section 7 is nonconstructive, but we show in Section 5 how the stars $\mathrm{Tan}^k X$ may be explicitly computed.

As regards the second criterion, we have worked out constructive proofs of the extension theorem in a number of cases. See the remarks in section 7.

Whitney himself [40] gave a constructive condition for the existence of a C^k extension in the one–dimensional case. This was later refined somewhat by Merrien [25]. This condition involves the uniform continuity of a constructively–defined divided difference $f[x_0, \ldots, x_k]$ on the $(k+1)$–st symmetric product $X \times \cdots \times X$. This kind of condition is less straightforward to verify in examples than the condition of the present paper, since the new condition involves only the examination of a finite–dimensional vector space at each point.

In more than one dimension, there have been some results for more–or–less "nice" or "fat" sets (e.g. [4, 5, 6, 7]) and for functions with extra properties (e.g. [10]), and there is a C^1 result for arbitrary dimensions in [28], but as far as we are aware the present result is the first to deal comprehensively with C^k extensions from arbitrary closed sets and for arbitrary k.

2. Notation and definitions.

Throughout the paper, d and k denote non–negative integers.

The open ball with centre a and radius r in $\mathbf{R}^d$ is denoted by $\mathbb{B}(a,r)$.

For a multi–index $i \in \mathbb{Z}_+^d$ of order $j = |i| = \sum_{m=1}^d i_m$ we denote

$$\partial^i = \frac{\partial^j}{\partial x^i} = \frac{\partial^j}{\partial x_1^{i_1} \cdots \partial x_d^{i_d}}.$$

and $x^i = (x_1, \ldots, x_d)^i = x_1^{i_1} \cdots x_d^{i_d}$. The factorial $i!$ is $i_1 \cdots i_d$.

If V is a real topological vector space, then $V^\dagger$ denotes the algebraic dual of V, that is, the space of linear functionals on V, and V^* denotes the topological vector space dual of V, that is, the space of continuous linear functionals on V.

By a *star* we mean a triple (X, V, S), where X is a set, V is a real vector space, and $S \subset V$ is a set of the form

$$S = \bigcup_{a \in X} S(a)$$

where each $S(a)$ is a subspace of V and $S(a) \cap S(b) = \{0\}$ whenever $a \neq b$. We also say that S is a *star in* V *over* X. We call the $S(a)$ the *rays* of the star S. We define $p : S \sim \{0\} \to X$ by specifying that $p(s) = a$ whenever $s \in S(a)$.

By a morphism of a star (X, V, S) to another star (Y, W, T), we mean a pair (f, g) of maps such that $f : X \to Y$ and $g : S \to T$ such that $g|(S(a))$ is a linear map of $S(a)$ into $T(f(a))$, for each $a \in X$.

Observe that this implies that the star (X, V, S) will be isomorphic to the star (X, W, S) whenever V is a subspace of W. We are not very interested in V, which is simply there in order to provide a place in which elements of different rays may be added together.

By the *span* of a star S, we mean its linear span in the ambient vector space V.

A *topological star* is a star (X, V, S) such that X is a topological space, V is a topological vector space, S is a closed subset of V, and the mapping $p : S \sim \{0\} \to X$ is continuous. These conditions imply that the rays $S(a)$ are closed. Another consequence is that

$$\limsup_{x \to a} \dim S(x) \leq \dim S(a),$$

i.e. the dimension of rays is an upper semi–continuous function of the base point. Thus the sets

$$\{x \in X : \dim S(x) \geq j\}$$

are closed in X.

The ring of all real–valued polynomial functions on $\mathbf{R}^d$ is denoted by $\mathbf{R}[x_1, \ldots, x_d]$ or just $\mathbf{R}[x]$, and the subspace consisting of all polynomials of degree less than or equal to k is denoted by $\mathbf{R}[x]_k$. For each $a \in \mathbb{R}^d$, we define the quotient algebra

$$\mathbf{R}[x]_{k,a} = \mathbf{R}[x]/\langle (x-a)^i : |i| = k+1 \rangle.$$

The restriction to $\mathbf{R}[x]_k$ of the quotient map $\mathbf{R}[x] \to \mathbf{R}[x]_{k,a}$ is a linear isomorphism of $\mathbf{R}[x]_k$ onto $\mathbf{R}[x]_{k,a}$.

For a C^k–manifold, M, $\mathcal{C}^k(M)$ denotes the algebra consisting of all k–times continuously differentiable real–valued functions on M, with pointwise operations. With the usual topology, $\mathcal{C}^k(M)$ is a separable Frechet (i.e. complete, metrizable, locally–convex topological) algebra. When there is no danger of confusion, we shall abbreviate $\mathcal{C}^k(M)$ to $\mathcal{C}^k$.

Let X be a subset of M. The *kernel* of X is the ideal

$$X_\perp = I(X) = I_k(X) = \{f \in \mathcal{C}^k(M) : f|X \equiv 0\}.$$

For any point $a \in M$, we write the kernel $I_k(\{a\})$ of the singleton $\{a\}$ as $I_k(a)$ or $I(a)$ or $a_\perp$.

For any subset S of $\mathcal{C}^k(M)$, the *annihilator* of S is the subspace

$$S^\perp = \{\gamma \in \mathcal{C}^k(M)^* : \gamma(f) = 0, \ \forall f \in S\}.$$

Let $a \in M$. The space of (at most) k–th *order tangents to* M *at* a or *continuous point differential operators on* M *at* a *of order at most* k, denoted by $\mathrm{Tan}^k(M,a)$, is the annihilator of $I(a)^{k+1}$ in $\mathcal{C}^k(M)^*$; that is,

$$\mathrm{Tan}^k(M,a) = \mathcal{C}^k(M)^* \cap (I(a)^{k+1})^\perp.$$

Let X be a subset of M, and $a \in X$. The space of k–th *order tangents to* (M,X) *at* a, is the set

$$\mathrm{Tan}^k(M,X,a) = \mathcal{C}^k(M)^* \cap I(X)^\perp \cap (I(a)^{k+1})^\perp$$

of k–th order tangents to M that depend only on the values of functions on X.

Since $I(\text{clos}X) = I(X)$, we have

$$\text{Tan}^k(M, X, a) = \text{Tan}^k(M, \text{clos}X, a).$$

Thus there is no loss in restricting attention to closed sets X. We do this from now on.

The dual $\mathcal{C}^k(M)^*$ becomes a $\mathcal{C}^k(M)$–module in the usual way: if $\gamma \in \mathcal{C}^k(M)^*$ and $f \in \mathcal{C}^k(M)$, then $f \bullet \gamma \in \mathcal{C}^k(M)^*$ is defined on all $h \in \mathcal{C}^k(M)$ by

$$(f \bullet \gamma)(h) = \gamma(fh).$$

Since the annihilator of any ideal of $\mathcal{C}^k(M)$ is a $\mathcal{C}^k(M)$–submodule of $\mathcal{C}^k(M)^*$, $\text{Tan}^k(M, X, a)$ is a $\mathcal{C}^k(M)$–submodule of $\mathcal{C}^k(M)^*$.

Lemma 2.1. *The set*

$$\text{Tan}^k(M, X) = \bigcup_{a \in M} \text{Tan}^k(M, X, a)$$

is a topological star in ${\mathcal{C}^k}^*$ over M, with respect to the weak–star topology on ${\mathcal{C}^k}^*$. It is a closed substar of the star

$$\mathit{Tan}^k M = \bigcup_{a \in M} \mathit{Tan}^k(M, a).$$

Proof. This is easily verified. QED

We call $\text{Tan}^k M$ the k–th *order tangent star of* M, and we call $\text{Tan}^k(M, X)$ the k–th *order tangent star of* (M, X). The reason for this terminology is that we shall have occasion to consider one and the same X with respect to several containing manifolds M. As we shall see, the dependence of $\text{Tan}^k(M, X)$ on M is weak.

All the stars of interest to us are substars of $\text{Tan}^k M$'s.

For a nonzero $\partial \in \text{Tan}^k M$ we denote by $\text{pt}\partial$ the unique point $a \in M$ such that $\partial \in \text{Tan}^k(M, a)$. As a matter of convenience we define pt0 to be some fixed point (it doesn't matter which) of M (—this convention enables us to avoid making an exceptional case of $\partial = 0$ all the time).

By a *k–jet* we mean a function

$$j : S \to \mathbf{R}$$

where S is a substar of a $\text{Tan}^k M$ and $j|S(a)$ is linear for each $a \in M$.

In terms of local coordinates, point differential operators are familiar–looking objects. It is enough to consider $M = \mathbb{R}^d$:

Lemma 2.2. *Let $\partial \in Tan^k(\mathbb{R}^d, a)$. Then there exist unique $\alpha_i \in \mathbf{R}$ such that*

$$\partial(f) = \sum_{|i| \le k} \alpha_i \frac{\partial^i f}{\partial x_i}(a)$$

whenever $f \in \mathcal{C}^k$.

