

WARPED PRODUCTS AND CONFORMAL BOUNDARIES OF CAT(0)-SPACES

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ABSTRACT. We discuss the conformal boundary of a warped product of two length spaces and provide a method to calculate this in terms of the individual conformal boundaries. This technique is then applied to produce CAT(0)-spaces with complicated conformal boundaries. Finally we prove that the conformal boundary of an Hadamard n -manifold is always simply connected for $n \geq 3$, thus providing a bound for the level of complication of the boundary of such a manifold.

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1. INTRODUCTION

There are various notions of boundaries at infinity of a metric space. In this paper we will discuss and compare two of these.

The ideal boundary is a classical concept usually defined as a set of equivalence classes of paths converging to infinity. The precise definition will be given in the next section. Conformal boundaries are defined via conformal distortions of the metric space, and thus depend on the choice of distortion function. In certain classes of spaces, e.g. Gromov hyperbolic spaces, there is a range of canonical choices of distortions, which produce homeomorphic boundaries, c.f. [3, Chapter 3]. These turn out also to be homeomorphic to the ideal boundary with a canonically defined topology, see [5, Theorem 2.4]. In the class of CAT(0)-spaces however, there is no canonical choice of conformal distortion, and the ideal and conformal boundary will typically be different, when the space is not Gromov hyperbolic. This is the case e.g. when there are flat “sections” in the space extending to infinity.

One of the main purposes of this paper is to determine the conformal boundary of a warped product with warping function in a class of functions of distance to a base point. This we do in Section 3 after some preliminaries in Section 2. In Section 4, we use our warped product characterization to produce CAT(0)-spaces where the two types of boundary differ in interesting ways.

In particular, we prove the following result. Note that the ideal boundary of every Hadamard n -manifold is homeomorphic to \mathbb{S}^{n-1} [4, II.8.11].

Theorem 1.1. *For each $n > 2$, there exists an Hadamard n -manifold whose conformal boundary is homeomorphic to a 1-point union of two $(n - 1)$ -spheres and an Hadamard n -manifold whose conformal boundary is homeomorphic to an $(n - 2)$ -disk.*

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Finally in Section 5, we show that even though the conformal boundary of an Hadamard manifold can be quite complicated from a topological viewpoint, it is at least simply connected, when the dimension of the manifold is greater than 2.

Theorem 1.2. *Let M be an Hadamard n -manifold, $n \geq 3$, then the conformal boundary $\partial_\rho M$ is simply connected.*

2. PRELIMINARIES

2.1. Length spaces and conformal distortions. We refer to [4] and [5] for more details on the concepts we briefly introduce below.

A length space is a metric space (X, d) , where the distance between two points is given as the infimum of lengths of rectifiable curves connecting points:

$$d(p, q) = \inf_{\gamma \in \Gamma(p, q)} \mathfrak{L}(\gamma),$$

where $\Gamma(p, q)$ denotes the set of rectifiable paths having p and q as endpoints. Given a continuous function $\rho : X \rightarrow (0, \infty)$, we can define the conformally distorted length metric:

$$\sigma_\rho(p, q) := \inf_{\gamma \in \Gamma(p, q)} \int_\gamma \rho ds,$$

see e.g. [3] for more details. One then defines the conformal boundary of X with respect to the distortion ρ as

$$\partial_\rho X := \overline{X}_\rho \setminus X,$$

where \overline{X}_ρ denotes the metric completion of X with respect to the length metric σ_ρ .

We will mainly focus attention on the case where the distortion is a function of the distance to a base point $o \in X$:

$$\rho(p) = g(d(o, p)),$$

and $g : [0, \infty) \rightarrow (0, \infty)$ is a continuous function with $g \in L^1([0, \infty))$. We use $\partial_g X$ as a short notation for the boundary with respect to such a conformal distortion. If X has an infinitely long geodesic ray emanating from o , e.g. if X is proper and unbounded, then we always have $\partial_g X \neq \emptyset$. See [5] for more details on conformal boundaries of this type.

The restriction to conformal distortions which are distance functions will simplify the analysis, but is not essential. In Theorem 1.2, proven in the final section, we also consider a slightly larger class of deformations. One easy observation, allowing for generalizations of the class of exact distance type deformations, is the following:

Proposition 2.1. *Let X be a length space with base point $o \in X$, $g : [0, \infty) \rightarrow (0, \infty)$ a continuous function and $\rho : X \rightarrow (0, \infty)$ a continuous function such that*

$$\frac{1}{C}g(d(o, p)) \leq \rho(p) \leq Cg(d(o, p)),$$

for some $C \geq 1$ and all $p \in X$. Then $\partial_\rho X$ is homeomorphic, in fact bi-Lipschitz equivalent, to $\partial_g X$.

