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A PROPERTY OF UNIVALENT FUNCTIONS IN A_p

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Abstract. The univalent functions in the diagonal Besov space A_p, where $1 < p < \infty$, are characterized in terms of the distance from the boundary of a point in the image domain. Here A_2 is the Dirichlet space. A consequence is that there exist functions in A_p, $p > 2$, for which the area of the complement of the image of the unit disc is zero.

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Introduction. The Dirichlet space A_2 consists of analytic functions on the disc whose images, counting multiplicity, have finite area. If one relaxes the condition on f by allowing f to belong to the somewhat larger space, the diagonal Besov space $A_p = A_{pp}^{1/p}$ with $p > 2$, what can one say about the area of the image of f? In this note we show that for such a function, the complement of the image of the unit disc may have zero area.

Let $1 < p < \infty$. Denote by A_p the space of functions $f(z)$ that are analytic on the open unit disc $D = \{z : |z| < 1\}$, and satisfy

$$\int_D |f'(z)|^p (1 - |z|^2)^{p-2}\, dA(z) < \infty.$$

The spaces A_p are called the *diagonal Besov spaces* to distinguish them from the more general class of Besov spaces A_{pq}^s, where $s > 0$, $1 < p, q < \infty$. See **[2]**. If we set $s = 1/p$, $q = p$ we obtain the space we call A_p. If $p < r$, then $A_p \subset A_r$, while A_2 is the Dirichlet space. On letting p tend to infinity we may identify A_∞ as the space of analytic functions f satisfying

$$(1 - |z|^2)|f'(z)| = O(1) \text{ as } |z| \to 1-.$$

This is the *Bloch space B*. The subspace of B, consisting of functions f for which

$$(1 - |z|^2)|f'(z)| \to 0 \text{ as } |z| \to 1-,$$

is denoted by B_0, and called the *little Bloch space*.

The distance function $d(w)$. An analytic function f on D which is one to one is said to be univalent. For a point $w = f(z)$ in the image domain an important notion is that of distance to the boundary:

$$d(w) = \inf\{|w - \zeta|,\ \zeta \in \partial f(D)\}.$$

We state the following corollary of Koebe's Distortion Theorem **[3]**.

THEOREM A. *Suppose that f is univalent in D. Then*

$$\frac{1}{4}d(w) \leq (1-|z|^2)|f'(z)| \leq d(w).$$

Thus for univalent f, we have

(i) $f \in B$ *if and only if* $\sup_w d(w) < \infty$,

(ii) $f \in B_0$ *if and only if* $\lim_{|z|\to 1} d(w) = 0$.

We can extend this result to A_p.

THEOREM 1. *Let f be univalent in D and* $1 < p < \infty$. *Then* $f \in A_p$ *if and only if*

$$\int_{f(D)} d(w)^{p-2}\, dA(w) < \infty.$$

Proof. From Theorem A we have, for $p > 2$,

$$\frac{1}{4^{p-2}} d(w)^{p-2} \leq (1-|z|^2)^{p-2}|f'(z)|^{p-2} \leq d(w)^{p-2}.$$

We observe that $dA(w) = |f'(z)|^2\, dA(z)$. Integrating the inequality above with respect to the measure $dA(w)$ over the image domain $f(D)$, we get

$$\frac{1}{4^{p-2}} \int_{f(D)} d(w)^{p-2}\, dA(w) \leq \int_D (1-|z|^2)^{p-2}|f'(z)|^p\, dA(z) \leq \int_{f(D)} d(w)^{p-2}\, dA(w).$$

For $1 < p < 2$, the inequalities are reversed. The result follows.

This simple result has useful consequences which we shall see in a moment. Pommerenke **[4]** has given a condition whereby a non-vanishing univalent function g in D has the property that $\log g$ belongs to B_0 (and consequently also to the space $VMOA$).

As above, for $w = g(z)$ we let $d(w) = \inf\{|w-\zeta|,\ \zeta \in \partial g(D)\}$. Then

$$\log g \in B_0 \text{ if and only if } \frac{d(w)}{|w|} \to 0 \text{ as } |w| \to 0, \infty.$$

We can extend this result to A_p.

THEOREM 2. *Suppose that g is univalent and non-vanishing in D, where* $1 < p < \infty$, *and let* $f(z) = \log g(z)$. *Then*

$$f \in A_p \text{ if and only if } \int_{g(D)} \frac{d(w)^{p-2}}{|w|^p}\, dA(w) < \infty.$$

Proof. As before, for $p > 2$, we have

$$\frac{1}{4^{p-2}} d(w)^{p-2} \leq (1 - |z|^2)^{p-2} |g'(z)|^{p-2} \leq d(w)^{p-2}.$$

This gives

$$\frac{1}{4^{p-2}} \frac{d(w)^{p-2}}{|w|^p} \, dA(w) \leq (1 - |z|^2)^{p-2} \frac{|g'(z)|^p}{|g(z)|^p} \, dA(z) \leq \frac{d(w)^{p-2}}{|w|^p} \, dA(w).$$

Observing that the middle term is $(1 - |z|^2)^{p-2} |f'(z)|^p \, dA(z)$ we get the result by integration. Again, for $1 < p < 2$ the inequalities are reversed.

