COMPARING THE FLOYD AND IDEAL BOUNDARIES OF A METRIC SPACE

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0. Introduction

There are various notions of "boundaries at infinity" of metric spaces in the literature. Perhaps the best known is the ideal boundary $\partial_I X$ defined using geodesic rays, and particularly studied for the classes of CAT(0) and proper geodesic Gromov hyperbolic spaces. Closely related to this is the concept of a Gromov boundary $\partial_G X$ defined using Gromov sequences (or "sequences converging to infinity"). For more on both of these concepts, see for instance [BH], [GH], [CDP], [BHK], and [V]. The Gromov boundary is usually defined only for Gromov hyperbolic spaces but we extend this concept to arbitrary metric spaces.

A third type of boundary at infinity is the *g-boundary* $\partial_g X$ of an unbounded length space which replaces an unbounded metric l by a bounded metric σ obtained via a conformal distortion involving a suitable function g. The *spherical* and *Floyd boundaries* are both just the g-boundary, but with g restricted to lie in certain nice classes of functions; many results involving these concepts are independent of the choice of g. The spherical boundary arises as a byproduct of sphericalization, a concept introduced in [BB2] in order to interpret results in [BB1] concerning the quasihyperbolizations of bounded length spaces in the context of unbounded spaces. The Floyd boundary has been studied for hyperbolic groups; see, for instance, [F], [CT], [T], [M], [K1], and [K2]. Both are special cases of the μ-boundary discussed in [G, Section 7.2] and [CDP, Chapter 11]. See also [H] for a study of uniformizing conformal distortions of hyperbolic spaces.

The current paper aims to shed more light on $\partial_g X$ by comparing and contrasting it with the ideal and Gromov boundaries. Understanding the spherical boundary is vital for deciding when sphericalization is an invertible process (see [BB2, Section 4]), but no detailed study of it was carried out there or elsewhere. Some results in this direction can be found in the literature (as we discuss in Section 2), but here we prove comparison results for larger classes of spaces X and functions g.

In Section 1, we define the above boundaries at infinity, along with their associated topologies. In Section 2, we state and prove the main comparison theorems. Finally, in Section 3, we give various examples to show that all assumptions in the comparison theorems are essential.

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1. Notation and Preliminaries

We write $a \vee b$ and $a \wedge b$ for the maximum and minimum, respectively, of two numbers a, b.

1.1. Metric spaces and paths.

Let (X, d) be a metric space which may have additional properties as specified. We denote by $B_d(x, r)$, $\overline{B}_d(x, r)$, and $S_d(x, r)$, the *open ball*, *closed ball*, and *sphere of radius* r *about* $x \in X$. If $r \le 0$, $B_d(x, r)$ is the empty set. A metric space is *proper* if all its closed balls are compact.

We denote by $\overline{X}_d$ the metric closure of (X, d) and, viewing X as a subset of $\overline{X}_d$, we write $\partial X_d = \overline{X}_d \setminus X$. If γ is a path in $\overline{X}_d$, $\mathrm{len}_d(\gamma)$ denotes the d-length of γ. Given $x, y \in \overline{X}_d$, we denote by $\Gamma_d(x, y; X)$, or simply $\Gamma_d(x, y)$, the class of rectifiable paths $\lambda : [0, T] \to \overline{X}_d$ parametrized by d-arclength for which $\gamma|_{(0,T)}$ is a path in X, $\lambda(0) = x$, and $\lambda(T) = y$. An *arc* in X is an injective path $\gamma : I \to X$. We do not distinguish notationally between paths and their images. If γ is an arc in X, and $u, v \in \gamma$, $\gamma[u, v]$ is the subarc of γ with endpoints u, v.

If Y is a rectifiably connected subset of a metric space (X, d), then the *intrinsic metric* in Y is defined by the rule that the distance between $x, y \in Y$ is the infimum of the d-lengths of paths in $\Gamma_d(x, y; Y)$. A rectifiably connected metric space (X, d) is a *length space*, and d is a length metric, if the intrinsic metric on X equals d, while it is a *geodesic space* if, for all $x, y \in X$, there exists a path $\gamma \in \Gamma_d(x, y)$ of length $d(x, y)$.

For a length space (X, l) and a function $f : X \to (0, \infty)$, we define the *conformal distortion* l' *of* l by

$$l'(x, y) = \inf_{\gamma \in \Gamma_l(x,y)} \int_\gamma f(z)\, dl(z), \qquad x, y \in X,$$

More briefly, we sometimes write $dl' = f\, dl$ or $dl'(z) = f(z)\, dl(z)$.

Given $\alpha \ge 1$, $h \ge 0$, an (α, h)-*quasi-isometry* f between metric spaces (X, d) and (X', d') is one satisfying the condition

$$\alpha^{-1} d(u, v) - h \le d'(f(u), f(v)) \le \alpha d(u, v) + h, \qquad u, v \in X.$$

If $\alpha = 1$, we say that f is a h-*rough isometry*; if $\alpha = 1$ and $h = 0$, we say that f is an *isometry*. We suppress the quasi-isometry parameters α, h whenever their exact values are irrelevant. *Quasigeodesics*, *rough geodesics*, and *geodesics* are quasi-isometric, rough isometric, and isometric images, respectively, of an interval $I \subset \mathbb{R}$; we append *segment*, *ray*, or *line* to these terms if I is of the form $[a, b]$, $[a, \infty)$, or $(-\infty, \infty)$, respectively, for some $a, b \in \mathbb{R}$, $a < b$. We also use the notation $[x, y]$, $x, y \in X$, to denote any geodesic segment from x to y if the metric is understood. Following Väisälä [V], we define a h-*short arc* to be an injective h-rough geodesic segment.

It is convenient to have two notations for sequences in metric spaces: we write $x = (x_i)$. We index sequences of sequences as follows: $x^j = (x^j_i)_{i=1}^\infty$, $j \in \mathbb{N}$.

1.2. CAT(0) and Gromov hyperbolic spaces.

Let us briefly recall some basic concepts and results concerning CAT(0) and Gromov hyperbolic spaces. For more on CAT(0) spaces, see [BH], and for more on hyperbolic spaces, see [CDP], [GH], and [V]. Below, (X, d) is a metric space.

A *h-short triangle $T \subset X$ with vertices x, y, z* is the union of three h-short arcs with endpoints x, y, z; we call these arcs the *sides* of T. In particular, a *geodesic triangle* is a 0-short triangle.

A *triangle map* is a function $f : T \to \mathbb{R}^2$ from a geodesic triangle T onto a Euclidean triangle $T' \subset \mathbb{R}^2$ whose sides are of the same length as those of T, and such that the restriction of f to any one side is an isometry. Note that triangle maps always exist, and are unique up to an isometry of $\mathbb{R}^2$. A *CAT(0) space* is a geodesic space (X, d) such that, $d(u, v) \le |f(u) - f(v)|$ for all $u, v \in T$ whenever $f : T \to \mathbb{R}^2$ is a triangle map for the geodesic triangle $T \subset X$.

The *Gromov product* with basepoint $p \in X$ is defined by

$$\langle x,\, y\rangle_p = (d(x,p) + d(p,y) - d(x,y))/2, \qquad x, y \in X.$$

We write $\langle x,\, y\rangle_{p;d}$ if the metric needs to be specified. The metric space (X, d) is *(Gromov) δ-hyperbolic*, $\delta \ge 0$, if

$$\langle x,\, z\rangle_p \ge \langle x,\, y\rangle_p \wedge \langle y,\, z\rangle_p - \delta, \qquad x, y, z, p \in X. \tag{1.3}$$

This definition makes it clear that if a space X is hyperbolic, then so are all spaces that are roughly isometric to X. A useful estimate [V, 2.33] says that if x, y lie in a δ-hyperbolic space (X, d), and $\gamma \in \Gamma_d(x, y)$ is h-short, then

$$\mathrm{dist}_d(p, \gamma) - 2\delta - h \le \langle x,\, y\rangle_p \le \mathrm{dist}_d(p, \gamma) + h/2. \tag{1.4}$$

Indeed only the lower bound requires hyperbolicity.

We say that (X, d) has *δ-thin triangles*, $\delta \ge 0$, if the distance from a point on a side of a geodesic triangle to the union of the other two sides is never more than δ. If a geodesic space (X, d) is δ-hyperbolic, then it has 3δ-thin triangles, and if it has δ-thin triangles, it is 3δ-hyperbolic; this follows from the $h = 0$ variants of 2.34 and 2.35 in [V]. Since every δ-hyperbolic space can be imbedded in a geodesic δ-hyperbolic space [BS], it follows that even nongeodesic δ-hyperbolic spaces have 3δ-thin triangles. However the thin triangles condition may then be much weaker than hyperbolicity due to a lack of geodesic segments.

A *tripod* $\tau \subset \mathbb{R}^2$ is the union of line segments from a single common endpoint p to three distinct endpoints a, b, c; we equip τ with its intrinsic metric l. A *h-tripod map* ϕ takes a h-short triangle $T \subset X$ to a tripod τ, in such a way that its restriction to any one side is a h-rough isometry. It is readily verified that if T is a h-short triangle with vertices x, y, z, then there exists a h-tripod map ϕ from T to a tripod with segments of length $\langle x,\, y\rangle_z$, $\langle x,\, z\rangle_y$, and $\langle y,\, z\rangle_x$. Furthermore, if (X, d) is δ-hyperbolic, then we can (and always do) choose $\phi : (T, d) \to (\tau, l)$ to be a $(4\delta + 4h)$-rough isometry; see 2.15 and 2.24 of [V].