The proof is trivial and is left to the reader.

2.2. Ideal Boundaries. A K -rough geodesic $\gamma : I \rightarrow X$ is a path, such that for all $s < t \in I$, we have

$$(2.2) \quad d(\gamma(s), \gamma(t)) \geq \mathfrak{L}(\gamma([s, t])) - K,$$

for some $K \geq 0$. Here we do not distinguish notationally between paths and their images. If (2.2) holds with $K = 0$, γ is called a geodesic. A length space is called *geodesic* if any two points can be joined by a geodesic. Any *proper* (i.e. where closed balls are compact) length space is geodesic by the Hopf-Rinow Theorem [4, I.3.7].

A K -rough ray γ in X is an infinitely long K -rough geodesic with one endpoint, i.e. when parametrized by arclength it can be defined on $[0, \infty)$. A *ray* is then a 0-rough ray, that is, an infinitely long geodesic with one endpoint.

Two paths γ_1, γ_2 are said to be *asymptotic* if the Hausdorff distance between their images is finite $d_{\mathcal{H}}(\gamma_1, \gamma_2) < \infty$. This defines an equivalence relation \sim on the set of paths. One easily checks that for two rough rays γ_1, γ_2 parametrized by arclength, we have $\gamma_1 \sim \gamma_2$ if and only if

$$\sup_{t>0} d(\gamma_1(t), \gamma_2(t)) < \infty.$$

We define the *ideal boundary* $\partial_I X$ to be the set of equivalence classes of (geodesic) rays. It will prove convenient to work with the slightly more general notion of a *rough ideal boundary*, $\partial_{RI} X$, defined to be the set of equivalence classes of rough rays. Clearly, we have $\partial_I X \subseteq \partial_{RI} X$. If X is Gromov hyperbolic then we have $\partial_I X = \partial_{RI} X$, however it is possible to have a strict inclusion:

Example 2.3. In general for an Hadamard manifold M^n , i.e. a complete, simply connected Riemannian manifold of nonpositive curvature, it is well known that $\partial_I M$ is homeomorphic to \mathbb{S}^{n-1} , when equipped with a canonically defined topology, c.f. [4, II.8.11]. Even in the simplest case $M = \mathbb{R}^2$, we have $\partial_I \mathbb{R}^2 \subsetneq \partial_{RI} \mathbb{R}^2$. Given a ray γ (a half-line going to infinity) it is possible to construct a rough ray, which “zig-zags” around γ , in a way such that the two paths are not asymptotic according to the definition above.

2.3. CAT(0)-spaces. A geodesic triangle T in a metric space X is the union of three geodesics, $\gamma_1 \in \Gamma(a, b)$, $\gamma_2 \in \Gamma(b, c)$, $\gamma_3 \in \Gamma(c, a)$. A geodesic triangle T is said to satisfy the CAT(0)-inequality if it is at least as *slim* as a comparison triangle with the same side lengths $\tilde{T} \subset \mathbb{R}^2$. See e.g. [4, II.1.1] for the precise definition. A CAT(0)-space is then a geodesic space in which all geodesic triangles satisfy the CAT(0)-condition.

2.4. Warped Products. We will define the warped product of two length spaces as in [1] and [2], see also [7]:

Let B and F be length spaces, and let $f : B \rightarrow (0, \infty)$ be a continuous function. Then define the following length structure on $B \times F$:

$$(2.4) \quad \mathfrak{L}_f(\gamma) = \int_{\gamma} \sqrt{v_B^2(t) + f^2(\gamma_B(t))v_F(t)^2} dt,$$

where $\gamma = (\gamma_B, \gamma_F)$ and v_B, v_F are the speeds of γ_B, γ_F respectively, which are defined almost everywhere. We will write

$$d_f(p, q) = \inf_{\gamma \in \Gamma(p, q)} \mathfrak{L}_f(\gamma)$$

for the length metric induced from the length structure \mathfrak{L}_f . The *warped product* $B \times_f F$ is then $B \times F$ equipped with the metric d_f .

We use standard terminology as for Riemannian warped products, c.f. [9]. B is called the base and F is called the fiber. Subsets of the form $B \times \{q\}$, $q \in F$, are called *leaves*, while the subsets $\{p\} \times F$, $p \in B$ are called *fibers*. A curve of the form $t \mapsto (\alpha(t), q)$ is called *horizontal*, while $t \mapsto (p, \beta(t))$ is called a *vertical* curve.