Proof. The ring $\mathbf{R}[x] = \mathbf{R}[x_1, \ldots, x_d]$ of polynomials in $x_1, \ldots, x_d$ is dense in $\mathcal{C}^k(\mathbb{R}^d)$. Thus ∂ is determined by its values on $\mathbf{R}[x]$. Since ∂ annihilates $(x-a)^i$ whenever $|i| > k$, it is in fact determined by its values on $\mathbf{R}[x]_{k,a}$, and hence by its values on $\mathbf{R}[x]_k$, and ∂f equals the value of ∂ on the Taylor polynomial of f at a. The existence follows. The uniqueness is clear. QED

To illustrate $\mathrm{Tan}^k(\mathbb{R}^d, a)$, consider $d = 2$ and fix $a \in \mathbf{R}^2$. We denote the evaluation functional $f \mapsto f(a)$ by δ_a, or just δ, and the functionals

$$f \mapsto \frac{\partial f}{\partial x}(a)$$

and

$$f \mapsto \frac{\partial f}{\partial y}(a)$$

by $\frac{\partial}{\partial x}|a$ and $\frac{\partial}{\partial y}|a$, or just $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. Similarly, $\frac{\partial^2}{\partial x^2}$ denotes $f \mapsto \frac{\partial^2 f}{\partial x^2}(a)$, etc. Then

$$\begin{aligned}
\mathrm{Tan}^0(\mathbf{R}^2, a) &= \mathbf{R}\delta, \\
\mathrm{Tan}^1(\mathbf{R}^2, a) &= \mathbf{R}\delta + \mathbf{R}\frac{\partial}{\partial x} + \mathbf{R}\frac{\partial}{\partial y}, \\
\mathrm{Tan}^2(\mathbf{R}^2, a) &= \mathbf{R}\delta + \mathbf{R}\frac{\partial}{\partial x} + \mathbf{R}\frac{\partial}{\partial y} \\
&\qquad + \mathbf{R}\frac{\partial^2}{\partial x^2} + \mathbf{R}\frac{\partial^2}{\partial x \partial y} + \mathbf{R}\frac{\partial^2}{\partial y^2},
\end{aligned}$$

etc.

A further consequence of the density of the polynomials in $\mathcal{C}^k(\mathbb{R}^d)$ is that the $\mathcal{C}^k(\mathbb{R}^d)$–modules $\mathcal{C}^k(\mathbb{R}^d)^*$ and $\mathrm{Tan}^k(\mathbb{R}^d, X, a)$ can be viewed as $\mathbf{R}[x]$–modules. The point is that the $\mathcal{C}^k$–action is completely determined by the $\mathbf{R}[x]$–action. Further, since $\mathrm{Tan}^k(\mathbb{R}^d, a)$ annihilates $(x-a)^i$ whenever the multi-index i has $|i| > k$, we may regard $\mathrm{Tan}^k(\mathbb{R}^d, a)$ as a module over the finite–dimensional algebra $\mathbf{R}[x]_{k,a}$.

A subset of $\mathrm{Tan}^k(\mathbb{R}^d, a)$ is a $\mathcal{C}^k$–module if and only if it is an $\mathbf{R}[x]_{k,a}$–module.

To illustrate the module action, consider $T = \mathrm{Tan}^2(\mathbb{R}^d, 0)$. We have

$$
\begin{aligned}
1 \bullet \frac{\partial^2}{\partial x \partial y} &= \frac{\partial^2}{\partial x \partial y}, \\
x \bullet \frac{\partial^2}{\partial x \partial y} &= \frac{\partial}{\partial y}, \\
y \bullet \frac{\partial^2}{\partial x \partial y} &= \frac{\partial}{\partial x}, \\
x^2 \bullet \frac{\partial^2}{\partial x \partial y} &= 0, \\
xy \bullet \frac{\partial^2}{\partial x \partial y} &= \delta, \\
x \exp{(x+y)} \bullet \frac{\partial^2}{\partial x \partial y} &= (x + x^2 + xy) \bullet \frac{\partial^2}{\partial x \partial y} \\
&= \frac{\partial}{\partial y} + \delta.
\end{aligned}
$$

As a module, T is generated, for instance, by the tangents $\frac{\partial^2}{\partial x^2}$, $\frac{\partial^2}{\partial x \partial y}$, and $\frac{\partial^2}{\partial y^2}$.

Lemma 2.2 shows that $\mathrm{Tan}^k(\mathbb{R}^d, a)$ is linearly isomorphic to $\mathbf{R}[x]_k$. Thus its dimension is $\binom{k+d}{k}$. For instance, $\mathrm{Tan}^2(\mathbf{R}^2, 0)$ has dimension 6, $\mathrm{Tan}^3(\mathbf{R}^7, 0)$ and $\mathrm{Tan}^7(\mathbf{R}^3, 0)$ have dimension 120, $\mathrm{Tan}^8(\mathbf{R}^8, 0)$ has dimension 12870, etc.

When M is a C^k manifold, and $X \subset M$, we may consider all the stars $T^0 = \mathrm{Tan}^0(M, X) \subset \mathcal{C}^0(M)^*$, $T^1 = \mathrm{Tan}^1(M, X) \subset \mathcal{C}^1(M)^*$, $\ldots$ $T^k = \mathrm{Tan}^k(M, X) \subset \mathcal{C}^k(M)^*$. Each is a C^k invariant. For a C^∞ manifold, we get an infinite sequence of invariants.

The star $T^0 = \mathrm{Tan}^0(M, X)$ is simple. The ray $T^0(a)$ is $\{0\}$ if $a \notin X$, and is one–dimensional if $a \subset X$. In fact,

$$\mathrm{Tan}^0(M, X, a) = \mathbf{R}\delta_a$$

whenever $a \in X$.

Formally, T^0 and T^1 are unrelated, since they lie in distinct vector spaces C^{0*} and C^{1*}, but there is a natural injection $C^{0*} \to C^{1*}$ (the restriction map, the adjoint to the inclusion $C^0 \subset C^1$). If we identify C^{0*} with its image in C^{1*}, then $\mathrm{Tan}^0(M, X)$ becomes a substar of $\mathrm{Tan}^1(M, X)$. Continuing, we may regard $\mathrm{Tan}^1(M, X)$ as a substar of $\mathrm{Tan}^2(M, X)$, and so on.

This allows us to define the *order* of an element ∂ of $\mathrm{Tan}^k(M, X)$, as the least j (necessarily $\leq k$) such that $\partial \in \mathrm{Tan}^j(M, X)$. However, as will appear from examples in Section 5, this order depends on X, in general; that is, if

$$\partial \in \mathrm{Tan}^k(M, X) \cap \mathrm{Tan}^k(M, Y),$$

then the order may be different if calculated with respect to X or to Y. Thus we must call it the X*–order*, to avoid confusion. In particular, there are two natural orders for an element $\partial \in \mathrm{Tan}^k(M, X)$, namely its X– order and its M–order. In what follows, we shall use the unqualified term order exclusively to refer to the M–order. For example, it might happen that the X–order of $\frac{\partial}{\partial y}|_a$ is 2, or 3, or more, but its order is always 1.

The order produces a grading of the module $\mathrm{Tan}^k M$ (and of its submodules), and the module action respects this grading, in the sense that multiplication of a tangent by a function does not increase the order.

The Frechet space $\mathcal{C}^k(M)$ is not normable unless M is compact, but there is a reasonable norm on the dual $\mathcal{C}^k(M)^*$ in case $M = \mathbb{R}^d$. Each $\mu \in \mathcal{C}^k(\mathbb{R}^d)^*$ has compact support, and we may define

$$\|\mu\|_{\mathcal{C}^{k*}} = \sup\{\mu(f) : f \in \mathcal{C}^k \text{ and } |\nabla^i f| \leq 1 \text{ on } \mathbb{R}^d,\ \forall |i| \leq k\}.$$

In the light of Whitney's extension theorem we can see that this norm is equivalent to

$$\mu \mapsto \sup\{\mu(f) : f \in \mathcal{C}^k \text{ and } |\nabla^i f| \leq 1 \text{ near } \mathrm{spt}\mu,\ \forall |i| \leq k\}.$$

3. Induced maps, and statement of main results.

Let M and N be C^k–manifolds, and let $F : M \to N$ be a C^k–map. Then F induces a continuous algebra homomorphism

$$F^\sharp : \begin{cases} \mathcal{C}^k(N) \to \mathcal{C}^k(M) \\ \qquad g \mapsto g \circ F, \end{cases}$$

and a continuous $\mathbf{R}$–linear map

$$F_\sharp : \mathcal{C}^k(M)^* \to \mathcal{C}^k(N)^*$$

defined by setting

$$\begin{aligned} F_\sharp(\gamma)(g) &= \gamma(F^\sharp(g)) \\ &= \gamma(g \circ F), \end{aligned}$$

for all $\gamma \in \mathcal{C}^k(M)^*$ and $g \in \mathcal{C}^k(N)$.

The algebra homomorphism $F^\sharp$ induces a $\mathcal{C}^k(N)$–action on any $\mathcal{C}^k(M)$–module: in particular the $\mathcal{C}^k(N)$–action on $\mathcal{C}^k(M)^*$ is defined by setting

$$g \underset{F}{\bullet} \gamma = F^\sharp(g) \bullet \gamma$$

for all $g \in \mathcal{C}^k(N)$ and $\gamma \in \mathcal{C}^k(M)^*$. Then $F_\sharp$ is a $\mathcal{C}^k(N)$–module homomorphism, that is,

$$F_\sharp[(F^\sharp g) \bullet \gamma] = g \underset{F}{\bullet} F_\sharp \gamma$$

for all $g \in \mathcal{C}^k(N)$ and $\gamma \in \mathcal{C}^k(M)^*$.

The associations

$$M \mapsto \mathcal{C}^k(M)^*$$
$$F \mapsto F_\sharp$$

described above define a covariant functor from the category of C^k–manifolds and C^k–maps to the category of topological vector spaces over $\mathbf{R}$ and continuous $\mathbf{R}$–linear maps.

Let the objects of the *category of pairs* be ordered pairs (M, X), where M is a C^k manifold and X is a closed subset of M; the morphisms from (M, X) to (N, Y) are C^k maps $F : M \to N$ such that $F(X) \subset Y$.