1.5. Ideal and Gromov boundaries.

Given a geodesic space (X, d), let $\mathrm{GR}_d(X)$ be the class of geodesic rays in X parametrized by d-arclength, and let $\mathrm{GR}_d(X, w)$ be the class of all rays in $\mathrm{GR}_d(X)$ with initial point w; we omit the d subscript if the metric is understood. The rays $\gamma, \nu \in \mathrm{GR}(X)$ are equivalent, $\gamma \sim \nu$, if $d_H(\gamma, \nu) < \infty$, where d_H is the Hausdorff distance associated with d, that is

$$d_H(\gamma, \nu) = \sup_{x\in\gamma} \mathrm{dist}_d(x, \nu) \vee \sup_{x\in\nu} \mathrm{dist}_d(x, \gamma).$$

It is clear that $\gamma \sim \nu$ if and only if $\sup_{t\ge 0} d(\gamma(t)), \nu(t)) < \infty$. We write $[\gamma]_X$, or simply $[\gamma]$, for the equivalence class of $\gamma \in \mathrm{GR}(X)$, and define the *ideal boundary* $\partial_I X$ to be $\mathrm{GR}(X)/\sim$. We also write $\overline{X}_I = X \cup \partial_I X$.

If (X, d) is a complete CAT(0) space, we attach the *cone topology* τ_{C} to $\overline{X}_I$. This topology is defined using a basepoint $o \in X$, but is independent of the choice of o. For a detailed definition, see [BH, II.8.5], but we briefly define the concept here. First, in any complete CAT(0) space, there is a unique geodesic γ_x from o to $x \in \overline{X}_I$ parametrized by d-arclength. This is rather obvious for $x \in X$, and is proven in [BH, II.8.2] for $x \in \partial_I X$; in this latter case, we mean that $\gamma_x \in \mathrm{GR}(X, o)$ and $[\gamma_x] = x$. Let $X_r := \partial_I X \cup (X \setminus \overline{B}_d(o, r))$, let $p_r : X_r \to S_d(o, r)$ be the "projection" defined by $p_r(x) = \gamma_x(r)$, and let the set $U(a, r, s)$, $r, s > 0$, consist of all $x \in X_r$ such that $d(p_r(x), p_r(a)) < s$. Then τ_{C} is the topology on $\overline{X}_I$ which coincides with the d-topology on X, and has as a local base at $a \in \partial_I X$ the sets $U(a, r, s)$, $r, s > 0$. It is easily verified that τ_{C} is Hausdorff and, since it can be defined as an inverse limit topology, τ_C is compact whenever X is proper.

Let (X, d, o) be a pointed metric space, and let $\langle\cdot, \cdot\rangle \equiv \langle\cdot, \cdot\rangle_o$ denote the Gromov product with respect to the basepoint o. A sequence $x = (x_i)$ in X is a *Gromov sequence* if $\langle x_i, x_j\rangle \to \infty$ as $i, j \to \infty$. We define a binary relation E on the set of Gromov sequences as follows:

$$x \mathrel{E} y \quad \Longleftrightarrow \quad \liminf_{i,j\to\infty} \langle x_i, y_j\rangle = \infty.$$

We say that the sequences x and y are equivalent, $x \sim y$, if there is a finite chain of sequences x^k, $0 \le k \le k_0$, such that

$$x = x^0, \quad y = x^{k_0}, \qquad \text{and} \qquad x^{k-1} \mathrel{E} x^k, \quad 1 \le k \le k_0.$$

This is easily seen to be an equivalence relation. The *Gromov boundary* $\partial_G X$ is the set of all equivalence classes $[x]$ of Gromov sequences x, and we write $\overline{X}_G = X \cup \partial_G X$. We shall use without comment the fact that every Gromov sequence is equivalent to each of its subsequences.

Our definition of $\partial_G X$ is nonstandard in two ways: $\partial_G X$ is usually defined only for hyperbolic spaces, and in that setting it is customary to say that x and y are equivalent if $\lim_{i\to\infty} \langle x_i, y_i\rangle = \infty$. In the hyperbolic setting, such a definition is equivalent to our definition according to 5.3 and 5.6 of [V], but in a nonhyperbolic setting the definitions differ. The reason we use a lim inf over i, j rather than a limit over $i = j$, is so that we can associate a geodesic ray with a Gromov sequence in our more general setting. Also note that, although E and $\sim$ are the same equivalence relation in the hyperbolic setting (as is clear from (1.3)), E is not necessarily an equivalence relation in a nonhyperbolic space. For instance, if v is any unit vector in the Euclidean plane, and we take $x_j = jv$ and $y_j = -jv$ for all $j \in \mathbb{N}$, then $\langle x_i,\, y_j\rangle = 0$ for all i, j, but nevertheless $x \sim y$ since $x\ E\ z$ and $z\ E\ y$ for $(z_j) = (jw)$, where w is a unit vector perpendicular to v.

In a proper geodesic hyperbolic space, it is well known that $\partial_G X$ and $\partial_I X$ can be identified as sets; see, for instance, [BH, III.H.3.13].

We extend the Gromov product with basepoint o to $\overline{X}_G \times \overline{X}_G$ via the equations

$$\langle a,\, b\rangle = \inf\,\{\liminf_{i,j\to\infty} \langle x_i,\, y_j\rangle :\ [x] = a,\ [y] = b\}, \qquad a, b \in \partial_G X,$$
$$\langle a,\, b\rangle = \inf\,\{\liminf_{i\to\infty} \langle x_i,\, b\rangle :\ [x] = a\}, \qquad a \in \partial_G X,\ b \in X.$$

Whenever $\partial_G X$ is nonempty, we equip it with the pseudometric d_ϵ for some $\epsilon > 0$. Here, the functions $\rho_\epsilon, d_\epsilon : \partial_G X \times \partial_G X \to [0, \infty)$ are defined by the equations

$$\rho_\epsilon(a, b) = \exp(-\epsilon\,\langle a,\, b\rangle), \qquad a, b \in \partial_G X,$$
$$d_\epsilon(a, b) = \inf \sum_{j=1}^{n} \rho_\epsilon(a_{j-1}, a_j), \qquad a, b \in \partial_G X,$$

where the infimum is taken over all finite sequences $a = a_0, \dots, a_b = b$, in $\partial_G X$. Clearly, d_ϵ is a pseudometric. If X is δ-hyperbolic and $\epsilon\delta \le 1/5$, then it follows from 5.13 and 5.16 of [V] that d_ϵ is actually a metric and

$$\rho_\epsilon(a, b)/2 \le d_\epsilon(a, b) \le \rho_\epsilon(a, b), \qquad a, b \in \partial_G X.$$

It is useful to define a Gromov product on the ideal boundary also. Let

$$\langle a,\, b\rangle = \inf\,\{\liminf_{s,t\to\infty} \langle \gamma_a(s),\, \gamma_b(t)\rangle\}, \qquad a, b \in \partial_I X,$$

where the infimum is taken over all $\gamma_a \in a$, $\gamma_b \in b$.

1.6. Spherical and Floyd boundaries.

The g-boundary $\partial_g X$ of a pointed unbounded length space (X, l, o) is $\partial X_\sigma \setminus \partial X_l$, where $d\sigma(z) = g(|z|)\, dl(z)$, $g : [0, \infty) \to (0, \infty)$ is a measurable function, and $|x| = l(x, o)$. We sometimes denote the metric σ by the more descriptive notation $\mathfrak{S}(l, o, g)$, and write $G(t) := \int_t^\infty g(s)\, ds$, $t \ge 0$. For the rest of this paper, (X, l, o), $\mathfrak{S}(l, o, g)$, σ, g, G, and $|x|$ are as defined here.

Not all measurable functions $g : [0, \infty) \to (0, \infty)$ are of interest to us. We now define the classes of such functions that do interest us. These classes involve the following conditions in which $C > 2$ is some parameter.

(F1) $g(t) \le Cg(s)$, whenever $s, t \ge 0$ and $s - 1 \le t \le 2s + 1$.

(S1) $g(t) \le Cg(s)$, whenever $s, t \ge 0$ and $(s - 1)/2 \le t \le 2s + 1$.

(F2) $G(0) \le Cg(0)$.

(S2) $G(t) \le C(1 + t)g(t)$, $t \ge 0$.

We say that g is a *weak C-Floyd function* if it satisfies (F1) and (F2), a *Floyd function* if it satisfies (S1) and (F2), and a *C-sphericalizing function* if it satisfies (S1) and (S2). Lastly, we say that g is *C-quasidecreasing* if $g(t) \le Cg(s)$ for all $0 \le s \le t$.

It is shown in [BB2] that if g is a C-sphericalizing function, there exist $C', \epsilon > 0$, dependent only on C, such that

(S3) $\dfrac{g(t)}{g(s)} \le C' \left(\dfrac{s}{t}\right)^{1+\epsilon}, \quad 1 \le s \le t.$

Thus sphericalizing functions are always quasidecreasing. However it is not hard to construct examples of Floyd functions that fail to be quasidecreasing.

As (S1) and (S3) indicate, sphericalization functions decay at some faster-than-linear polynomial rate. The prototypical sphericalization function is $g(t) \equiv 2/(1+t^2)$, associated with the process of obtaining the Riemann sphere from Euclidean space. Floyd functions can decay more slowly (but not any faster) than sphericalization functions: $g(t) \equiv 1/t\log^2(2+t)$ is a Floyd function, but not a sphericalization function. There is no limit on how fast a weak Floyd function can decay. For instance, $g(t) \equiv \exp(-\epsilon t)$ is a (quasidecreasing) weak Floyd function for all $\epsilon > 0$.

It is easy to see that a C-Floyd function g satisfies the following properties:

(F3) $(1+t)g(t) \le CG(t), \quad$ for all $t \ge 0$.
(F4) $\lim_{t\to\infty} tg(t) = 0$.
(F5) $\dfrac{g(t)}{g(0)} \le C^2/(1+t), \quad t \ge 0$.
(F6) $G(t) \le (C_g+1)G(t+1), \quad t \ge 0$.