From the definition of the warped product metric, it is clear that the projection onto the base coordinate $\pi_B : B \times_f F \rightarrow B$ is 1-Lipschitz,

$$d_B(\pi_B(p), \pi_B(q)) \leq d_f(p, q).$$

It is also evident that the leaves $B \times \{q\}$, $q \in F$ are all isometric to B . Furthermore, if $f : B \rightarrow (0, \infty)$ has a global minimum at $p \in B$, one easily checks that the fiber $\{p\} \times F$ is isometric to F with metric scaled by $f(p)$.

Lemma 2.5. *The identity map $B \times_f F \rightarrow B \times F$ is a homeomorphism when $B \times F$ is equipped with the product topology. If B and F are complete, then so is $B \times_f F$. Thus if B and F are proper, so is $B \times_f F$.*

Proof. Given $p, q \in B \times F$, we have

$$(2.6) \quad d_f(p, q) \leq f(\pi_B(p))d_F(\pi_F(p), \pi_F(q)) + d_B(\pi_B(p), \pi_B(q)),$$

by first moving in the fiber through p and then in the leaf through q . This shows that the identity is continuous from $B \times F$ to $B \times_f F$.

Now choose a ball $\mathcal{B} \subset B$ of radius r around $\pi_B(p)$ such that $f(x) \geq \frac{1}{2}f(\pi_B(p))$ for $x \in \mathcal{B}$. Given $\epsilon > 0$, choose $0 < \delta < \min\{\epsilon, f(\pi_B(p))\frac{\epsilon}{2}, r\}$. Then if $d_f(p, q) < \delta < r$ we can ensure that for an almost minimizing curve $\gamma = (\gamma_B, \gamma_F)$ connecting p and q , we have $\gamma_B \subset \mathcal{B}$, since $\mathfrak{L}_B(\gamma_B) \leq \mathfrak{L}_f(\gamma)$. Thus

$$\mathfrak{L}_f(\gamma) \geq (\min_{\gamma} f) \mathfrak{L}_F(\gamma_F) \geq \frac{1}{2}f(\pi_B(p)) \mathfrak{L}_F(\gamma_F) \geq \frac{1}{2}f(\pi_B(p))d_F(\pi_F(p), \pi_F(q))$$

and we conclude that $d_f(p, q) \geq \frac{1}{2}f(\pi_B(p))d_F(\pi_F(p), \pi_F(q))$, hence $d_F(\pi_F(p), \pi_F(q)) < \epsilon$ and $d_B(\pi_B(p), \pi_B(q)) < d_f(p, q) < \epsilon$, which shows that the identity $B \times_f F \rightarrow B \times F$ is continuous.

Since the projection π_B is 1-Lipschitz, we get a Cauchy sequence $\{\pi_B(p_n)\} \subset B$, when $\{p_n\}$ is Cauchy in $B \times_f F$. An easy variation of the argument given above shows that also $\{\pi_F(p_n)\} \subset F$ is Cauchy, and thus the d_f -Cauchy sequence $\{p_n\}$ is convergent in the product topology, hence also in $B \times_f F$.

Finally, if B and F are both proper, hence complete, then $B \times_f F$ is complete. $B \times_f X$ is also locally compact since the product topology is, hence $B \times_f F$ is proper by the Hopf-Rinow Theorem, c.f. [4, I.3.7]. \square

3. BOUNDARIES OF WARPED PRODUCTS

To simplify the analysis, we will from now on consider warping functions, which are functions of the distance to a point. We assume that a base point $o_B \in B$ is chosen, and for $p \in B$ introduce the notation $|p| := d_B(o_B, p)$. For $f : [0, \infty) \rightarrow (0, \infty)$ a continuous function, we consider warping functions of the form

$$f \circ |\cdot| : B \rightarrow (0, \infty)$$

and also use the shorthand notation $B \times_f F$ for warped products with this type of composite warping functions. The meaning should be clear from context.

The restriction to warping functions of this type is not essential, and the results below could be generalized by considering warping functions, which are comparable (in the right way) to functions of distance. However the class of warping functions we consider, provides us with sufficient flexibility to construct the examples of CAT(0)-spaces, we are looking for.