Similarly, let $\mathcal{T}$ denote the category of *pointed pairs*, with the following objects and morphisms: an object is a triple (M, X, a), where M is a C^k–manifold, X a closed subset of M and a an element of X; and a morphism from (M, X, a) to (Y, N, b) is a C^k–map $F : M \to N$ such that $F(X) \subseteq Y$ and $F(a) = b$.

Such a $\mathcal{T}$–morphism F from (M, X, a) to (Y, N, b) induces, as described above, a continuous $\mathbf{R}$–linear map $F_\sharp : \mathcal{C}^k(M)^* \to \mathcal{C}^k(N)^*$. Let F_* denote the restriction of $F_\sharp$ to $\mathrm{Tan}^k(M, X, a)$, so, for any $\partial \in \mathrm{Tan}^k(M, X, a)$, $F_*(\partial) \in \mathcal{C}^k(N)^*$ is defined by

$$F_*(\partial)(g) = \partial(g \circ F),$$

for $g \in \mathcal{C}^k(N)$. In fact, $F_*(\partial) \in \mathrm{Tan}^k(Y, N, b)$, as is readily seen.

Thus a $\mathcal{T}$–morphism F induces a continuous $\mathbf{R}$–linear map

$$F_* : \mathrm{Tan}^k(M, X, a) \to \mathrm{Tan}^k(N, Y, b)$$

and there is a functor from the category $\mathcal{T}$ to the category of topological real vector spaces, and continuous $\mathbf{R}$–linear maps, given by

$$(M, X, a) \mapsto \mathrm{Tan}^k(M, X, a)$$
$$F \mapsto F_*.$$

Similarly, a morphism $F : (M, X) \to (N, Y)$ induces a morphism

$$F_* : \mathrm{Tan}^k(M, X) \to \mathrm{Tan}^k(N, Y)$$

of topological stars, and the maps

$$\begin{cases} (M,X) \mapsto \mathrm{Tan}^k(M,X), \\ \qquad F \mapsto F_* \end{cases}$$

give a covariant functor from pairs to topological stars.

Recall that $\mathrm{Tan}^k(M,X,a)$ is a $\mathcal{C}^k(M)$–submodule of $\mathcal{C}^k(M)^*$, with action defined by

$$[f \bullet \partial](h) = \partial(fh)$$

for all $f,h \in \mathcal{C}^k(M)$ and $\partial \in \mathrm{Tan}^k(M,X,a)$. Further, for any $\mathcal{T}$–morphism F from (M,X,a) to (N,Y,b), the induced map F_* is a $\mathcal{C}^k(N)$–module homomorphism:

$$F_*[(F^\sharp g) \bullet \partial] = g \underset{F}{\bullet} F_*\partial$$

for all $g \in \mathcal{C}^k(N)$ and $\partial \in \mathrm{Tan}^k(M,X,a)$. Since any $\mathcal{C}^k(M)$–modulebecomes, via $F_\sharp$, a $\mathcal{C}^k(N)$–module, the image under F_* of any $\mathcal{C}^k(M)$–submodule of $\mathrm{Tan}^k(M,X,a)$ is a $\mathcal{C}^k(N)$–submodule of $\mathrm{Tan}^k(N,Y,b)$.

To illustrate induced maps F_*, consider the map

$$F : \begin{cases} \qquad \mathbf{R}^2 \to \mathbf{R}^3, \\ (x,y) \mapsto (u,v,w) = (x+y, x^2+y^2, xy^2), \end{cases}$$

If $\partial \in \mathrm{Tan}^2(\mathbf{R}^2, 0)$ is

$$\partial = \frac{\partial}{\partial x} + \frac{\partial^2}{\partial x \partial y},$$

then for $g = g(u,v,w) \in \mathcal{C}^2$ we compute

$$\begin{aligned} (F_*\partial)g &= \partial g(x+y, x^2+y^2, xy^2) \\ &= g_u + 2xg_v + y^2 g_w + g_{uu} + 2xg_{uv} + y^2 g_{uw} + 2xg_{uv} + 4x^2 g_{vv} \\ &\quad +2xy^2 g_{vw} + 2yg_w + y^2 g_{uw} + 2xy^2 g_{vw} + y^4 g_{ww}\big|_0 \\ &= g_u(0) + g_{uu}(0), \end{aligned}$$

so the image of ∂ under F_* is $\frac{\partial}{\partial u}|0 + \frac{\partial^2}{\partial u^2}|_0$.

In general, the value of $F_*\partial$ will have order less than or equal to the order of ∂, and depends only on the k–th order Taylor polynomial of F at $\mathrm{pt}\partial$. A sharper statement is true, as is easily seen:

Lemma 3.1. *Let $F : M \to N$ be a C^k map, let r be the degree of the lowest nonconstant term in the Taylor expansion of F at a point $a \in M$, and let $\partial \in \mathrm{Tan}^k(M, a)$. Then*

$$\mathit{order} F_* \partial \leq \mathit{order} \partial - r + 1.$$

We are now in a position to give a precise statement of the main result of this paper.

Theorem 1. *Let M be a C^k manifold, $X \subset M$ be closed, and $f : X \to \mathbf{R}$ be continuous. Let G denote the graph of f. Let*

$$\pi : \begin{cases} M \times \mathbf{R} \to M \\ (x, y) \mapsto x \end{cases}$$

be the projection and denote the point $(a, f(a))$ by $\tilde{a}$. Then f has a C^k extension to M if and only if the map

$$\pi_* : \mathit{Tan}^k(M \times \mathbf{R}, G, \tilde{a}) \to \mathit{Tan}^k(M, X, a)$$

is bijective for each $a \in X$.

Throughout the paper, we reserve the notation π for such projection maps.

Remarks. 1. The hypothesis that f be continuous cannot be removed. For instance, the function $f : \mathbf{R} \to \mathbf{R}$, given by

$$f(x) = \begin{cases} 0, x \leq 0, \\ 1/x, x > 0, \end{cases}$$

has closed graph, and π_* gives a bijection from $\mathrm{Tan}^k(\mathrm{graph} f)$ onto $\mathrm{Tan}^k\mathbf{R}$ for each k (see below for the calculation of such Tan^k's), although f is not continuous at 0.

2. There is a simple criterion for injectivity of π_*, which is given in the following lemma.

Lemma 3.2. *Let $x_1, \ldots, x_d$ be local coordinates for a neighbourhood U of a, and let $x_1, \ldots, x_d$, y be local coordinates for $U \times \mathbf{R}$. Then the map*

$$\pi_* : \mathit{Tan}^k(M \times \mathbf{R}, G, \tilde{a}) \to \mathit{Tan}^k(M, X, a)$$

is injective if and only if the tangent

$$\left.\frac{\partial}{\partial y}\right|_{\tilde{a}} : g \mapsto \frac{\partial g}{\partial y}(\tilde{a})$$

does not belong to $\mathrm{Tan}^k(M \times \mathbf{R}, G, \tilde{a})$.

Proof: Let $\partial \in \mathrm{Tan}^k(M \times \mathbf{R}, G, \tilde{a})$. We have $\pi_* \partial = 0$ if and only if ∂ is a sum of terms

$$\frac{\partial^{|i|+j}}{\partial x^i \partial y^j}$$

(— evaluated at $\tilde{a}$), all of which actually involve a $\frac{\partial}{\partial y}$, i.e. have $j \geq 1$. Thus, taking account of the way the module action works, if we have a $\partial \neq 0$ with $\pi_* \partial = 0$, then a suitable polynomial multiple $p \bullet \partial$ is exactly $\frac{\partial}{\partial y}$, and thus $\frac{\partial}{\partial y} \in \mathrm{Tan}^k(M \times \mathbf{R}, G, \tilde{a})$. QED

3. We say that a ray $\mathrm{Tan}^k(M, X, a)$ is of *e–dimensional type* if there is a coordinate map (i.e. a C^k diffeomorphism) $x : U \to \mathbb{R}^d$ of a neighbourhood U of a onto $\mathbb{R}^d$ such that x_* maps $\mathrm{Tan}^k(M, X, a)$ into $\mathrm{Tan}^k(\mathbb{R}^d, \mathbf{R}^e, 0)$. It is easy to see that Theorem 1 may be rephrased in the following form:

Corollary 3.3. *A closed subset X of a C^k manifold M is contained in some e–dimensional C^k submanifold of M if at each point $a \in X$ the ray $\mathrm{Tan}^k(M, X, a)$ is of e–dimensional type.*

4. Proof of Necessity.

Throughout this section, let M be a C^k–manifold, X a closed subset of M and $a \in X$.