In fact, (S1) readily implies (F3), which in turn implies (F4) because of Lebesgue's monotone convergence theorem. Also (F2) and (F3) immediately give (F5). It follows that $G(t)$ and $(1+t)g(t)$ are mutually comparable if g is a sphericalization function. Finally (F6) follows immediately from the estimate $G(t)-G(t+1) \le C_g(G(t+1)-G(t+2)) \le C_gG(t+1)$.

Of these four properties only (F3) fails for weak Floyd functions, although we have to work a little harder to verify (F4). Suppose for the sake of contradiction that g is a C-weak Floyd function and $\limsup_{t\to\infty} tg(t) > \epsilon > 0$. We write $t_0 := 0$ and choose positive numbers t_n, $n \in \mathbb{N}$, with $t_ng(t_n) > \epsilon$ and $t_n \ge 2t_{n-1}$ for all $n \in \mathbb{N}$. Using the quasidecreasing property, we see that

$$G(t_{n-1}) - G(t_n) \ge \int_{t_n/2}^{t_n} g(s)\,ds \ge \frac{\epsilon(t_n/2)}{Ct_n} = \frac{\epsilon}{2C}.$$

By summing the telescoping series, we get a contradiction to the integrability of g.

Suppose g is a weak Floyd function. It is straightforward to see that σ is a bounded metric. Furthermore, if $\gamma \in \Gamma_l(x,y)$, $x \in X$, $y \in \partial X_l$, then clearly γ is also of finite σ-length. Thus ∂X_l can be viewed in a natural way as a subset of ∂X_σ. We define the *g-boundary of X*, $\partial_g X$ to be $\partial X_\sigma \setminus \partial X_l$, and the *$g$-closure of X* to be $\overline{X}_\sigma$. We use terms such as *Floyd/spherical boundary* when we wish to restrict the class of functions to which g belongs. A simple example of a spherical boundary is the North Pole of the Riemann sphere, obtained by sphericalizing $\mathbb{R}^n$, $n > 1$, using $g(t) = 2/(1+t^2)$.

Property (S2) is needed for the comparability of the quasihyperbolic metrics associated with (X, l) and (X, σ) investigated in [BB2]. When comparing $\partial_g X$ and $\partial_G X$, however, more general functions g can be handled. Dealing with (quasidecreasing) weak Floyd functions, where possible, has the advantage of unifying the treatment of the spherical boundary with that of the Floyd boundary (which seems to have been previously studied only in the context of group theory) and that of the g-boundary for exponential decay functions in hyperbolic spaces (studied for instance in [G, 7.2.M], [CDP, Proposition 11.1.9], and [BHK]). Let us also note that the concept of a "Floyd boundary" in the group theory literature is not fixed. Floyd's original concept in [F] is roughly what we call a quasidecreasing Floyd function, while that in [K1] is roughly what we call a weak Floyd function.

For all general results in this paper, the choice of g is irrelevant as long as it is a sphericalization function; in fact, most results allow g to belong to a more general class such as quasidecreasing weak Floyd functions. However, Example 3.10 shows that in general, different choices of sphericalization functions g can lead to different g-boundaries.

It is useful to define a Gromov product on Floyd boundaries in the obvious way:

$$\langle a,\, b\rangle = \inf\,\{\liminf_{i,j\to\infty} \langle x_i,\, y_j\rangle\}, \qquad a, b \in \partial_g X,$$

where the infimum is taken over all sequences x and y in X that σ-converge to a and b, respectively.

2. Main comparison results

The literature contains quite a few equivalent models of the Gromov boundary of a proper geodesic δ-hyperbolic space based, for instance, on equivalence classes of quasigeodesic rays, geodesic rays, or Gromov sequences; see [GH, Section 3.1] and [BH, Section III.H.3]. Two of these notions lead to what we have called the ideal and Gromov boundaries, and they can be identified as sets in this context [BH, Lemma III.H.3.13]. Additionally, it is known that the Gromov boundary coincides with $\partial_g X$ when $g(t) = \exp(-\epsilon t)$ and $0 < \epsilon < \epsilon_0(\delta)$; see [G, 7.2.M], [CDP, Proposition 11.1.9], or [BHK, Proposition 4.13].

In this section, we give variants of these results for more general weak Floyd functions, such as Theorem 2.1(c), and also results for larger classes of spaces. Along the way, we prove estimates linking σ-distance and the Gromov product that may have some independent interest.

There are several reasons to study more general results of this type, the most obvious being that the assumptions in the earlier results are quite strong, so it is natural to see what can be proved using much weaker assumptions. But, more importantly, these generalizations are essential if we wish to use ideal and Gromov boundaries to better understand spherical boundaries, since functions of the form $g(t) = \exp(-\epsilon t)$ are not sphericalization functions and cannot give analogues of the sort of quasihyperbolic metric comparison results found in [BB2]. Additionally, sphericalization of a given metric l is designed to help determine whether the associated quasihyperbolic metric k_l is hyperbolic but, even if k_l is a proper geodesic hyperbolic metric, there is no reason to expect l to have any of these properties.

Standing assumptions for this section: (X, l, o) is an unbounded pointed length space, g is a weak Floyd function, $\sigma = \mathfrak{S}(l, o, g)$, and $\partial_g X$ is the associated g-boundary. We write $|x| = l(x, o)$, and denote by $\langle \cdot, \cdot \rangle$ the Gromov product with respect to the metric l and basepoint o. We use the other notation introduced in §1.2 and §1.5 without comment.

We will say that there is a natural map from $\partial_G X$ to $\partial_g X$, if every Gromov sequence is also a σ-Cauchy sequence. It is easy to see that if this is the case, the boundary element of $\partial_g X$ defined by a Gromov sequence x is independent of the representative in the equivalence class $[x] \in \partial_G X$; thus we have a well defined map $J_1 : \partial_G X \to \partial_g X$. The natural map from $\partial_I X$ to $\partial_G X$ is the one that associates with a geodesic ray γ in X the Gromov sequence $(\gamma(t_i))$ where $t_i \to \infty$ as $i \to \infty$; this is independent of the representative γ and the sequence (t_i).

Two of our main boundary comparison results are as follows; the assumption $\epsilon\delta \le 1/5$ is made solely to ensure the comparability of ρ_ϵ and d_ϵ.

Theorem 2.1. *Suppose that the numbers $\delta \ge 0$ and $\epsilon > 0$ are such that $\epsilon\delta \le 1/5$.*

(a) *If $\partial_G X$ is nonempty and g is quasidecreasing, then $\partial_g X$ is nonempty, and there is a natural map $J_1 : \partial_G X \to \partial_g X$.*
(b) *If (X, l) is proper then $\partial_G X$ is nonempty, and if the natural map $J_1 : \partial_G X \to \partial_g X$ exists, then it is surjective.*
(c) *If (X, l) is δ-hyperbolic, $\partial_G X$ is nonempty, and g satisfies the decay condition $g(t)\exp(\epsilon_0 t) \ge K > 0$ for sufficiently small $\epsilon_0 = \epsilon_0(\delta) > 0$, then we have a natural map $J_1 : (\partial_G X, d_\epsilon) \to (\partial_g X, \sigma)$, which is a homeomorphism.*

Theorem 2.2. *Suppose that the numbers $\delta \ge 0$ and $\epsilon > 0$ are such that $\epsilon\delta \le 1/5$. Suppose also that (X, l) is geodesic.*

(a) *If $\partial_I X$ is nonempty, then so is $\partial_G X$, and there is a natural map $J_2 : \partial_I X \to \partial_G X$.*
(b) *If (X, l) is proper, then $\partial_I X$ is nonempty and $J_2 : \partial_I X \to \partial_G X$ is surjective.*
(c) *If (X, l) is δ-hyperbolic, and $\partial_I X$ is nonempty, then $J_2 : \partial_I X \to \partial_G X$ is injective.*
(d) *If (X, l) is δ-hyperbolic, complete, and CAT(0), and $\partial_I X$ is nonempty, then J_2 is a homeomorphism from $(\partial_I X, \tau_C)$ to its image in $(\partial_G X, d_\epsilon)$.*

It is easy to see that the decay condition of Theorem 2.1(c) is necessary to ensure that the natural map J_1 is injective. For example, in hyperbolic space of curvature -1 and dimension n, where the Riemannian mteric in polar coordinates is $dt^2 + \sinh(t)^2 d\theta^2$ on $(0, \infty) \times \mathbb{S}^{n-1}$, it is easy to see that curves with constant distance to the origin $\gamma_r : t \mapsto (r, \omega(t))$ have length $\sinh(r)\,\mathrm{len}_{\mathbb{S}^{n-1}}(\omega)$. Thus if we choose g such that $g(r)\sinh(r) \to 0$, then the g-boundary is a single point.

Note that Floyd functions automatically satisfy the decay condition of Theorem 2.1(c) for every $\epsilon_0 > 0$, so as an immediate corollary of the above theorems, we get the following result for proper hyperbolic spaces. The second statement of this corollary was already known; see, for instance, [BH, section III.H.3.7].

Corollary 2.3. *Suppose that (X,l) is proper and δ-hyperbolic for some $\delta \ge 0$, that g is a Floyd function, and that $\epsilon > 0$ is such that $\epsilon\delta \le 1/5$. Then $J_1 : (\partial_G X, d_\epsilon) \to (\partial_g X, \sigma)$ is a homeomorphism. If (X,l) is also CAT(0), then $J_2 : (\partial_I X, \tau_C) \to (\partial_G X, d_\epsilon)$ is also a homeomorphism.*

There is a natural map $J_3 : \partial_I X \to \partial_g X$, defined like J_2 by taking sequences along rays. It is easy to see that this map is well defined. Let us state for the record an analogue of Theorems 2.1 and 2.2 for the natural map J_3.