Lemma 3.1. *Let $f : [0, \infty) \rightarrow [1, \infty)$ be a homeomorphism satisfying exponential growth conditions of the form*

$$(3.2) \quad \exists a > 0 \forall t > 0 : \quad f(t+a) \geq 2f(t),$$

$$(3.3) \quad \exists A > 0 \forall t > 0 : \quad f(t+1) \leq Af(t),$$

Then the rough ideal boundary of $X := B \times_f F$ is given by the set equation

$$(3.4) \quad \partial_{RI}X = (\partial_{RI}B \times F) \cup \partial_{RI}F$$

Proof. First of all, since all leaves $B \times \{q\}$ are isometric to B , and likewise because the fiber $\{o_B\} \times F$ is isometric to F it is clear that $(\partial_{RI}B \times F) \cup \partial_{RI}F \subseteq \partial_{RI}X$. So we need to show that any rough ray $\gamma = (\gamma_B, \gamma_F)$ in X is asymptotic to either a horizontal or a vertical rough ray.

Assume that $\gamma = (\gamma_B, \gamma_F)$ is a rough ray parametrized by arclength, so $v_F \leq 1/f(|\gamma_B|)$ a.e. We begin by proving a couple of useful estimates for γ . By assumption there exists $K \geq 0$ such that for all $t \geq 0$:

$$(3.5) \quad \begin{aligned} t - K &\leq d_f((o_B, \gamma_F(0)), \gamma(t)) \\ &\leq d_f((o_B, \gamma_F(0)), (o_B, \gamma_F(t))) + d_f((o_B, \gamma_F(t)), \gamma(t)) \\ &\leq \mathfrak{L}_F(\gamma_F([0, t])) + |\gamma_B(t)| \end{aligned}$$

$$(3.6) \quad \leq \int_0^t \frac{1}{f(|\gamma_B(s)|)} ds + |\gamma_B(t)|$$

as is seen by first moving in the fiber $\{o_B\} \times F$, then in the base, and applying the triangle inequality.

If $\gamma_B(t) \not\rightarrow o_B$ for $t \rightarrow \infty$ then there is some $\epsilon > 0$ such that $|\gamma_B(t)| > \epsilon$ for some arbitrarily large values of t . Since the speed of γ_B is bounded by 1, it follows that $I := \{t \geq 0 : |\gamma_B(t)| > \epsilon/2\}$ has infinite Lebesgue measure, and so

$$(3.7) \quad t - \int_0^t \frac{ds}{f(|\gamma_B(s)|)} = \int_0^t \left(1 - \frac{1}{f(|\gamma_B(s)|)}\right) ds \geq \mu(I \cap [0, t]) \left(1 - \frac{1}{f(\epsilon/2)}\right) \rightarrow \infty.$$

Assume that $|\gamma_B(t)|$ is bounded, hence γ_F and thus F must be unbounded. It immediately follows from (3.5) that γ_F must be a rough ray, so that γ is asymptotic to the vertical ray $t \mapsto (o_B, \gamma_F(t))$. In fact we have $\lim_{t \rightarrow \infty} \gamma_B(t) = o_B$, since if this were false then (3.7) and (3.6) together would contradict the boundedness of $|\gamma_B(t)|$.

Suppose instead that $|\gamma_B(t)|$ is unbounded. Again combining (3.7) and (3.6), we see that

$$|\gamma_B(t)| \geq t - K - \mathfrak{L}_F(\gamma_F([0, t])) \geq t - K - \int_0^t \frac{1}{f(|\gamma_B(s)|)} ds \rightarrow \infty$$

Thus $I := \{t : |\gamma_B(t)| < c_1\}$ has finite measure, while $J := \{t : |\gamma_B(t)| > 2c_1\}$ has infinite Lebesgue measure. Thus:

$$\begin{aligned} \frac{1}{f(c_1)}t - \mathfrak{L}_F(\gamma_F([0, t])) &\geq \frac{1}{f(c_1)}t - \int_0^t \frac{ds}{f(|\gamma_B(s)|)} \\ &= \int_0^t \left(\frac{1}{f(c_1)} - \frac{1}{f(|\gamma_B(s)|)} \right) ds \\ &\geq \mu(J \cap [0, t]) \left(\frac{1}{f(2c_1)} - \frac{1}{f(c_1)} \right) - \mu(I \cap [0, t])f(0) \rightarrow \infty. \end{aligned}$$

Since $f(t) \rightarrow \infty$, this last estimate and (3.5) together imply that $|\gamma_B(t)| - c_2t \rightarrow \infty$ for any $c_2 \in [0, 1)$. By a change of parameters, we deduce that $\mathfrak{L}_F(\gamma_F) \leq C \int_0^\infty \frac{1}{f(|\gamma_B(t)|)} dt$ is finite, and thus that $|\gamma_B(t)| > t - K_2$ for some $K_2 > 0$, i.e. γ_B is a rough ray in B .