Lemma 4.1. *$I_k(a)^{k+1}$ contains all the functions in $\mathcal{C}^k(M)$ that vanish near a.*

Proof: Let $g \in \mathcal{C}^k(M)$ be zero on a neighbourhood U of a. Let W be a compact neighbourhood of a that is included in U. Let $h \in \mathcal{C}^k(M)$ satisfy $h \equiv 0$ on W, and $h \equiv 1$ off U. Then $h \in I(a)$, and $g = h^k g$. Thus $g \in I(a)^{k+1}$. QED

Corollary 4.2. *Let $f, g \in \mathcal{C}^k(M)$. If $f = g$ on a neighbourhood of a, then*

$$\partial f = \partial g$$

for all $\partial \in \mathrm{Tan}^k(M, X, a)$. QED

Lemma 4.3 (Localness of Tangent Lemma). *Let U be an open subset of M such that $a \in U$. Then the map*

$$i_* : \mathrm{Tan}^k(U, X \cap U, a) \to \mathrm{Tan}^k(M, X, a),$$

induced by the inclusion i: $U \to M$*, is an isomorphism.*

Proof: For $\epsilon \in \mathrm{Tan}^k(U, X \cap U, a)$, $\mathrm{i}_*(\epsilon)$ is defined by

$$\mathrm{i}_*(\epsilon)(f) = \epsilon(f|U)$$

for $f \in \mathcal{C}^k(M)$. To define a function inverse to i_*, let $W \subseteq U$ be a compact neighbourhood of a and let $\phi \in \mathcal{C}^k(M)$ be a function such that $\phi \equiv 1$ on W, $\phi \equiv 0$ off U. The function $E : \mathcal{C}^k(U) \to \mathcal{C}^k(M)$ defined by setting, for $g \in \mathcal{C}^k(U)$,

$$E(g)(x) = \begin{cases} g(x)\phi(x), \ x \in U \\ \qquad\quad 0, \ x \notin U \end{cases}$$

is linear and continuous. Let j denote the restriction to $\mathrm{Tan}^k(M, X, a)$ of the map from $\mathcal{C}^k(M)^*$ to $\mathcal{C}^k(U)^*$ induced by E. Then, for $\partial \in \mathrm{Tan}^k(M, X, a)$, $j(\partial) \in \mathcal{C}^k(U)^*$ is defined by

$$j(\partial)(g) = \partial(E(g)), \ \forall g \in \mathcal{C}^k(U).$$

Clearly,

$$j : \mathrm{Tan}^k(M, X, a) \to \mathrm{Tan}^k(U, X \cap U, a),$$

and (making use of Corollary 4.2) it is easy to show that j is inverse to i. QED

Remark. Using this Lemma one also sees that the image $F_*\partial$ of a tangent under a C^k map F depends only on the germ of the map F at $\mathrm{pt}\partial$.

Lemma 4.4. *Let $e \leq d$, let X be a closed subset of $\mathbf{R}^e$, and let $a \in X$. Then the map*

$$i_* : \mathit{Tan}^k(\mathbf{R}^e, X, a) \to \mathit{Tan}^k(\mathbb{R}^d, X, a)$$

induced by inclusion is an isomorphism.

Proof: There is no loss of generality in assuming that $a = 0$. In view of the identification

$$\mathrm{Tan}^k(\mathbb{R}^d, X, 0) = \left\{ \partial = \sum_{|i| \leq k} \alpha_i \partial^i\big|_0 : \partial \perp I(X) \right\},$$

it is sufficient to show surjectivity. Fix

$$\partial = \sum_{|i| \leq k} \alpha_i \partial^i \in \mathrm{Tan}^k(\mathbb{R}^d, X, 0).$$

If $i_j \neq 0$ for some $j > e$, then $x^i \in I(\mathbf{R}^e)$. It follows that $x^i \in I(X)$, and so $\alpha_i = (i!)^{-1}\partial x^i = 0$. Thus $\partial \in \mathrm{Tan}^k(\mathbf{R}^e, X, 0)$. QED

Lemma 4.5. *Let N be a closed C^k–submanifold of M such that $X \subseteq N$. Then the inclusion map of N into M induces an isomorphism*

$$Tan^k(M, X, a) \cong Tan^k(N, X, a).$$

Proof: If U is any open subset of M such that $a \in U$, then $N \cap U$ is an open subset of N and by Lemma 4.3,

$$\mathrm{Tan}^k(M, X, a) \cong \mathrm{Tan}^k(U, X \cap U, a)$$
$$\mathrm{Tan}^k(N, X, a) \cong \mathrm{Tan}^k(N \cap U, X \cap U, a).$$

The required isomorphism is established by showing that the map

$$\mathrm{i}_* : \mathrm{Tan}^k(N \cap U, X \cap U, a) \to \mathrm{Tan}^k(U, X \cap U, a)$$

induced by inclusion is bijective, for the open subset U of M chosen in the following way.

Let $d = \dim M$ and $e = \dim N$. Take U to be an open neighbourhood of a in M such that there exists a bijective map $F : U \to \mathbb{R}^d$, F and F^{-1} have class C^k,

$$\begin{aligned} F(N \cap U) &= \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_{e+1} = \ldots = x_d = 0\} \\ &= \mathbf{R}^e, \text{say}, \end{aligned}$$

and $F(a) = 0 \in \mathbb{R}^d$. This is possible because N is a C^k–submanifold of M.

Let $Y = F(X \cap U)$. Then Y is a closed subset of $\mathbf{R}^e$, and

$$\mathrm{Tan}^k(U, X \cap U, a) \cong \mathrm{Tan}^k(\mathbb{R}^d, Y, 0).$$

Since $G = F|(N \cap U)$ maps $N \cap U$ bijectively onto $\mathbf{R}^e$ and G and G^{-1} are of class C^k, we have

$$\mathrm{Tan}^k(N \cap U, X \cap U, a) \cong \mathrm{Tan}^k(\mathbf{R}^e, Y, 0).$$

Thus, using Lemma 4.4, we have the diagram of isomorphisms:

$$\begin{array}{ccc}
\mathrm{Tan}^k(M, X, a) & & \\
\big\uparrow \ \mathrm{i} & & \\
\mathrm{Tan}^k(U, X \cap U, a) & \xrightarrow{F_*} & \mathrm{Tan}^k(\mathbb{R}^d, X, 0) \\
\big\uparrow \ \mathrm{i} & & \big\uparrow \ \mathrm{i} \\
\mathrm{Tan}^k(N \cap U, X \cap U, a) & \xrightarrow{G_*} & \mathrm{Tan}^k(\mathbf{R}^e, Y, 0) \\
\big\downarrow \ \mathrm{i} & & \\
\mathrm{Tan}^k(N, X, a) & &
\end{array}$$

and the result follows. QED

In the light of this lemma, it is not important which containing manifold M is used to compute $\mathrm{Tan}^k(M,X)$. Consequently, **we normally suppress reference to** M, and denote $\mathrm{Tan}^k(M,X)$ simply by $\mathrm{Tan}^k X$ (and $\mathrm{Tan}^k(M,X,a)$ by $\mathrm{Tan}^k(X,a)$).

Proof of Necessity. In view of functoriality (section 3) and localness (Lemma 4.3), it suffices to prove the theorem for the case where $M = \mathbb{R}^d$.

Fix $a \in X$, and let $\tilde{a} = (a, f(a))$ be the point above it on $\mathrm{graph} f$.

Inclusion i: $\mathbb{R}^d \to \mathbf{R}^{d+1}$ induces an isomorphism

$$\mathrm{i}_* : \mathrm{Tan}^k(\mathbb{R}^d, X, a) \to \mathrm{Tan}^k(\mathbf{R}^{d+1}, X, a),$$

by Lemma 4.4. Let $g \in \mathcal{C}^k(\mathbb{R}^d)$ have $g|X = f$. Define $F : \mathbf{R}^{d+1} \to \mathbf{R}^{d+1}$ by

$$F(x,y) = (x, y + g(x))$$

for all $x \in \mathbb{R}^d$, $y \in \mathbf{R}$. Then F is a C^k–map, F is bijective, and the inverse map F^{-1} defined by

$$F^{-1}(x,y) = (x, y - g(x))$$

is also a C^k–map. Further, F and F^{-1} are morphisms in the category $\mathcal{T}$. By functoriality, the induced map

$$F_* : \mathrm{Tan}^k(\mathbf{R}^{d+1}, X, a) \to \mathrm{Tan}^k(\mathbf{R}^{d+1}, \mathrm{graph} f, \tilde{a})$$

is an isomorphism. Now

$$\pi \circ F \circ \mathrm{i} = 1_{\mathbb{R}^d},$$

so

$$\pi_* \circ F_* \circ \mathrm{i}_* = 1.$$

Since $\mathrm{Tan}^k(\mathbb{R}^d, X, a)$ is a finite–dimensional vector space, and $\mathrm{Tan}^k(\mathbf{R}^{d+1}, \mathrm{graph} f, \tilde{a})$ has the same dimension, and π_* has a right inverse, it follows that π_* is bijective. QED

The Localness of Tangent Lemma shows that the condition that π_* be bijective is a local one. So also is the extension problem:

Lemma 4.6 (Localness of Extension). *Let X be a closed subset of a C^k manifold M, and let $f : X \to \mathbf{R}$ be continuous. Suppose that for each point $a \in X$ there exists an open neighbourhood U_a of a and a C^k function $g_a : U_a \to \mathbf{R}$ such that $g_a|U_a \cap X = f|U_a \cap X$. Then f has a C^k extension to M.*

Proof. Take a C^k (locally–finite) partition of unity $\{\phi_n\}$ on a neighbourhood of X in M, subordinate to the covering

$$\{U_a : a \in X\}.$$

Each ϕ_n maps some U_{a_n} to $\mathbf{R}$ and is supported on a compact subset of U_{a_n}. The functions $h_n = g_{a_n} \cdot \phi_{a_n}$, extended by zero, are C^k functions on M, so the function $g = \sum_n h_n$ provides a C^k extension of f to M. QED

It follows from this that we need only prove Theorem 1 in the case $M = \mathbb{R}^d$.

5. Calculation of $\mathbf{Tan}^k$.

In this section, X is a closed subset of $\mathbb{R}^d$, $a \in X$, and $\mathrm{Tan}^k(X, a)$ means $\mathrm{Tan}^k(\mathbb{R}^d, X, a)$. We will also find it convenient to use the notation

$$\langle \partial_1, \ldots, \partial_m \rangle$$

to represent the $\mathbf{R}[x]$–module generated by the tangents ∂_1 ,..., ∂_m. By Lemma 4.3, $\mathrm{Tan}^k(X, a)$ depends only on the germ of X at a. Also, if X agrees near a with an m-dimensional $\mathcal{C}^k$-submanifold of $\mathbb{R}^d$, then by applying Lemma 4.4 and the method of proof of Theorem 1, we see that a suitable perpendicular projection p will induce an isomorphism of $\mathrm{Tan}^k(X, a)$ onto $\mathrm{Tan}^k(\mathbf{R}^m, 0)$, so we may compute $\mathrm{Tan}^k(X, a)$, using $(p_*)^{-1}$.