Theorem 2.4. *Suppose (X,l) is geodesic.*

(a) *If $\partial_I X$ is nonempty , then so is $\partial_g X$, and there is a natural map $J_3 : \partial_I X \to \partial_g X$.*
(b) *If (X,l) is proper, then $\partial_I X$ is nonempty and $J_3 : \partial_I X \to \partial_g X$ is surjective.*
(c) *If (X,l) is complete and CAT(0), and $\partial_I X$ is nonempty, then $J_3 : (\partial_I X, \tau_C) \to (\partial_g X, \sigma)$ is continuous.*
(d) *If (X,l) is δ-hyperbolic, complete, and CAT(0), $\partial_I X$ is nonempty, and g satisfies the decay condition $g(t)\exp(\epsilon_0 t) \ge K > 0$ for sufficiently small $\epsilon_0 = \epsilon_0(\delta) > 0$, then J_3 is a homeomorphism from $(\partial_I X, \tau_C)$ to its image in $(\partial_g X, \sigma)$.*

The proofs of (a) and (b) are similar to the proofs of (a) and (b) of Theorem 2.2. We also omit the easy proof of part (c). Part (d) follows from Theorems 2.1 and 2.2, since $J_3 = J_1 \circ J_2$ whenever the natural map J_1 is well defined, e.g. when g is quasidecreasing or (X,l) is Gromov hyperbolic.

Note that for proper hyperbolic spaces, the ideal boundary is often identified with the Gromov boundary via the natural bijection J_2, and is then endowed with the metric d_ϵ for sufficiently small $\epsilon > 0$, rather than the cone metric that we use. In this case, the ideal and Floyd boundary are homeomorphic without any need to assume that the space is CAT(0); of course this also follows from Theorems 2.1 and 2.2.

Before proving Theorems 2.1 and 2.2, let us state and prove three auxiliary results that link σ-distance with the Gromov product.

Lemma 2.5. *Suppose that $u, v \in X$, $\langle u, v\rangle \ge s \ge 2$, and that g is a C_g-weak Floyd function.*

(a) *If g is C-quasidecreasing, we have $\sigma(u,v) < 2CG(s-1)$.*
(b) *If (X,l) is δ-hyperbolic, we have $\sigma(u,v) < C'G(s-1)$, for some constant $C' = C'(\delta, C_g)$.*

Proof. Let $L := l(u,v)$ and choose $\gamma \in \Gamma_l(u,v)$ with l-length L', where $L' - L \le 1$. Writing $\eta = (|u| + |v| - L')/2$, we see that $1 \le \langle u, v\rangle - 1 \le \eta \le \langle u, v\rangle$, and it follows from the triangle inequality that $\eta \le |u|$. We split γ into two parts: $\gamma^1 = \gamma|_{[0,|u|-\eta]}$ and $\gamma^2 = \gamma|_{[|u|-\eta,L']}$. We have $|u| - t \le |\gamma^1(t)|$, and in the quasidecreasing case (a) it follows that $g(|\gamma^1(t)|) \le C_g g(|u| - t)$. In the other case (b), it follows from (1.4) that there is a constant $K_0(\delta)$ such that also $|\gamma^1(t)| \le |u| - t + K_0(\delta)$ and then (F1) implies that there is a constant $C' = C'(\delta, C_g)$ such that $g(|\gamma^1(t)|) \le C'g(|u| - t)$.

Let K denote either C or C' depending on the case. Then

$$\mathrm{len}_\sigma(\gamma^1) \le K \int_0^{|u|-\eta} g(|u|-t)\,dt = K \int_\eta^{|u|} g(t)\,dt < KG(\eta).$$

Since $|u| - \eta = L' - (|v| - \eta)$, a similar argument yields the estimate $\mathrm{len}_\sigma(\gamma^2) < KG(\eta)$. Since $\eta \ge s - 1$, we are done. □

Theorem 2.6. *Suppose g is a C_g-weak Floyd function, (X, l) is δ-hyperbolic, $x, y \in X$, and $s := \langle x,\, y\rangle$.*

(a) *If $\gamma \in \Gamma_l(x, y)$ is a h-short arc with respect to l, $h \ge 0$, then γ is an $(\alpha, \beta g(0))$-quasigeodesic with respect to σ, where α, β depend only on δ, h and C_g. In particular, for fixed δ and C_g, and $0 \le h \le 1$, we can take $\beta = \beta_0 h$, with α and β_0 being absolute constants.*

(b) *There exists $c = c(\delta, C_g) > 0$ such that*

$$\sigma(x, y) \ge c(G(s) - G(s + \epsilon)), \quad \text{where } \epsilon := l(x, y) \wedge ((1 + s)/2), \tag{2.7}$$

Thus if g is a C_g-Floyd function, then there exists $c' = c'(\delta, C_g) > 0$ such that

$$\sigma(x, y) \ge c' g(s)[(1 + s) \wedge l(x, y)]. \tag{2.8}$$

Proof. The assumptions and conclusions are invariant under multiplication of g by a constant, so we assume without loss of generality that $g(0) = 1$. Recall that a δ-hyperbolic space can be imbedded in a geodesic δ-hyperbolic space according to a result of Bonk and Schramm [BS]. Thus we can regard (X, l) as a subspace of a geodesic δ-hyperbolic space (X', l'). We write $|x|' = l'(x, o)$, $x \in X'$, and define $\sigma' = \mathfrak{S}(l', o, g)$. Since $|\cdot|'$ is an extension of $|\cdot|$, it follows that $\mathrm{len}_{\sigma'}$ is an extension of len_σ. Thus $\sigma'(x, y) \le \sigma(x, y)$, but equality may fail since there are typically more paths in X' than in X.

To prove that γ is an (α, β)-quasigeodesic, it suffices to prove that $\mathrm{len}_\sigma(\gamma) \le \alpha\sigma(x, y) + \beta$, since we can apply this to arbitrary subsegments of γ. Suppose first that $l(x, y) < 1$, and so $\mathrm{len}_l(\gamma) < 1 + h$. Since all values $g(|z|)$, $z \in \gamma$, are mutually comparable, we have $\mathrm{len}_\sigma(\gamma) \le C_0 g(|x|)(l(x, y) + h)$, where C_0 depends only on C_g and h; in particular, we can take $C_0 = C_g^2$ if $h \le 1$. If $\lambda \in \Gamma_l(x, y)$ is arbitrary, we similarly have $\mathrm{len}_\sigma(\lambda) \ge C_g^{-1} g(|x|) l(x, y)$. Using also (S1), we deduce that γ is an $(\alpha, \beta g(0))$-quasigeodesic, where $\alpha = C_0 C_g$ and $\beta = C_0 h$.

We next consider the case $l(x, y) \ge 1$. We appeal to Proposition 11.1.6 of [CDP], which says that geodesic segments with respect to l are $(C, 0)$-quasigeodesics with respect to σ. This works for every so-called μ-metric σ, and the class of μ-metrics contains all Floyd metrics. Throughout Chapter 11 of [CDP], it is assumed that the space is proper and geodesic as well as hyperbolic, but a careful reading of Proposition 11.1.6 and the results it uses shows that the assumption that the space is proper is not needed here. We may therefore apply this result to (X', l') to deduce that in the case $h = 0$, γ must be a $(C, 0)$-quasigeodesic with respect to σ' in X', and so also with respect to σ in X.

The case $h > 0$ follows from the following phenomenon which is usually referred to as *geodesic stability*: all (α, h)-quasigeodesic segments in the δ-hyperbolic space (X', l') between our given points x, y lie within a bounded distance of each other, with the bound depending only on α, h, and δ. For a proof, see [V, 3.7]. Let us cut both our given path γ and some geodesic path $\lambda \in \Gamma_l(x, y; X')$ into n equal length subpaths, where n is the least integer not less than $l(x, y)$. By geodesic stability, and the properties of λ and γ, we see that the union of any one subpath of γ together with the corresponding subpath of λ is contained in some l-ball of uniformly bounded diameter. Thus by (F1), the σ-lengths of these subpaths are both comparable to the same multiple of their l-lengths. Since their l-lengths differ by at most a factor $2h + 1$, it follows that their σ-lengths are also comparable. Summing over subpaths, we get comparability of $\operatorname{len}_\sigma(\gamma)$ and $\sigma(x, y)$.

It remains to prove (2.8). Let $h \in (0, 1]$. Suppose $\gamma \in \Gamma_l(x, y)$ is a h-short arc, and $z \in \gamma$ minimizes distance to o. By (1.4), $s - 1/2 \le |z| \le s + 2\delta + 1$. If $\operatorname{len}_l(\gamma) \le 1 + s$, then it is clear from (F1) that $\operatorname{len}_\sigma(\gamma) \gtrsim g(s) \operatorname{len}_l(\gamma) \gtrsim G(s) - G(s + \operatorname{len}_l(\gamma))$. If $\operatorname{len}_l(\gamma) \ge 1 + s$, then taking a subpath γ' of γ that includes z and has length $(1 + s)/2$, we see that

$$\operatorname{len}_\sigma(\gamma) \ge \operatorname{len}_\sigma(\gamma') \gtrsim G(s) - G(2s + 1).$$

Putting together both cases, we get that $\operatorname{len}_\sigma(\gamma) \gtrsim G(s) - G(s + \epsilon)$. Letting $h \to 0$, and using the fact that γ is an $(\alpha, \beta_0 h)$-quasigeodesic with respect to σ, we deduce (2.7). Finally (2.8) follows immediately from (2.7) and (S1). □

Remark 2.9. If x, y, z are as in the last paragraph of the proof of Theorem 2.6, then by routine analysis (as in [BB2, Remark 2.8]), it follows that $\sigma(x, y) \ge 2G(|z|) - G(|x|) - G(|y|)$. Assuming $h < 1$, we know that $G(|z|) \approx G(s)$, where $s := \langle x,\, y\rangle$ and the comparability constants depend only on δ and C_g. Without loss of generality, we assume that $|x| \le |y|$. Since $G(|z|) \ge G(|x|)$, it follows that exists $c = c(\delta, C_g) > 0$ such that

$$\sigma(x, y) \ge cG(s) - G(|y|). \tag{2.10}$$

But $G(|y|) \to 0$ as $|y| \to \infty$, so it follows that $\sigma(x, y) \ge cG(s)/2$ whenever $|x| \vee |y| > R$, and R is some radius dependent only on δ, C_g, and $G(s)$.