Two applications of Theorem 1. According to a theorem of Richter and Shields [**5**], every function f in the Dirichlet space A_2 can be written as the quotient of two bounded functions in A_2. This result depends on the fact that there is a compact set K having positive two-dimensional Lebesgue measure lying in the complement of $f(D)$. Their proof is such that if we could prove the last statement above for any $f \in A_p$, then an analogue of their conclusion would hold: if $f \in A_p$ then $f = g/h$, where $g, h \in A_p \cap H^\infty$. We need only take $p > 2$ since if $p \leq 2$ the area of $f(D)$ is finite. However we shall now show that there exists a univalent function $f \in A_p$ such that the complement of the image $f(D)$ has zero two-dimensional Lebesgue measure.

The following construction uses an idea from an unpublished manuscript of Douglas M. Campbell. Consider for each integer $m \geq 0$, the half-strip

$$S_{m,1} = \{x + iy : m < x < m + 1, \ 0 < y < \infty\}.$$

We perform a countable number of operations the n^{th} of which is the removal from $S_{m,1}$ of 2^{n-1} infinite vertical slits whose initial points are $2\pi in/(m + 1)^2 + k/2^n + m$, $(k = 1, 3, \ldots 2^n - 1)$. In the lower half strip

$$S_{m,2} = \{x + iy : m < x < m + 1, \ -\infty < y < 0\},$$

we carry out operations which are the mirror image of those above; that is we perform a countable number of operations the n^{th} of which is the removal from $S_{m,2}$ of 2^{n-1} infinite vertical slits whose initial points are $-2\pi in/(m + 1)^2 + k/2^n + m$, $(k = 1, 3, \ldots 2^n - 1)$. We have now made a countable number of slits in the right half plane. Finally we extend the slitting procedure to the left half plane by reflection in the y axis. We denote the resulting simply connected domain by G. Note that $0 \in G$ and also each line $\Re z = m$ for each integer m. Now let f be the conformal mapping of D onto G with $f(0) = 0, f'(0) > 0$.

We shall now show that $\int_G d(w)^{p-2} dA(w) < \infty$ and invoke Theorem 1, thereby showing that $f \in A_p$. We may confine attention to the first quadrant. Consider the half-strip $S_{m,1}$ and, for $n \geq 1$, consider the subset

$$L(m, n) = \{w \in S_{m,1} : 2\pi n/(m + 1)^2 < \Im w < 2\pi(n + 1)/(m + 1)^2\}$$

with area $2\pi/(m+1)^2$. It is easy to see that $d(w) < 1/2^n$ for each $w \in L(m, n)$. It follows that

$$\int_{S_{m,1}} d(w)^{p-2} dA(w) \leq \sum_{n=0}^{\infty} \frac{1}{2^{n(p-2)}} \frac{2\pi}{(m+1)^2} = 2\pi C_p/(m+1)^2.$$

Summing over m now gives the desired result. It is clear that the area of the complement of $f(D)$ is zero.

REMARK. Under the assumption $2 \leq p < \infty, 0 < q < \infty$ and $0 < s < 1/2$, K. Dyakonov [**1**] has shown that every function in A^s_{pq} is the ratio of two bounded functions in A^s_{pq}. We noted above that if $1 < p < 2$ then the proof of Richter and Shields can be adapted to give the result for A_p. Thus the conclusion holds for A_p, for all $p > 1$.

For a second application suppose that f is a bounded univalent function on D. Clearly $f \in A_2$, since the area of $f(D)$ is finite. We show that f need not belong to A_p for any $p < 2$. Consider the open unit square $Q = \{(x, y) : 0 < x < 1,\ 0 < y < 1\}$. For each $n \geq 1$ we make 2^{n-1} vertical slits in Q each of height $1/(n+1)$ with base points $(k/2^n,\ k = 1, 3, \ldots 2^n - 1)$. The resulting simply connected domain is called G. Let f be a conformal map of D onto G and let $w = u + iv$ be a point in G. Consider the points w of G lying in a strip $\frac{1}{n+2} < v < \frac{1}{n+1}$. We readily check that $d(w) \leq 1/2^{n+1}$ for all points w in the strip. Choose $p < 2$. It follows that

$$\int_G d(w)^{p-2} dA(w) \geq 2^{2-p} \sum_{n=1}^{\infty} \frac{2^{(2-p)n}}{(n+1)(n+2)} = \infty,$$

which implies by Theorem 1 that f is not in A_p.

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