For example, if we consider

$$X = \{(x, y, z) \in \mathbf{R}^3 : z = x + xy + x^3\}$$

at $a = 0$, then $\pi : (x, y, z) \mapsto (x, y)$ induces $\pi^{-1} : (x, y) \mapsto (x, y, x + xy + x^3)$. If we take, for instance, $\mathrm{Tan}^3(X, 0)$, it equals

$$\begin{aligned}\pi_*^{-1}\mathrm{Tan}^3(\mathbf{R}^2, 0) &= \pi_*^{-1}\left\langle \frac{\partial^3}{\partial x^3}, \frac{\partial^3}{\partial x^2 \partial y}, \frac{\partial^3}{\partial x \partial y^2}, \frac{\partial^3}{\partial y^3} \right\rangle \\ &= \left\langle \pi_*^{-1}\frac{\partial^3}{\partial x^3}, \pi_*^{-1}\frac{\partial^3}{\partial x^2 \partial y}, \pi_*^{-1}\frac{\partial^3}{\partial x \partial y^2}, \pi_*^{-1}\frac{\partial^3}{\partial y^3} \right\rangle,\end{aligned}$$

and, for instance,

$$\pi_*^{-1}\frac{\partial^3}{\partial x^3}f(x,y,z) = \frac{\partial^3}{\partial x^3}f(x,y,x+xy+x^3)$$
$$= f_{xxx} + 3f_{zxx} + 3f_{zzx} + f_{zzz} + 6f_z,$$

so that

$$\pi_*^{-1}\frac{\partial^3}{\partial x^3} = \frac{\partial^3}{\partial x^3} + 3\frac{\partial^3}{\partial x^2\partial z} + 3\frac{\partial^3}{\partial x\partial z^2} + 6\frac{\partial}{\partial z}.$$

However, these X are very special, and we need a method for dealing with general sets X.

We denote by $\mathrm{Tan}^k(X,a)_\perp$ the vector space

$$\{p(x_1,\dots,x_d) \in \mathbf{R}[x]_k : \partial p = 0,\ \forall \partial \in \mathrm{Tan}^k(X,a)\}.$$

Since $\mathrm{Tan}^k(X,a)$ is linearly isomorphic to a subspace of $\mathbf{R}[x]_k^\dagger$, and $\dim \mathbf{R}[x]_k < \infty$, we obtain:

Lemma 5.1. *$Tan^k(X,a) = [Tan^k(X,a)_\perp]^\perp$ whenever $X \subseteq \mathbf{R}^d$ is closed and $a \in X$.*

Thus, if we know that polynomials $p_1,\dots,p_n$ belong to $\mathrm{Tan}^k(X,a)_\perp$, then we deduce that

$$\mathrm{Tan}^k(X,a) \subseteq \{p_1,\dots,p_n\}^\perp.$$

This provides a way to come at $\mathrm{Tan}^k(X,a)$ from above, as long as we can test polynomials for membership in $\mathrm{Tan}^k(X,a)_\perp$. We shall give such a test, shortly.

We denote by $J_k(a)$ the $\mathcal{C}^k$–ideal consisting of all those functions $f \in \mathcal{C}^k$ that are *flat of order k* at the point a, i.e.

$$\partial^i f(a) = 0, \qquad \forall |i| \le k.$$

It is easily deduced from Taylor's theorem that $J_k(a)$ may also be described as the closure in $\mathcal{C}^k$ of the ideal of all those C^k functions that vanish on a neighbourhood of a. It may thus be described in Function Algebra terms as the minimal closed ideal with hull $\{a\}$.

Lemma 5.2. *$I_k(X) + J_k(a)$ is a closed subspace of $\mathcal{C}^k$.*

Proof: $\mathcal{C}^k/J_k(a)$ is a finite dimensional Hausdorff topological vector space, hence all its subspaces are closed in the quotient topology. The quotient map

$$q : \mathcal{C}^k \to \mathcal{C}^k/J_k(a)$$

is continuous, hence

$$I_k(X) + J_k(a) = q^{-1}\frac{I_k(X) + J_k(a)}{J_k(a)}$$

is closed. QED

Lemma 5.3. *Let $p \in \mathbf{R}[x]_k$. Then the following are equivalent:*
(1) $p \in \mathrm{Tan}^k(X,a)_\perp$;
(2) there exists $g \in J_k(a)$ such that $p + g \in I_k(X)$.

Proof: Let (1) hold. Since $\mathrm{Tan}^k(X,a) = \mathcal{C}^* \cap (I_k(X)+J_k(a))^\perp$, the Separation Theorem yields $p \in \mathrm{clos}_{\mathcal{C}^k}(I_k(X)+J_k(a))$. The last lemma then yields $p \in I_k(X)+J_k(a)$, whence (2) holds. The converse is obvious. QED

This lemma provides a workable way to identify polynomials in $\mathrm{Tan}^k(X,a)_\perp$. Next, we give a way to come at $\mathrm{Tan}^k(X,a)$ from below. For a sequence of sets $A_n \subset \mathbf{R}^d$, and $a \in \mathbf{R}^d$, we write $A_n \to a$ if for all $r > 0$ there exists $N \in \mathbf{N}$ such that $n > N \Rightarrow A_n \subset \mathbb{B}(a,r)$. For $\mu = \sum_{i=1}^n \lambda_i \partial_i \in \mathrm{span}\mathrm{Tan}^k(\mathbb{R}^d)$, with $\lambda_i \neq 0$ and $\mathrm{pt}(\partial_i) \neq \mathrm{pt}(\partial_j)$, we set

$$\mathrm{spt}\mu = \{\mathrm{pt}(\partial_i) : i = 1, \ldots, n\}.$$

Lemma 5.4. *Let $\partial \in \mathrm{Tan}^k(\mathbb{R}^d, a)$. Suppose there exists $\mu_n \in \mathrm{span}\mathrm{Tan}^k(X)$ such that $\mathrm{spt}\mu_n \to a$ and $\mu_n \to \partial$ weak-star in $\mathcal{C}^{k*}$. Then $\partial \in \mathrm{Tan}^k(X,a)$.*

Proof: Since $\partial \in \mathrm{Tan}^k(\mathbb{R}^d, a)$, we have $\partial \perp J_k(a)$. Since $\mu_n \in I_k(X)^\perp$, we have $\partial \in I_k(X)^\perp$. Thus $\partial \in J_k(a)^\perp \cap I_k(X)^\perp$. QED

Corollary 5.5. *Let $\partial \in \mathrm{Tan}^k(\mathbb{R}^d, a)$. Suppose there exists $\mu_n \in \mathrm{span}\{\delta_x : x \in X\}$ such that $\mathrm{spt}\mu_n \to a$ and $\mu_n \to \partial$ weak-star in $\mathcal{C}^{k*}$. Then $\partial \in \mathrm{Tan}^k(X,a)$. QED*

It is perhaps tempting to suppose that all $\partial \in \mathrm{Tan}^k(X,a)$ might be obtained as in this corollary, but in general this fails. It may be necessary to iterate the construction several times, using limits of limits, limits of limits of limits, and so on. There is, however, a limit (!) to the number of iterations needed. Fix X and a, and let

$$V_0 = \bigcup\{\mathrm{span}\{\delta_x\} : x \in X\},$$

and, inductively,

$$V_{n+1} = \{\partial : \exists \mu_n \in \mathrm{span}V_n \text{ with } \mathrm{spt}\mu_n \to \mathrm{pt}(\partial) \text{ and } \mu_n \to \partial \text{ weak-star}\}.$$

Then the V_n are stars and the vector spaces $V_n(a)$ are increasing subspaces of the finite-dimensional vector space $\mathrm{Tan}^k(\mathbb{R}^d, a)$, hence they stabilize after a finite number $N(k, X, a)$ of steps. This argument does not yield an effective bound on N, since it is *a priori* conceivable that the $V_n(a)$ might be stationary for a range of n's, and then increase. Thus the following is interesting:

Problem. *Find, if possible, $N(k)$ such that for all X, $V_{N(k)} = \mathit{Tan}^k(X)$.*

We know (see below) that $N(1) = 2$. There is no known control for $N(2)$.

We can now see a practical way to completely determine $\mathrm{Tan}^k(X, a)$. If we establish that $\partial_1, \ldots, \partial_m \in \mathrm{Tan}^k(X, a)$ (by using Lemma 5.4) and that $p_1, \ldots, p_n \in \mathrm{Tan}^k(X, a)_\perp$ (by using Lemma 5.3), and if

$$\mathrm{span}\{\partial_1, \ldots, \partial_m\} = \{p_1, \ldots, p_n\}^\perp,$$

then $\mathrm{Tan}^k(X, a)$ is the common value.

Example 1. *Let $X \subset \mathbf{R}$.*
(a) If $a \in X$ is isolated, then $\mathit{Tan}^k(X, a) = \mathbf{R}\delta_a, \forall k$.
(b) If $a \in X$ is an accumulation point, then $\mathit{Tan}^k(X, a) = \mathit{Tan}^k(\mathbf{R}, a)$.

Proof: (a) is clear.