Theorem 2.11. *Suppose g is a C_g-weak Floyd function and (X, l) is δ-hyperbolic. There exists $C = C(\delta, C_g) > 0$ such that for all $a, b \in \partial_g X$, $a \ne b$, and $s := \langle a,\, b\rangle$,*

$$C^{-1} G(s) \le \sigma(a, b) \le CG(s). \tag{2.12}$$

If g is actually a C_g-sphericalization function then there exists $C' = C'(\delta, C_g)$ such that

$$\liminf_{i,j\to\infty} \langle u_i,\, v_j\rangle \le C(1 + \langle a,\, b\rangle) \tag{2.13}$$

whenever u and v are sequences in X that are σ-convergent to a and b, respectively.

Proof. By a subsequence argument, we may choose sequences x and y in X that σ-converge to a and b, respectively, with

$$s \leq \liminf_{i\to\infty} \langle x_i,\, y_i\rangle < s+1.$$

Let us choose i so that $s-1 \leq \langle x_i,\, y_i\rangle < s+1$, with i being so large that $|x_i| \wedge |y_i| > 2s+2$, $\sigma(x_i,a) \vee \sigma(y_i,b) < \sigma(a,b)/4$, and $\sigma(x_i,y_i) \geq cG(s)/2$; the last inequality is given by Remark 2.9. Note that the inequality $\sigma(x_i,a) \vee \sigma(y_i,b) < \sigma(a,b)/4$ ensures the comparability of $\sigma(x_i,y_i)$ and $\sigma(a,b)$, so it suffices to show that $\sigma(x_i,y_i)$ and $G(s)$ are comparable.

Now $l(x_i,y_i) = |x_i| + |y_i| - 2\langle x_i,\, y_i\rangle > 2s+2$. If $s \geq 2$, then Lemma 2.5, the estimate $\sigma(x_i,y_i) \geq cG(s)/2$, and (F6) together ensure the comparability of $\sigma(x_i,y_i)$ and $G(s)$. If instead, $s \leq 2$, then according to (F6), $G(s) \approx G(0)$. Moreover by Theorem 2.6 and (F2), we see that $\sigma(x_i,y_i) \gtrsim g(0) \gtrsim G(0)$ and, since $\operatorname{dia}_\sigma(X) \lesssim 2G(0)$, we again get comparability of $\sigma(x_i,y_i)$ and $G(s)$.

Finally if g is a sphericalization function and u and v are arbitrary sequences in X that are σ-convergent to a and b, then it follows essentially by the above argument that

$$\sigma(a,b) \leq CG(t),$$

where $t = \liminf_{i,j\to\infty} \langle u_i,\, v_j\rangle$. Thus $G(s) \lesssim G(t)$ which, since R is a sphericalization function, is equivalent to the inequality $(1+s)g(s) \lesssim (1+t)g(t)$. Now by (S3), we deduce that $1+t \lesssim 1+s$, as required. □

Proof of Theorem 2.1. Using Lemma 2.5(a), we see that every Gromov sequence x in X is σ-convergent. Since the tail ends of a Gromov sequence exit every ball $B_l(o,r)$, $r>0$, the σ-limit of x cannot be in $\overline{X}_l$. Thus x is σ-convergent to some $z \in \partial_g X$, and $\partial_g X$ is nonempty whenever $\partial_G X$ is nonempty. It also follows from Lemma 2.5(a) that equivalent Gromov sequences σ-converge to the same $z \in \partial_g X$. Thus the natural map J_1 is simply the σ-limit of any representative of the equivalence class, and we have proven (a).

We next prove (b). Let (x_i) be a sequence of points in X with $|x_i| \geq i$, and let $A(x_i)$ be a 1-short arc from o to x_i, $i \in \mathbb{N}$. We also define $P_k(x_i)$, for all $k,i \in \mathbb{N}$, $k \leq i$, to be the first point of intersection of $A(x_i)$ with $S_l(o,k)$. Writing $x_i^0 = x_i$, we will inductively define a sequence of nested subsequences. Given a subsequence $(x_i^{k-1})_{i=1}^\infty$ of (x_i), where k is the inductive index, compactness ensures the existence of a subsequence $(x_i^k)_{i=1}^\infty$ of $(x_i^{k-1})_{i=1}^\infty$ such that $x_1^k = x_n$ for some $n \geq k$, and such that all the points $P_k(x_i^k)$, $i \in \mathbb{N}$, lie within an l-distance 1 of each other. Now let y be the diagonal sequence, i.e. $y_i = x_i^i$. Since all the arcs $A(x_i)$ are 1-short, it follows that $l(x_i, P_k(x_i)) \leq |x_i| - k + 1$, and so if $i \leq j$ then

$$\begin{aligned}
2\langle y_i,\, y_j\rangle &= |y_i| + |y_j| - l(y_i,y_j) \\
&\geq |y_i| + |y_j| - [l(y_i,P_i(y_i)) + l(P_i(y_i),P_i(y_j)) + l(P_i(y_j),y_j)] \\
&\geq |y_i| + |y_j| - [(|y_i| - i + 1) + 1 + (|y_j| - i + 1)] = 2i - 3.
\end{aligned}$$

Thus (y_i) is a Gromov sequence and $\partial_G X$ is nonempty.

Given $x \in \partial_g X$, we can find a sequence x in X which is σ-convergent to x. Since the tail of x must exit every ball $B_l(o,r)$, we can assume without loss of generality that $|x_i| > i$. Applying the above argument to x, we get a Gromov sequence y which is a subsequence of x. Clearly $J_1([y]) = x$ and so we have proven (b).

Next, we prove (c). That there is a natural map $J_1 : \partial_G X \to \partial_g X$ follows as for (a), using Lemma 2.5(b). As in the proof of Theorem 2.6, we embed (X,l) in a geodesic δ-hyperbolic space (X',l') and get that $\sigma'(x,y) \le \sigma(x,y)$, $x,y \in X$, where $\sigma' = \mathfrak{S}(l',o,g)$. We can now use the second half of the argument of [BHK, Lemma 4.10]. This is built on the Gehring-Hayman theorem [BHK, Theorem 5.1], which holds for geodesic spaces; properness is not necessary. We then have the second inequality of [BHK, Lemma 4.10]:

$$\frac{\exp(-\epsilon' \langle x,\, y\rangle)}{\epsilon'} \le C\sigma_{\epsilon'}(x,y), \quad l(x,y) \ge \frac{1}{\epsilon'}$$

for $\epsilon' \le \epsilon_0(\delta)$ sufficiently small, $\sigma_{\epsilon'} = \mathfrak{S}(l',o,h)$, and $h(t) = \exp(-\epsilon' t)$.

However if $g(t) \ge K\exp(-\epsilon_0 t)$ it is clear that

$$\sigma(x,y) \ge \sigma'(x,y) \ge K\sigma_{\epsilon_0}(x,y) \ge C' \exp(-\epsilon_0' \langle x,\, y\rangle), \quad l(x,y) \ge \frac{1}{\epsilon_0}.$$

It follows that every σ-Cauchy sequence converging to a point in $\partial_g X$ is also a Gromov sequence. Thus J_1 has an inverse map, and must be bijective. It also follows from the estimate above, that J_1 is open. Continuity of J_1 follows from Lemma 2.5(b). □

Remark 2.14. There is an alternative proof of part (c) when g is a Floyd function. As above, we know that there is a natural map $J_1 : \partial_G X \to \partial_g X$. It follows easily from (2.8) that every σ-Cauchy sequence converging to $\partial_g X$ is also a Gromov sequence. Thus J_1 has an inverse and is bijective. Continuity and openness follow immediately from Theorem 2.11, the fact that $G(s) \to 0$ as $s \to \infty$, and the comparability of d_ϵ and ρ_ϵ.

Proof of Theorem 2.2. Given a geodesic ray $\gamma \in \mathrm{GR}_l(X)$, we simply define $x_i = \gamma(t_i)$, $i \in \mathbb{N}$, where (t_i) is any fixed sequence of non-negative numbers with limit infinity. It follows immediately from the triangle inequality that x is a Gromov sequence, and that $[x]$ is independent of the choice of sequence (t_i). Thus $\partial_G X$ is nonempty whenever $\partial_I X$ is nonempty. If $\gamma, \nu \in \mathrm{GR}_l(X)$, with $d_H(\gamma,\nu) < \infty$, then the triangle inequality again implies easily that any pair of Gromov sequences constructed from γ and ν are equivalent to each other. Thus we have a natural map J_2 from $\partial_I X$ to $\partial_G X$, and (a) follows.

The proof of (b) is similar to that of Theorem 2.1(b). Let (x_i) be a sequence of points in X with $|x_i| \ge i$, let $A(x_i) = [o, x_i]$ be associated geodesic segments, and let $P_k(x_i)$ be the point of intersection of $A(x_i)$ with $S_l(o,k)$. Writing $x_i^0 = x_i$, we will inductively define a sequence of nested subsequences. Given a subsequence $(x_i^{k-1})_{i=1}^\infty$ of (x_i), where $k \in \mathbb{N}$ is the inductive index, compactness ensures the existence of a subsequence $(x_i^k)_{i=1}^\infty$ of $(x_i^{k-1})_{i=1}^\infty$ such that $x_1^k = x_n$ for some $n \ge k$, and such that all the points $P_k(x_i^k)$, $i \in \mathbb{N}$, lie within an l-distance 1 of each other and converge to some point $z_k \in S_l(o,k)$ as $i \to \infty$.

We also write $z_0 = o$. Now let y be the diagonal sequence, i.e. $y_i = x_i^i$, $i \in \mathbb{N}$. It follows that $l(P_{k-1}(x_j), P_k(x_j)) = 1$ whenever $k \le j$, and so if $k \le i, j$, then

$$\begin{aligned} l(P_{k-1}(y_i), P_k(y_j)) &\le l(P_{k-1}(y_i), P_k(y_i)) + l(P_k(y_i), P_k(y_j)) \\ &= 1 + l(P_k(y_i), P_k(y_j)) \to 1 \quad (i, j \to \infty). \end{aligned}$$

Consequently, $l(z_{k-1}, z_k) = 1$, and so if we string together geodesic segments $[z_{k-1}, z_k]$, $k \in \mathbb{N}$, we get a geodesic ray γ. Thus $\partial_I X$ is nonempty.

As in the proof of Theorem 2.1(b), we see that the sequence y constructed above is a Gromov sequence and that $y \sim z$. Thus $J_2([\gamma]) = [y]$. If the original sequence x was already a Gromov sequence, we have constructed a geodesic ray γ such that $J_2([\gamma]) = [x]$. Part (b) follows immediately.

We next prove (c). Suppose $\gamma, \nu \in \mathrm{GR}_l(X)$ are not equivalent, and let $x_0 = \gamma(0)$, $y_0 = \nu(0)$. Let ν' be the rough geodesic ray emanating from x_0 obtained by concatenating a geodesic segment $[x_0, y_0]$ with ν. This ray might have a single loop but, by eliminating this loop if it exists, we get an injective rough geodesic ray λ parametrized by l-arclength. Note that all initial segments of λ are h-short arcs, where $h := 2l(x_0, y_0)$. For each $r > 0$, we construct a h-short triangle with vertices x_0, $\gamma(r)$, and $\lambda(r)$, with sides given by initial segments of γ and λ, together with some geodesic segment from $\gamma(r)$ to $\lambda(r)$. Since γ and ν are nonequivalent, there exists $r_0 > 0$ such that $l(\gamma(r_0), \lambda(r_0)) > 4\delta + 4h$.

Recall that a tripod map ϕ for a h-short triangle is a $(4\delta + 4h)$-rough isometry. It follows that if ϕ_r is a tripod map for T_r, $r > r_0$, then $\phi_r(\gamma(r_0)) \ne \phi_r(\lambda(r_0))$, and so

$$l(\gamma(r), \lambda(r)) > 2(r - r_0) - 4(\delta + h).$$

Therefore

$$\begin{aligned} 2\,\langle \gamma(r),\, \lambda(r) \rangle &= |\gamma(r)| + |\lambda(r)| - l(\gamma(r), \lambda(r)) \\ &< (r + |x_0|) + (r + |y_0|) - 2(r - r_0) + 4(\delta + h) \\ &= |x_0| + |y_0| + 2r_0 + 4(\delta + h), \end{aligned}$$

and so any Gromov sequences associated with γ and ν are nonequivalent. Thus J_2 is injective.

We next prove the continuity part of (d). Suppose $a, b \in \partial_I X$, and suppose that b lies in the basic open neighbourhood $U(a, r, s)$. Let γ_a, γ_b be the geodesic rays from o to a, b, respectively, as defined for the cone topology in §1.2. We associate with these rays the Gromov sequences $(\gamma_a(i))_{i=1}^\infty$ and $(\gamma_b(i))_{i=1}^\infty$. For $i, j > r$,

$$\begin{aligned} 2\,\langle \gamma_a(i),\, \gamma_b(j) \rangle &= i + j - l(\gamma_a(i), \gamma_b(j)) \\ &\ge i + j - [l(\gamma_a(i), \gamma_a(r)) + l(\gamma_a(r), \gamma_b(r)) + l(\gamma_b(r), \gamma_b(j))] \\ &= i + j - [i - r + l(\gamma_a(r), \gamma_b(r)) + j - r] > 2r - s. \end{aligned}$$

It follows readily from the definition of hyperbolicity (or by appealing to [V, 5.6]) that

$$\liminf_{i,j\to\infty} \langle a'_i,\, b'_j\rangle > r - s/2 - 2\delta, \qquad a' \in J_2(a),\ b' \in J_2(b),$$

and so $\langle a,\, b\rangle > r - s/2 - 2\delta$. Thus $\langle a,\, b\rangle > R$ whenever $b \in U(a, r, 1)$ for $r = R + 2\delta + 1/2$. In view of the definition of ρ_ϵ, and the comparability of d_ϵ and ρ_ϵ, it follows that $J_2 : (\partial_I X, \tau_C) \to (\partial_G X, d_\epsilon)$ is continuous.

To finish the proof of (d), we must show that J_2 is relatively open. Thus for fixed $a \in \partial_I X$, and $r, s > 0$, we need to show that there exists $R > 0$ such that $b \in U(a, r, s)$ whenever $b \in \partial_I X$ satisfies $\langle J_2(a),\, J_2(b)\rangle > R$. We claim that this is true whenever $R > (6r + 3s)\delta/s$.

Because X is a complete CAT(0) space, we can choose $\gamma_a, \gamma_b \in \mathrm{GR}_l(X, o)$ such that $[\gamma_a] = a$, $[\gamma_b] = b$; see [BH, II.8.2]. Let $x_i = \gamma_a(i)$, $y_i = \gamma_b(i)$ for each non-negative integer i. We choose k so large that $\langle x_i,\, y_j\rangle > R$ for all $i, j \ge k$. Let λ_a, λ_b be the initial segments of γ_a, γ_b, respectively, of length $R - 3\delta$. Since $\langle x_i,\, y_j\rangle > R$, geodesic segments $[x_i, y_j]$ must remain outside the ball $B_l(o, R)$. By the 3δ-thin triangles property applied to the geodesic triangle with vertices o, x_i, y_j, we see that if we take $t = R - 3\delta$, then $\lambda_a(t)$ must be within a distance 3δ of some point $\lambda_b(t')$, $0 \le t' \le R - 3\delta$. Since λ_a, λ_b are geodesic segments with the same initial point, it follows that $l(\lambda_a(t), \lambda_b(t)) \le 6\delta$. The CAT(0) property now implies that $l(\lambda_a(r), \lambda_b(r)) \le 6\delta r/(R - 3\delta) < s$, which finishes the proof of the claim. □

Remark 2.15. It follows from our analysis that if (X, l) is a δ-hyperbolic space and $\delta\epsilon < 1/5$, then $J_1 : (\partial_G X, d_\epsilon) \to (\partial_g X, \sigma)$ is a quasisymmetry. In fact if a, b are distinct points in $\partial_G X$ and $t := \epsilon^{-1} \log^+(1/d_\epsilon(a, b))$, then $\sigma(J_1(a), J_1(b)) \approx G(t)$. However, since a Floyd $g(t)$ decays no faster than a power of t for large t (according to (S1)), the same is true of $G(t)$, and so the map J_1 fails to be a power quasisymmetry in the sense of [BS, Section 6] whenever $(\partial_g X, \sigma)$ has any accumulation points and g is a Floyd function.

3. Set-theoretic counterexamples

In this section, we show that each part of Theorems 2.1, 2.2, and 2.4 fails if any of its assumptions about (X, l) are dropped. We take as a standing assumption that g is a quasidecreasing weak Floyd function; all our examples are independent of the choice of g in this class.

Let us call the 3-tuple of cardinalities $(\#(\partial_I X), \#(\partial_G X), \#(\partial_g X))$ the *cardinality triple of X*. Our first goal is to show that there are practically no constraints on the possible values of cardinality triples. In fact we have the following result.

Theorem 3.1. *The cardinality triple (a, b, c) of an unbounded pointed length space (X, l, o) can take on any value subject only to the two constraints given by Theorems 2.1 and 2.2, namely that if $a > 0$ then $b > 0$, and if $b > 0$ then $c > 0$.*

We begin by discussing the constructions used in the examples required to prove this result. Let us denote by $\mathrm{Join}_o(\{(X_i, o_i) \mid i \in I\})$, or simply $\mathrm{Join}_o(\{X_i \mid i \in I\})$, the *one-point join* of a family of pointed geodesic spaces (X_i, l_i, o_i). This space (X, o) consists of the quotient of the disjoint union of $\{X_i \mid i \in I\}$ under the identification of the points o_i. This common origin is denoted o, and we identify X_i as a subset of X in the natural manner. We attach the natural geodesic metric l to X, i.e. $l(x,y) = l_i(x,y)$ if $x, y \in X_i$, while $l(x,y) = l_i(x,o) + l_j(o,y)$ if $x \in X_i$, $y \in X_j$, $i \neq j$. Clearly the ideal, Gromov, and g-boundaries of a one-point join are each given as the disjoint unions of the corresponding boundaries of the constituent spaces.

Most of the examples we consider are what we call bridge spaces (Y, l). Here, we begin with pointed geodesic spaces (X_i, l_i, o_i), $i \in I$; we call these the *base pieces*. Then we form the *base space* $(X, l_X) = \mathrm{Join}_o(\{X_i \mid i \in I\})$, and write $|x| = l_X(x, o)$, $x \in X$. Next, let $\mathcal{B}$ be the collection of all pairs (u, v) such that $|u| = |v| \geq 1$ and u, v lie in different base pieces. For each $(u, v) \in \mathcal{B}$, we have a *bridge* $\beta[u, v]$ with metric $l_{u,v}$ which is isometric to an interval of length $h(|u|) \geq 1$, where the *bridge function* $h : [1, \infty) \to [1, \infty)$ must be specified. The *(continuous) bridge space* Y *on* X *(with bridge function* h*)* is now defined by joining the bridges and the base space using a set of *two-point joins* indexed by $\mathcal{B}$; more precisely, Y is the quotient of the disjoint union of X and all bridges $\beta[u, v]$, under the identification of the endpoints of $\beta[u, v]$ with the points u and v, respectively. This induces a natural length metric l on Y which restricts to the metric l_X on X and $l_{u,v}$ on each bridge $\beta[u, v]$. This metric l is defined using paths that can be broken into a finite number of subpaths each of which is contained in either X or a single bridge; it does not degenerate because all bridges have length at least 1. We shall also make occasional use of *discrete bridge spaces* in which the set $\mathcal{B}$ is replaced by its subset $\mathcal{B}' = \{(u, v) \in B : |u| \in \mathbb{N}\}$. The continuous version makes some proofs a little simpler, but the discrete version is just as useful, with the added property that it is sometimes proper whereas the continuous version fails to be proper in all interesting cases. Note that in both the discrete and continuous cases, $l(x, o) = l_X(x, o)$ for all $x \in X$, so we may unambiguously write $|y| = l(y, o)$ for all $y \in Y$. We always use $\langle \cdot, \cdot \rangle_o$ to denote the Gromov product in Y with respect to l. When we need the Gromov product in X, we write $\langle \cdot, \cdot \rangle_{o, l_X}$. We identify $\beta[u, v]$ with its image in Y and we also use $\beta[u, v]$ to denote any arc in $\Gamma_l(u, v; Y)$ with image $\beta[u, v]$.