(b) Choose $x_n \to a$, $x_n \neq a$. Then $(x_n - a)^{-1}(\delta_{x_n} - \delta_a) \to \frac{d}{dx}$, weak-star in $\mathcal{C}^{k*}$, so the Corollary yields $\frac{d}{dx} \in \mathrm{Tan}^k(X, a)$. If $k > 1$, then

$$2(x_n - a)^{-2}\left\{\delta_{x_n} - \delta_a - (x_n - a)\frac{d}{dx}\right\} \to \frac{d^2}{dx^2},$$

weak-star in $\mathcal{C}^{k*}$, so Lemma 5.4 yields $\frac{d^2}{dx^2} \in \mathrm{Tan}^k(X, a)$. Continuing, we obtain

$$\mathrm{Tan}^k(\mathbf{R}, a) = \mathrm{span}\left\{\delta_a, \frac{d}{dx}, \ldots, \frac{d^k}{dx^k}\right\},$$

hence the result. QED

Example 2.

$$X = \left\{(x, y) \in \mathbf{R}^2 : y^2 = x^3\right\}, \quad a = (0, 0).$$

For $f \in \mathcal{C}^1$, we have

$$f(h^2, h^3) = f(0, 0) + f_x(0, 0)h^2 + \mathrm{o}(h^2)$$

whence $\delta_a,\ \frac{\partial}{\partial x}|_a \in \mathrm{Tan}^1(X,a)$. Also,

$$f(h^2,h^3) - f(h^2,-h^3) = 2h^3 f_y(h^2,-h^3) + \mathrm{o}(h^3)$$

whence $\frac{\partial}{\partial y}|_a \in \mathrm{Tan}^1(X,a)$. Thus

$$\mathrm{Tan}^1(X,a) = \mathrm{Tan}^1(\mathbf{R}^2,a).$$

Moving on to $\mathrm{Tan}^2(X,a)$, we note first that for $f \in \mathcal{C}^2$ we have

$$f(h^2,h^3) = f(0,0) + f_x(0,0)h^2 + f_y(0,0)h^3 + \frac{1}{2}f_{xx}(0,0)h^4 + \mathrm{o}(h^4)$$

so $\delta_a,\ \frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial^2}{\partial x^2} \in \mathrm{Tan}^2(X,a)$. Also

$$\begin{aligned} f(h^2,h^3) - f(h^2,-h^3) &= f_y(h^2,0)2h^3 + \frac{1}{2}f_{yy}(h^2,0)h^6 + \mathrm{o}(h^6) \\ &= f_y(0,0)h^3 + f_{yx}(0,0)h^5 + \mathrm{o}(h^5) \end{aligned}$$

so $\frac{\partial^2}{\partial x \partial y} \in \mathrm{Tan}^2(X,a)$. In the other direction, $y^2 \in \mathrm{Tan}^2(X,a)_\perp$, because $-x^3 \in J_2(a)$ and $y^2 - x^3 \in I_2(X)$. Since

$$\left\{y^2\right\}^\perp = \mathrm{span}\left\{\delta_a, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial x \partial y}\right\},$$

we conclude that $\mathrm{Tan}^2(X,a)$ equals the common value. QED

Example 3.

$$X = \left\{(x,y) \in \mathbf{R}^2 : y^2 = x^3, y \geq 0\right\}, \quad a = (0,0).$$

As in the last example, we see that $\delta_a,\ \frac{\partial}{\partial x} \in \mathrm{Tan}^1(X,a)$ and $\delta_a,\ \frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial^2}{\partial x^2} \in \mathrm{Tan}^2(X,a)$. But this time, $\frac{\partial}{\partial y} \notin \mathrm{Tan}^1(X,a)$. In fact, the nonnegative function $|x|^{\frac{3}{2}}$ belongs to $J_1(a)$ and $y - |x|^{\frac{3}{2}} = 0$ on X, hence $y \in \mathrm{Tan}^1(X,a)_\perp$, and we conclude that

$$\mathrm{Tan}^1(X,a) = \mathrm{span}\left\{\delta_a, \frac{\partial}{\partial x}\right\}.$$

Similarly, $|x|^{\frac{5}{2}}$ belongs to $J_2(a)$ and $xy - |x|^{\frac{5}{2}} = 0$ on X, so xy belongs to $\mathrm{Tan}^2(X,a)_\perp$, as does y^2, so

$$\mathrm{Tan}^2(X,a) = \mathrm{span}\left\{\delta_a, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial^2}{\partial x^2},\right\}.$$

QED

6. Tan^k and other tangent concepts.

First, we compare Tan^k with the classical tangent concepts of Whitney, Denjoy and Zariski.

Let $X \subset \mathbb{R}^d$ be closed, and $a \in X$. A vector $u \in \mathbb{R}^d$ is called

(1) a *Whitney unit tangent* if there exist $x_n \in X, x_n \neq a, x_n \to a$, with

$$|x_n - a|^{-1}(x_n - a) \to u;$$

(2) a *Denjoy unit tangent* if there exist $x_n, y_n \in X, x_n \neq y_n, x_n \to a, y_n \to a$, with

$$|x_n - y_n|^{-1}(x_n - y_n) \to u;$$

(3) a *Zariski tangent vector* if

$$u \cdot \nabla f(a) = 0, \quad \forall f \in I_1(X).$$

Let us denote the span of the Whitney unit tangents by $W(X,a)$, the span of the Denjoy unit tangents by $D(X,a)$, and the space of Zariski tangent vectors by $Z(X,a)$. Let

$$\begin{aligned} w(X,a) &= \dim W(X,a), \\ d(X,a) &= \dim D(X,a), \\ z(X,a) &= \dim Z(X,a). \end{aligned}$$

Each Whitney unit tangent is a Denjoy unit tangent, but the converse fails; consider, for instance, the set $xy = 0$ in $\mathbf{R}^2$. In general we have

$$W \subset D \subset Z.$$

Each inclusion may be proper. The pair (X,a) of Example 2 above has $w = 1$ and $d = 2$. It takes a little more effort to arrange $d \neq z$:

Example 4.

$$\begin{aligned} X = \{(0,0)\} &\cup \left\{ \left(\frac{1}{n}, 0 \right) : n \in \mathbf{N} \right\} \\ &\cup \left\{ \left(\frac{1}{n} + \frac{1}{(2m)!}, 0 \right) : n, m \in \mathbf{N}, m > n \right\} \\ &\cup \left\{ \left(\frac{1}{n} + \frac{1}{(2m+1)!}, \frac{1}{n(2m+1)!} \right) : n, m \in \mathbf{N}, m > n \right\}, \end{aligned}$$

and $a = (0,0)$.

It is straightforward to check that at a, the set of Denjoy unit tangents is $\{(\pm 1, 0)\}$, so $d(a) = 1$. At the point $a_n = (\frac{1}{n}, 0)$, the set of Whitney unit tangents is

$$\left\{(1,0), \left(\frac{n}{\sqrt{n^2+1}}, \frac{1}{\sqrt{n^2+1}}\right)\right\},$$

hence $w(a_n) = 2$, hence $d(a_n) = z(a_n) = 2$. The dimension $z(x)$ cannot jump down, so

$$\dim z(a) \geq \lim_{n \uparrow \infty} z(a_n) = 2,$$

and $z(a) = 2$.

In general, the Zariski tangent space is essentially the pure first-order part of $\mathrm{Tan}^1(X, a)$:

$$\begin{aligned} \mathrm{Tan}^1(X,a) &= \mathbf{R}\delta_a + \{f \mapsto u \cdot \nabla f(a) : u \in Z(X,a)\} \\ &\simeq \mathbf{R}\delta_a \oplus Z(X,a). \end{aligned}$$

So $\mathrm{Tan}^k(X, a)$ may be described as the natural extension of the Zariski tangent space to higher orders.

The space $D(X, a)$ may also be regarded as the space of continuous point derivations on a certain algebra. In [28] it was shown that $D(X, a)$ is isomorphic to the space of (pure) first-order continuous point derivations on the space $D^1(X) = \mathrm{clos}_{\mathrm{Lip}(1,X)}\mathcal{C}^1$. It was also shown that the star associated to $D(X)$ is dense in the star associated to $Z(X)$. See below.

Each compact subset of the unit circle $\mathbb{S}^1$ is the set of Whitney unit tangents for some (X, a). Each compact subset of $\mathbb{S}^1$ symmetric under reflection in the origin is the set of Denjoy unit tangents for some (X, a).

Regarded as differential invariants of the pair (X, a), the numbers w, d and z have different powers of discrimination. For instance, w distinguishes $y^2 - x^3 = 0$ from $y^2 - x^2 = 0$, whereas d and z do not; d and z distinguish $y^2 - x^3 = 0$ from $y = 0$ whereas w does not; d distinguishes the pair (X, a) of Example 4 from $(\mathbf{R}^2, a)$, whereas z does not.

For the set $xy(x^2 - y^2) = 0$, the set of Whitney unit tangents distinguishes it from $y = 0$ and from $\mathbf{R}^2$ at (0,0), and none of the other classical invariants does this.

The collection of $\mathrm{Tan}^k(X, a)$'s and the associated invariant numbers such as

$$\tau_k(X,a) = \dim \mathrm{Tan}^k(X,a)$$

are superior to all the classical invariants. For instance, Tan^2 distinguishes $y^2 - x^3 = 0$ from $y^2 - x^2 = 0$; τ_1 distinguishes $y^2 - x^3 = 0$ from $y = 0$; τ_2 distinguishes the pair (X, a) of Example 4 from $(\mathbf{R}^2, a)$; τ_1 distinguishes $xy(x^2 - y^2) = 0$ from $y = 0$; τ_4 distinguishes $xy(x^2 - y^2) = 0$ from $\mathbf{R}^2$.

A couple of these remarks will bear elaboration. First, we have the following:

Lemma 6.1. *If u is a Whitney unit tangent to X at a, and $k \in \mathbf{N}$, then there exists $\partial \in Tan^{k-1}(\mathbb{R}^d, a)$ such that $(u \cdot \nabla)^k + \partial \in Tan^k(X, a)$.*

Remark: We call a tangent of the form $(u \cdot \nabla)^k + \partial$ a *top-order pure* tangent.

Proof of Lemma. There exist $x_n \in X$ such that $x_n \to a$ and $|x_n - a|^{-1}(x_n - a) \to u$. Let $h_n = x_n - a$. Then for $f \in \mathcal{C}^k$ we have

$$f(x_n) = f(a) + h_n \cdot \nabla f(a) + \ldots + \frac{1}{k!}(h_n \cdot \nabla)^k f(a) + \mathrm{o}(|h_n|^k).$$

Let $u_n = |h_n|^{-1} h_n = u + \epsilon_n$, $d_n = |h_n|$. Then

$$\begin{aligned} f(x_n) = f(a) &+ u \cdot \nabla f(a) d_n + \epsilon_n \cdot \nabla f(a) d_n \\ &+ \frac{1}{2}(u \cdot \nabla)^2 f(a) d_n^2 + \frac{1}{2}(u \cdot \nabla)(\epsilon_n \cdot \nabla) f(a) d_n^2 + \frac{1}{2}(\epsilon_n \cdot \nabla)^2 f(a) d_n^2 \\ &+ \cdots \\ &+ \frac{1}{k!}(u \cdot \nabla)^k f(a) d_n^k + \mathrm{o}(d_n^k). \end{aligned}$$

From this relation we can peel off various tangents at a. First, we get $u \cdot \nabla f(a)$. Next, it depends on the relative sizes of $|\epsilon_n|$ and d_n. Removing some x_n, if need be, we may assume that $\frac{|\epsilon_n|}{d_n} \to 0$ or $\frac{d_n}{|\epsilon_n|} \to 0$ or $\frac{\epsilon_n}{d_n} \to v \in \mathbb{R}^d$. If $\frac{|\epsilon_n|}{d_n} \to 0$, we get $(u \cdot \nabla)^2 \in \mathrm{Tan}^2(X, a)$. If $\frac{d_n}{|\epsilon_n|} \to 0$, we may assume that $\frac{\epsilon_n}{|\epsilon_n|} \to v \in \mathbb{R}^d$, and we get $v \cdot \nabla \in \mathrm{Tan}^2(X, a)$ and then $(u \cdot \nabla)^2 \in \mathrm{Tan}^2(X, a)$. If $\frac{\epsilon_n}{d_n} \to v \in \mathbb{R}^d$, then we get $(u \cdot \nabla)^2 + 2(v \cdot \nabla) \in \mathrm{Tan}^2(X, a)$. With the third order terms, it is necessary to consider further cases, but the pattern continues: we may pick up additional first or second order pdo's, but we get, at least, $(u \cdot \nabla)^3 + \partial$ for a ∂ of order less than 3. We omit further details. QED

One of the numerical invariants we may associate to $\mathrm{Tan}^k(X, a)$ is the dimension p_k of the span of those $u \in \mathbb{R}^d$ such that $(u \cdot \nabla)^k + \partial \in \mathrm{Tan}^k(X, a)$ for some ∂ of order $k - 1$. This is an invariant, because a $\mathcal{T}$-morphism $\phi: (\mathbb{R}^d, X, a) \to (\mathbb{R}^d, Y, b)$, when restricted to and projected on the k-th order homogeneous tangents, is k-linear, equal to the k-th symmetric product of its derivative. Hence it maps objects of the form

$$(u \cdot \nabla)^k + \partial$$

with order $\partial < k$ to objects of the form

$$(v \cdot \nabla)^k + \partial'$$

where $v = D\phi(a)u$ and order $\partial' < k$.

In the cases of $y^2 - x^3 = 0$ and $y^2 - x^2 = 0$, at the origin, the first has $p_2 = 1$ and the second has $p_2 = 2$.

As regards $xy(x^2 - y^2) = 0$ and $\mathbf{R}^2$, observe that the fourth order polynomial $xy(x^2 - y^2) \in \mathrm{Tan}^4(X, 0)_\perp$, so $\mathrm{Tan}^4(X, 0) \neq \mathrm{Tan}^4(\mathbf{R}^2, 0)$. More generally, we have the following:

Example 5. *Let X be contained in the union of k nonsingular C^k-curves in $\mathbf{R}^2$ that meet at a. Then $\mathrm{Tan}^k(X, a) \neq \mathrm{Tan}^k(\mathbf{R}^2, a)$.*

To see this, choose $g_1, \ldots, g_k \in \mathcal{C}^k$ such that $g_i = 0$ on X and $\nabla g_i(a) \neq (0, 0)$. Set $u_i = \nabla g_i(a)$, and let ∂ denote the tangent

$$f \mapsto [(u_1 \cdot \nabla) \ldots (u_k \cdot \nabla) f](a).$$

Then the product $g = g_1 \ldots g_k$ belongs to $\mathcal{C}^k$, and is annihilated by $\mathrm{Tan}^k(X, a)$, but

$$\partial f = |u_1|^2 \ldots |u_k|^2 \neq 0.$$

Thus $\partial \in \mathrm{Tan}^k(\mathbf{R}^2, a) \sim \mathrm{Tan}^k(X, a)$. QED

Example 3 shows that Tan^2 can distinguish the top half of $y^2 - x^3 = 0$ from the right half axis. It thus possesses a discriminating power superior to all the classical tangent spaces.

Next, let us compare Tan^k with the vector bundles of Pohl and Feldman.

The star $\mathrm{Tan}^k M$ is

$$\bigcup_{a \in M} \mathrm{Tan}^k(M, a).$$

The corresponding *disjoint* union

$$T^k(M) = \dot{\bigcup_{a \in M}} \mathrm{Tan}^k(M, a)$$

is a vector bundle. The subsets

$$T^k(M, X) = \dot{\bigcup_{a \in M}} \mathrm{Tan}^k(M, X, a)$$

corresponding to closed subsets $X \subset M$ are not, in general, vector bundles, because they are not locally–trivial, but if X is a C^k–submanifold of M, then $T^k(M,X)$ is a sub–bundle of $T^k(M)$. These bundles were introduced and studied by Pohl in his thesis, and have been used by Feldman to study higher–order connections [16, 30, 31]. One could expound the results of the present paper using the framework of the bundle–like objects $T^k(M,X)$. Our main reason for adopting the present exposition is that the set $\mathrm{Tan}^k(M,X)$ is in any case unavoidable, because it is essential at various points to associate elements of C^{k*} to tangents (— each nonzero element $t \in T^k(M,X)$ corresponds uniquely to a functional $\partial \in \mathrm{Tan}^k(M,X)$). The present approach avoids an extra layer of notation.

Finally, let us describe the relationship between $\mathrm{Tan}^k(M,X,a)$ and Glaeser's space of *paratangents of order* k [17].

Following Glaeser, we say that a C^k submanifold N of M gives a *minimal* C^k *imbedding for* X *near* a if (1) the germ of X at a is contained in N, and (2) no C^k submanifold of smaller dimension than N contains the germ of X at a. It is clear that minimal C^k imbeddings always exist, for any X and a.

Lemma 6.2. *(Glaeser) Suppose that N is a minimal C^k imbedding for X near a, and N' is another C^k submanifold of M that contains X. Then $T_aN \subset T_aN'$.*

We repeat the simple proof, for the reader's convenience.

Proof. Suppose there existed $\partial \in T_aN \sim T_aN'$. Then, near a, $N \cap N'$ would be a C^k submanifold of smaller dimension than N, and would contain the germ of X at a, contradicting the minimality of N. QED

Thus T_aN is the same for all minimal embeddings N of X near a. The *paratangent of order* k *of* X *at* a is defined to be this common T_aN. We denote it by $\mathrm{ptan}^k(M,X,a)$.

Evidently, from the lemma, we also have that $\mathrm{ptan}^k(M,X,a)$ is the intersection of all the T_aN, taken over those C^k submanifolds N of M that contain X.

The relationship between $\mathrm{Tan}^k(M,X,a)$ and $\mathrm{ptan}^k(M,X,a)$ is the following.

Lemma 6.3. *For each C^k manifold M, each closed set $X \subset M$, and each point $a \in M$, we have*
$\mathbf{R}\delta_a \oplus \mathit{ptan}^k(M,X,a) = \mathit{Tan}^1(M,a) \cap \mathit{Tan}^k(M,X,a)$.

Proof. One direction of this is explicitly shown by Glaeser [17, p.55, Corollary 1] (apart from the unwinding of the definitions): If g is a C^k function on M that vanishes on

X, and there existed a $\partial \in \mathrm{ptan}^k(M, X, a)$ with $\partial f \neq 0$, then a is not a critical point of g, so the implicit function theorem tells us that near a, the set $N = g^{-1}(0)$ is a C^k submanifold, and thus by Lemma 6.2,

$$\mathrm{ptan}^k(M, X, a) \subset \mathrm{Tan}^1(M, N, a),$$

so $\partial \in \mathrm{Tan}^1(M, N, a)$, so $\partial g = 0$, which is a contradiction. Thus

$$\mathrm{ptan}^k(M, X, a) \subset \mathrm{Tan}^k(M, X, a),$$

and that shows that the lhs is contained in the rhs.