We are especially interested in bridge spaces with three particular choices of bridge function h. The *short bridge space* $Y = S(X)$ corresponds to taking $h(t) \equiv 1$, the *medium bridge space* $Y = M(X)$ corresponds to taking $h(t) \equiv t$, and the *long bridge space* $Y = L(X)$ corresponds to taking $h(t) \equiv 2t$. We also denote by $S'(X)$, $M'(X)$, and $L'(X)$ the discrete versions of these bridge spaces.

Before stating a result on the various boundaries of the three types of bridge spaces, let us next define three properties of a pointed geodesic space (Z, l, o) that we shall require for all of our base pieces, i.e. for $(Z, l, o) = (X_i, l_i, o_i)$, $i \in I$. These properties are clearly true for all intervals on the real line, and we shall see that short or medium bridge spaces inherit them from their base pieces. It follows that all our examples satisfy these conditions, since we always use as base pieces either intervals or short or medium bridge spaces in which the base pieces are intervals. We say that (Z, l, o) has the *geodesic segment property* if there is a geodesic segment from the origin to every $x \in Z$. We say that (Z, l, o) has the *geodesic ray property* if for every $a \in \partial_I Z$, there exists $\gamma \in \mathrm{GR}(Z, o)$ such that $[\gamma] = a$, and we say that (Z, l, o) has the *Gromov product property* if $\langle x, y \rangle_o \geq l(x, o)/2$ whenever

$x, y \in Z$, $|x| \le |y|$. This last property has some useful consequences. It implies that the class of Gromov sequences in Z coincides with the class of *sequences that tend to infinity*, i.e. sequences (x_n) in Z such that $|x_n| \to \infty$. It also implies that all such Gromov sequences are equivalent, and that

$$l(x, y) \le |y|, \qquad x, y \in Z, \ |x| \le |y|. \tag{3.2}$$

Lemma 3.3. *Suppose (Y, l) is a bridge space on $(X, l_X) = \mathrm{Join}_o(\{(X_i, o_i) \mid i \in I\})$, where at least two of the base pieces exceeds any given finite diameter, and all base pieces have the the geodesic segment, geodesic ray, and Gromov product properties.*

(a) *If $Y = S(X)$ or $Y = S'(X)$, then $\partial_I Y$ is empty if $\partial_I X$ is empty, and $\partial_I Y$ is a singleton set if $\partial_I X$ is nonempty. Also, $\partial_G Y$ is a singleton set.*
(b) *If $Y = M(X)$ or $Y = M'(X)$, then $\partial_I Y$ can be identified with $\partial_I X$, while $\partial_G Y$ is a singleton set.*
(c) *If $Y = L(X)$ or $Y = L'(X)$, then $\partial_I Y$ can be identified with $\partial_I X$, and $\partial_G Y$ can be identified with $\partial_G X$.*

In all three cases, $\partial_g X$ is a singleton set and (Y, l, o) has the geodesic segment and geodesic ray properties. Also (Y, l, o) has the Gromov product property in cases (a) and (b).

We postpone the proof of Lemma 3.3, and instead give a sequence of examples that we need to prove Theorem 3.1. The fact that each example has the stated cardinality triple follows with little difficulty from Lemma 3.3 and the previously mentioned fact that the ideal, Gromov, and Floyd boundaries of a one-point join are the disjoint unions of the corresponding boundaries of the constituent spaces.

Example 3.4. Let $X = \mathrm{Join}_o(\{(X_i, o_i) \mid i \in \mathbb{N}\})$, where $X_i = [o_i, b_i]$ is an interval of length i. Then X has cardinality triple $(0, 0, 0)$.

Example 3.5. Let $V_1 = S(X)$ where X is as in Example 3.4. Then V_1 has cardinality triple $(0, 1, 1)$.

Example 3.6. Let $W_1 = L(X)$, where X is as in Example 3.4. Then W_1 has cardinality triple $(0, 0, 1)$.

Example 3.7. For each cardinality $c > 0$, let $U_c = M(X_c)$, where $X_c = \mathrm{Join}_o(\{H_i \mid i \in I\})$, each space (H_i, l_i, o_i) is isometric to the half-line $[0, \infty)$ with origin 0, and I has cardinality c. We also define U_0 to be the space V_1 in Example 3.5. Then U_c has cardinality triple $(c, 1, 1)$, $c \ge 0$.

Example 3.8. For each cardinality $c > 0$, let $V_c = L(X_c)$, where X_c is the one-point join of a collection of c copies of the set V_1 in Example 3.5. We also define V_0 to be the space W_1 in Example 3.6. Then V_c has cardinality triple $(0, c, 1)$, for each $c \ge 0$.

Example 3.9. Let W_c be the one-point join of a collection of c copies of W_1, where W_1 is as in Example 3.6 and $c > 0$ is some cardinal. We also define W_0 to be the space X in Example 3.4. Then W_c has cardinality triple $(0, 0, c)$, $c \geq 0$.

Proof of Theorem 3.1. Putting together Examples 3.7, 3.8, and 3.9, we see that the space $\mathrm{Join}_o(U_{c_1}, V_{c_2}, W_{c_3})$ has cardinality triple $(c_1, c_2 + 1, c_3 + 2)$, where c_1, c_2, c_3 are arbitrary cardinals. This covers most cases. The missing case $a = b = 0$ is covered by Example 3.9, so it remains only to consider the case $c = 1$. We may assume that $b > 0$, since otherwise $a = 0$ and we can appeal to Example 3.6. If b' is the cardinal defined by $b' + 1 = b$, then $Z = \mathrm{Join}_o(U_a, V_{b'})$ has cardinality triple $(a, b, 2)$. Finally, $A := L(Z)$ has cardinality triple $(a, b, 1)$. □

We are now ready to prove that all assumptions about (X, l) in Theorems 2.1, 2.2, and 2.4 are needed. First note that the nonemptiness assumption is essential where it occurs, since the space in Example 3.4 is unbounded, geodesic, 0-hyperbolic, CAT(0), and complete (but not proper), and yet it has cardinality triple $(0, 0, 0)$.

Consider next the proper assumption in parts (b) of these theorems. If X is the one-point join of a half-line $[0, \infty)$ with the space W_1 of Example 3.6, then X is a complete geodesic space and $\partial_G X$ is nonempty, but J_1 and J are not surjective. If instead X is the one-point join of a half-line $[0, \infty)$ (with $o = 0$) and the space V_1 of Example 3.5, then X is a complete geodesic space in which $\partial_I X$ is nonempty, and moreover X is hyperbolic (since it is 1-roughly isometric to the real line), but yet J_2 and J are not surjective.

We next show that injectivity fails for nonhyperbolic spaces. Let H_i be an isometric copy of $[0, \infty)$ for $i = 1, 2, 3$, with o_i corresponding to 0 in each case. Let X_1 be the one-point join of H_1 and H_2, let Y_1 be the discrete bridge space $M'(X_1)$, let X_2 be the one-point join of Y_1 and H_3, and let Y_2 be the discrete bridge space $L'(X_2)$. Then $\partial_I Y_2$ has three elements corresponding to the three rays H_i. Two of them are identified by J_2, and $\partial_G Y_2$ has only two elements. Finally both of these elements are identified by J_1, and $\partial_g Y_2$ is a singleton set. Note that injectivity fails even though Y_1 and Y_2 are proper geodesic spaces (we need to use the discrete versions of these bridge spaces to get proper spaces).

The assumption that X is hyperbolic is the standard assumption in the literature to ensure that d_ϵ is a metric, and the assumption that X is complete and CAT(0) is the standard assumption to ensure that τ_C is worthy of study, so we do not need to justify these assumptions in the parts of these theorems that have topological conclusions.

Example 3.10. So far in this paper, the choice of g was unimportant, at least if it was a sphericalization function. Note however that it is rather easy to give examples where $\partial_g X$ depends on g. To see this, suppose that g_1 and g_2 are distinct weak Floyd functions with g_1 decreasing, and $\lim_{t\to\infty} g_2(t)/g_1(t) = 0$. We also normalize so that $g_1(1) = g_2(1) = 1$. For instance, the sphericalization functions $g_1(t) \equiv 2/(1+t^2)$ and $g_2(t) \equiv 2/(1+t^3)$ satisfy these conditions. Viewing $(\mathbb{R}, 0)$ as the one-point join of $(-\infty, 0]$ and $[0, \infty)$, we let (Y, l_Y) be the bridge over $\mathbb{R}$ with bridge function $h(t) = 1/g_1(t)$. Finally, we define a new space (Z, l) by making a further set of two-point joins: we join 0 to every point on every bridge

$\beta[-t,t]$, $t \ge 1$, via a path of length t. It is readily verified that $\partial_g Z$ is either isometric to an interval or is a singleton set, depending on whether we take $g = g_1$ or $g = g_2$, respectively.