To see the other direction, suppose that

$$\partial \in \mathrm{Tan}^1(M, a) \cap \mathrm{Tan}^k(M, X, a),$$

and let N be a minimal C^k immersion for X near a. If it could happen that $\partial \notin \mathbf{R}\delta_a \oplus T_a N$, then there would exist a function $g \in C^k(M)$ with $g|N = 0$ and $\partial g \neq 0$. But then $g \in I_k(X)$ and $\partial g \neq 0$, which is impossible. QED

We have the following corollary to Lemma 3.2, where, as usual, we use the notation $\tilde{a}$ for the point $(a, f(a))$ on the graph G of the continuous function $f : X \to \mathbf{R}$.

Corollary 6.4. *The map*

$$\pi_* : Tan^k(M \times \mathbf{R}, G, \tilde{a}) \to Tan^k(M, X, a)$$

is injective if and only if the restriction of π_* to $ptan^k(M \times \mathbf{R}, G, \tilde{a})$ is injective.

Note that π_* (in common with all induced maps) does not increase order, and hence *a priori* maps $\mathrm{ptan}^k(M \times \mathbf{R}, G, \tilde{a})$ into the span of $\mathbf{R}\delta_a$ and $\mathrm{ptan}^k(M, X, a)$. In fact, since it is a projection map, it is easy to see that no point masses can occur in the image, so it actually maps ptan^1 to ptan^1.

Proof. It is clear that the condition is necessary. For the converse, it suffices to note that $\frac{\partial}{\partial y}|_{\tilde{a}} \in \mathrm{Tan}^1(M, a)$, and apply Lemma (3.2). QED

We are now in a position to prove sufficiency.

7. Proof of Sufficiency.

It is convenient to state and prove an amplified version of Theorem 1.

Theorem $1'$. *Let M be a C^k manifold, X a closed subset of M, and $f : X \to \mathbf{R}$ be a continuous function. Let G denote the graph of f. Then the following four conditions are equivalent:*
(1) f has a C^k extension to M;
(2) $\pi_* : Tan^k(M \times \mathbf{R}, G) \to Tan^k(M, X)$ is a bijection.
(3) $\pi_* : Tan^k(M \times \mathbf{R}, G) \to Tan^k(M, X)$ is an injection.
(4) $\pi_* : ptan^k(M \times \mathbf{R}, G, a) \to ptan^k(M, X, a)$ is injective for each $a \in X$.

Proof: We showed that (1) implies (2) in section 4. That (2) implies (3) is obvious, and Corollary 6.4 states that (3) implies (4). It remains to prove that (4) implies (1).

Suppose that (4) holds. Fix $a \in X$. Let $d = \dim M$. The space $\mathrm{ptan}^k(M, X, a)$ has dimension at most d, so the same is true for $\mathrm{ptan}^k(M \times \mathbf{R}, G, \tilde{a})$. Let $\tilde{N}$ be a minimal C^k imbedding for G near $\tilde{a}$, and suppose its dimension is r. Since $\frac{\partial}{\partial y}|_{\tilde{a}} \notin T_a\tilde{N}$, the projection π maps a neighbourhood of a bijectively onto an r–dimensional submanifold $N \subset M$, which must then be a minimal imbedding for X near a. Replacing $\tilde{N}$ by $\tilde{N} \cap \pi^{-1}(N)$, if need be, we see that $\tilde{N}$ is the graph of a C^k function $f_1 : N \to \mathbf{R}$, which extends f from $N \cap X$ to N. Let f_a be a C^k extension of f_1 to a tubular neighbourhood U_a of N in M (— the existence of such extensions of smooth functions from smooth submanifolds is immediate from one of the standard alternative definitions of smooth manifold). We have now extended f to a neighbourhood U_a of a.

The existence of a global extension of f now follows from Lemma 4.6. QED

Remarks.

1. It remains an interesting problem to give a constructive proof of the theorem. We have worked out explicit constuctions for the case when $k = 1$ (C^1 extensions in any dimension) and the case $k = 2$, $d = 1$ (C^2 extensions in 1 dimension). These will appear elsewhere.

The first author has previously provided a couple of explicit C^1 extension theorems [27,28], but the methods used there have no hope of dealing with C^2 extensions. The matter is closely connected with the lack of anything corresponding to a Besicovitch structure theory ([15], section 3.3) for smoothness 2 or greater.

We believe that the Tan^k structure will prove crucial to the successful completion of the constructive extension programme, and that, for instance, the paratangent alone does not carry sufficient information to allow a reasonable extension formula.

2. In section 5 we described practical procedures for computing $\mathrm{Tan}^k(M, X, a)$ in explicitly–described examples. In view of the theorem just proved, it is of interest to derive procedures for computing the paratangent spaces $\mathrm{ptan}^k(M, X, a)$.

The definition of $\mathrm{ptan}^k(M, X, a)$ is completely nonconstructive, but its constructibility is a consequence of Lemma 6.3 and the constructibility of
$\mathrm{Tan}^k(M, X, a)$. The description of $\mathrm{ptan}^k(M, X, a)$ as the space of first–order distributions supported at a that are weak–star limits in $C^k(M)*$ of linear combinations of point masses from X [17, p.56] is not constuctive.

We recall two results from section 5. Corollary 5.5 allows us to extend progressively our stock of elements of
$\mathrm{Tan}^k(M, X, a)$. Lemma 5.3 allows us to develop a list of annihilators of $\mathrm{Tan}^k(M, X, a)$. If linearly–independent k–order tangents $\partial_1,\ldots,\partial_n$ belong to $\mathrm{Tan}^k(M, X, a)$ and if functions $p_1,\ldots,p_m$ annihilate $\mathrm{Tan}^k(M, X, a)$ and have linearly–independent cosets in
$C^k(M)/J^k(a)$, and if

$$n + m = \dim Tan^k(M, a) \left(= \binom{k+d}{k} \right)$$

then

$$\mathrm{Tan}^k(M, X, a) = \mathrm{span}_{\mathbf{R}}\{\partial_1, \ldots, \partial_n\}.$$

Recall that $C^k(M)/J_k(a)$ is essentially the finite–dimensional local algebra $\mathbf{R}[x]_{0,k}$. We can apply these results to construct elements of $\mathrm{ptan}^k(M, X, a)$ and of $\mathrm{ptan}^k(M, X, a)^\perp$. If ∂_1, ∂_2, …, ∂_n are linearly independent elements of $\mathrm{ptan}^k(M, X, a)$ and p_1, …, p_m

annihilate $\mathrm{ptan}^k(M,X,a)$ and have linearly–independent cosets in $C^k(M)/J_1(M)$, and if $n+m=\dim M$, then

$$\mathrm{ptan}^k(M,X,a)=\mathrm{span}_{\mathbf{R}}\{\partial_1,\ldots,\partial_n\}.$$

For instance, take $M=\mathbf{R}^2$, $a=(0,0)$, $k=2$, and

$$X=\{(x,0):x<0\}\cup\{(x,x^2):x\geq 0\}.$$

Then $\delta_{(x,0)}$ $(x<0)$ and $\delta_{(x,x^2)}$ $(x\geq 0)$ belong to $\mathrm{Tan}^2(M,X)$, so Lemma 5.5 tells us that

$$\begin{aligned}\lim_{x\uparrow 0}\frac{\delta_{(x,0)}-\delta_{(0,0)}}{x}&=\frac{\partial}{\partial x}|_a,\\ \lim_{x\uparrow 0}\frac{\delta_{(2x,0)}-2\delta_{(x,0)}+\delta_{(0,0)}}{x^2}&=\frac{\partial^2}{\partial x^2}|_a,\\ \lim_{x\downarrow 0}\frac{\delta_{(2x,4x^2)}-2\delta_{(x,x^2)}+\delta_{(0,0)}}{x^2}&=\frac{\partial^2}{\partial x^2}|_a+2\frac{\partial}{\partial y}|_a\end{aligned}$$

belong to $\mathrm{Tan}^2(M,X,a)$, hence $\frac{\partial}{\partial x}|_a$ and $\frac{\partial}{\partial y}|_a$ belong to $\mathrm{ptan}^2(M,X,a)$, hence

$$\mathrm{ptan}^2(M,X,a)=T_a\mathbf{R}^2.$$

Thus the function π_*, induced by $\pi:(x,y)\mapsto x$, is not injective, as is to be expected, since X is the graph of a non–C^2 function.

3. For a non–closed set X, when

$$\mathrm{Tan}^k(M,X)=\mathrm{Tan}^k(M,\mathrm{clos}X),$$

Theorem $1'$ remains true if we replace the assumption of continuity on f by boundedness.

4. It is worth remarking that one can quite easily formulate a superficially reasonable–looking necessary and sufficient condition for the existence of a C^k extension, which involves only the values of f on X.

Regarding X as a subset of $\mathcal{C}^k(M)^*$ (via the map $a\mapsto\delta_a$), we may refer to $\mathrm{span}X$. Since the δ_a are linearly independent, f determines a linear map $f_\sharp:\mathrm{span}X\to\mathbf{R}$. We have the following:

Proposition. *f has a C^k extension to M if and only if $f_\sharp$ is weak–star continuous on span$X$.*

Proof. Only if is obvious. The converse is a simple application of the Hahn–Banach theorem for locally–convex spaces: given that $f_\sharp$ is weak–star continuous on $\mathrm{span}X$, it has a weak–star continuous extension to ${\mathcal{C}^k}^*$, and this extension is of the form $T \mapsto T(g)$ for some $g \in \mathcal{C}^k$, whence $g|X = f$. QED

The reason this is not really interesting is that it is useless without a way to check weak–star continuity of $f_\sharp$. In effect, useful theorems like Whitney's, and those of the present paper, provide manageable sufficient conditions for this weak–star continuity.

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