Proof of Lemma 3.3. For each type of boundary, we shall see that there is a natural map from the X-version of this boundary to the Y-version. To avoid having to repeatedly consider the empty set as a special case, let us agree that the "empty map" is the "natural map" from the empty set to any other set, and that this map is always injective and is surjective if and only if the other set is empty. Also, we omit the analysis for discrete bridge spaces (Y', l') which is essentially the same as that for the corresponding continuous bridge spaces (Y, l), because l' is roughly isometric to the restriction of l to Y'.

We consider each type of boundary separately, beginning with the ideal boundary. First note that every geodesic ray in X is equivalent to a geodesic ray in $X \setminus \{o\}$: we simply remove an initial segment if the ray goes through o. It follows that every $\gamma \in \mathrm{GR}(X)$ is equivalent to a ray $\gamma' \in \mathrm{GR}(X_i, o)$ for some $i \in I$. Since $l(x,o) = l_i(x,o)$ for all $x \in X_i$, γ' is also geodesic in Y. It is easy to deduce that Y inherits the geodesic segment property from X. Also equivalent rays in X remain equivalent in Y. We conclude that there is a natural map $J_I : \partial_I X \to \partial_I Y$, and that Y has the geodesic ray property.

Given any Y-geodesic ray γ_0, we can find an equivalent Y-geodesic ray $\gamma \subset X$ simply by cutting off a sufficiently long initial segment of γ; indeed, since bridges join points equidistant from the origin and have length at least 1, a Y-geodesic ray $\gamma : [0,\infty) \to Y$ can intersect no more than $|\gamma(0)| + 2$ bridges. We may assume that $\gamma \in \mathrm{GR}(X_i, o)$, and so γ is also an X-geodesic ray. Thus J_I is surjective.

It remains to consider whether or not inequivalent X-geodesic rays γ, ν are equivalent in Y. We may assume that $\gamma \in \mathrm{GR}(X_i, o)$ and $\nu \in \mathrm{GR}(X_j, o)$ for some $i, j \in I$. Suppose first that $Y = S(X)$. If $i \ne j$, then the Hausdorff distance between γ and ν is at most 1, and so γ and ν are equivalent in Y. If $i = j$, γ and ν are still equivalent since $d_H(\gamma(t), \nu(t)) \le 2$, as can be seen by concatenating the bridges $\beta[\gamma(t), y]$ and $\beta[y, \nu(t)]$ for some point y in another base piece X_j; note that by assumption there exists a base piece X_j containing a point y' with $|y'| \ge t$, so there exists a point y with $|y| = t$ on any path from o to y'. However, in the case of medium or long bridge spaces, γ, ν must remain inequivalent in Y since bridges far from the origin are very long.

We next wish to consider Gromov boundaries. Since the Gromov product in any bridge space Y dominates the Gromov product in the base space X, Gromov sequences in X remain Gromov sequences in Y, and equivalent sequences remain equivalent. Thus the identity map from $\mathrm{GR}_{l_X}(X)$ to $\mathrm{GR}_l(Y)$ yields a natural map $J_G : \partial_G X \to \partial_G Y$.

If u, v lie in distinct base pieces with $1 \le |u| \le |v|$, then

$$l(u,v) = h(|u|) + |v| - |u|. \tag{3.11}$$

since if $\gamma \in \Gamma_l(o,v)$ is a geodesic path and $v' := \gamma(|u|)$, then a minimal length path from u to v (for any of our three choices of h) consists of the concatenation of $\beta[u,v']$ and $\gamma[v',v]$. Inequality (3.11) immediately implies that

$$2\,\langle u,\, v\rangle_o = 2|u| - h(|u|) \tag{3.12}$$

whenever u, v are as above. For $Y = L(X)$, these estimates imply that $\langle u, v\rangle_o = 0$ for points u, v as above, and that distances and inner products in Y between points in X are the same as their analogues in X. It follows that inequivalent Gromov sequences in X remain inequivalent in $Y = L(X)$.

By contrast, if (Y, l) is a short or medium bridge space, then (3.12) implies that $\langle u, v\rangle_o \geq |u|/2$, and so (Y, l) inherits the Gromov product property from its base pieces. Thus all Gromov sequences in Y are equivalent in this case. Since X is unbounded, the Gromov product property ensures that $\partial_G X$ is a singleton set.

It remains to show that J_G is surjective when $Y = L(X)$. Suppose therefore that $Y = L(X)$ from now on. There are two potential obstacles to surjectivity: sequences (x_n) in X which are Gromov sequences in Y but not in X, and sequences (y_n) in Y which are Gromov sequences in Y but are not contained in X. The first obstacle does not arise since $\langle \cdot, \cdot\rangle_o$ in is an extension of $\langle \cdot, \cdot\rangle_{o,l_X}$.

We claim that the second obstacle never produces new elements of $\partial_G Y$, i.e. every $a \in \partial_G Y$ is represented by a Y-Gromov sequence in X. It suffices by a subsequence argument to consider Gromov sequences (y_n), where $y_n \in \beta[u_n, v_n]$ and $|u_n| \to \infty$.

Let us call a sequence (z_n) *pure* if it lies in a single base piece. Since X is a one-point join, each equivalence class $a \in \partial_G X$ contains a pure sequence (z_n). By another subsequence argument, we can assume our sequence (y_n) is one of three types categorized according to which base pieces contain the associated points u_n and v_n: in *Case A*, no base piece contains more than one of the points $\{u_n, v_n \mid n \in \mathbb{N}\}$; in *Case B*, (u_n) and (v_n) are both pure sequences; in *Case C*, (u_n) is a pure sequence but the points v_n all lie in distinct base pieces.

Case A never occurs since it implies that $\langle y_n, y_m\rangle_o = 0$ for all $m, n \in \mathbb{N}$, contradicting the fact that (y_n) is a Gromov sequence. We next consider Case B. By yet another subsequence argument and symmetry, we can reduce to the case where $l(y_n, u_n) \leq l(y_n, v_n)$ for all $n \in \mathbb{N}$. Suppose $|u_m| \leq |u_n|$. By the Gromov product property, $l(u_m, u_n) \leq |u_n|$, and so $l(y_m, u_n) \leq c_m + |u_n|$. Since $|y_m| = |u_m| + c_m$, it follows that $\langle y_m, u_n\rangle_o \geq |u_m|/2$. If instead $|u_m| \geq |u_n|$, it follows similarly that $\langle y_m, u_n\rangle_o \geq |u_n|/2$. Thus (y_n) and (u_n) are equivalent.

As for Case C, let $d_n := l(y_n, u_n) - l(y_n, v_n)$. Another subsequence argument allows us to assume either that (d_n) is a positive sequence that tends to infinity or that (d_n) is bounded above. In the former case, we claim that for each $m \in \mathbb{N}$ and all $n \geq n_0 = n_0(m)$, we have $\langle y_m, y_n\rangle_o = 0$. Assuming this (secondary) claim, we get a contradiction to the assumption that (y_n) is a Gromov sequence, and so we call rule out this possibility. To prove the claim, let us assume that $n \geq m$. Note that the only route from y_n to y_m that could be strictly shorter than going directly to the origin and back out again is to via u_m and u_n. Comparing $l(u_m, u_n)$ with $l(v_m, v_n) = |u_m| + |u_n|$, the reduction in distance travelled is at most $2|u_m|$, but the increase in distance caused by going along the longer parts of the initial and final bridges is $d_m + d_n$. Once d_n is at least $2|u_m| - d_m$, the Gromov product is zero and so we have proven our secondary claim.

It remains to consider the case where (d_n) is bounded above by some number K. In this case,

$$l(y_m, y_n) \le l(y_m, u_m) + l(u_m, u_n) + l(u_n, y_n) \le l(y_m, u_m) + |u_n| + l(u_n, y_n),$$

whereas

$$|y_m| + |y_n| \ge (l(y_m, u_m) + |u_m| - K) + (l(u_n, y_n) + |u_n| - K),$$

and so

$$2\, \langle y_m,\, y_n \rangle_o \ge |u_m| - 2K \to \infty.$$

Thus the fact that $|u_n| \to \infty$ as $n \to \infty$ ensures that (y_n) is indeed a Gromov sequence in this case. However, it is equivalent to (u_n). To see this, note that if $|u_m| \le |u_n|$ then the Gromov product property ensures that

$$l(y_m, u_n) \le l(y_m, u_m) + l(u_m, u_n) \le l(y_m, u_m) + |u_n|$$

whereas

$$|y_m| + |u_n| \ge (l(y_m, u_m) + |u_m| - K) + |u_n|,$$

and so

$$2\, \langle y_m,\, u_n \rangle_o \ge |u_m| - K \to \infty \quad (m, n \to \infty).$$

In the case $|u_m| > |u_n|$, we similarly get $\langle y_m,\, u_n \rangle_o \ge |u_n| - K$. Thus (y_n) and (u_n) are equivalent, as desired, and we have proven in all cases that every $a \in \partial_G Y$ is represented by a Y-Gromov sequence in X.

Finally, we consider the g-boundary. Since $l(x, o) = l_X(x, o)$, $x \in X$, the metric $\sigma_X = \mathfrak{S}(l_X, o, g)$ coincides with the restriction of $\sigma = \mathfrak{S}(l, o, g)$ to $X \times X$. It readily follows that there is a natural map $J_S : \partial_g X \to \partial_g Y$. Using (F4), it is easy to see that the σ-diameter of a short, medium, or long bridge $\beta[u, v]$ tends to zero as $|u| \to \infty$. Moreover the Gromov product property and (F4) also imply that $\sigma(u, v)$ is arbitrarily small if u, v are in the same bridge space with $|u| \wedge |v|$ sufficiently large. It follows that in all cases, there is exactly one equivalence class in $\partial_g Y$ and it contains all sequences (y_n) in Y that tend to infinity. □

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