Since $\mathfrak{L}_F(\gamma_F) < \infty$, we see that $\gamma_F(t)$ is convergent to some $p \in F$. Thus γ is asymptotic to the horizontal rough ray $t \mapsto (\gamma_B(t), p)$, since by moving vertically it follows that:

$$(3.8) \quad \sup_{t>0} d(\gamma(t), (\gamma_B(t), p)) \leq \sup_{t>0} f(|\gamma_B(t)|) \int_t^\infty \frac{1}{f(|\gamma_B(s)|)} ds < \infty.$$

In fact, in view of (3.3) and the double inequality $t - K_3 \leq |\gamma_B(t)| \leq t + K_3$ for some $K_3 > 0$ and all $t \geq 0$, the second inequality in (3.8) reduces to the inequality

$$\sup_{t>0} f(t) \int_t^\infty \frac{ds}{f(s)} < \infty,$$

and this follows from (3.2). \square

Let us return to the conformal boundary $\partial_g X$, defined with respect to a base point $o \in X$ and a continuous function $g : [0, \infty) \rightarrow (0, \infty)$, with $g \in L^1([0, \infty))$. We will call g k -quasidecreasing if

$$g(t) \leq kg(s)$$

for all $0 \leq s \leq t$ and some $k \geq 1$.

There is a map

$$J_X : \partial_{RI} X \rightarrow \partial_g X,$$

defined by choosing sequences going to infinity along each rough ray, e.g. $x_n = \gamma(n)$, $n \in \mathbb{N}$. We will need the following result:

Lemma 3.9. *If X is a proper unbounded length space and $g : [0, \infty) \rightarrow (0, \infty)$ a quasidecreasing distortion function in $L^1([0, \infty))$, then $J_X : \partial_{RI} X \rightarrow \partial_g X$ is well defined and surjective. In fact $J_X(\partial_I X) = \partial_g X$.*

Proof. First of all, we need to see that given a rough ray γ , $\gamma(t_n)$ will define an element in $\partial_g X$, for t_n a parameter sequence converging to ∞ . Since γ is a rough ray we have $d(o, \gamma(t)) \geq t - K$ for some fixed $K > 0$. Thus $g(d(o, \gamma(t))) \leq kg(t - K)$, if g is k -quasidecreasing. Since $g \in L^1([0, \infty))$ it follows that the σ_g -length of γ is finite, and thus that $\gamma(t_n)$ will define a σ_g -Cauchy sequence for any $t_n \rightarrow \infty$; here σ_g is the conformally distorted metric.

That J_X is well defined, i.e. that it does not depend on the choice of representative of an equivalence class of rough rays nor on the choice of sequence $\{\gamma(t_n)\}$ for $t_n \rightarrow \infty$, follows also from $g \in L^1([0, \infty))$ and so $g(t) \rightarrow 0$ for $t \rightarrow \infty$. So for any two sequences

$x = \{x_n\}, y = \{y_n\}$ chosen along equivalent rays, we have $\sigma_g(x_n, x_m) \rightarrow 0, \sigma_g(y_n, y_m) \rightarrow 0$ and $\sigma_g(x_n, y_m) \rightarrow 0$ for $n, m \rightarrow \infty$. Thus x and y will define equivalent σ_g -Cauchy sequences converging to some point in $\partial_g X$.

That J_X is surjective, in fact already as a map from $\partial_I X \subseteq \partial_{RI} X$ follows from Theorem 2.1 and Theorem 2.2 in [5]. In short an argument goes as follows: given a σ_g -Cauchy sequence $\{x_n\} \subset X$ converging to a point in $\partial_g X$, we must have $d(o, x_n) \rightarrow \infty$, for any fixed point $o \in X$. In the proof of Theorem 2.2(b) a geodesic ray γ with $\gamma(0) = o$ is constructed such that $d(\gamma, y_n) \leq 1$, where $y = \{y_n\}$ is a subsequence of $\{x_n\}$. Clearly $J_X(\gamma)$ is then equivalent to y , since $g(t) \rightarrow 0$ for $t \rightarrow \infty$. Furthermore a subsequence of a σ_g -Cauchy sequence is equivalent to the original sequence. \square

Now we fix also a base point $o_F \in F$ and thus a base point $o_X := (o_B, o_F) \in X = B \times_f F$. We use the notation $|p|_B := d_B(o_B, p), |q|_F := d_F(o_F, q), |r|_X := d_X(o_X, r)$ for points $p \in B, q \in F, r \in X$. σ_B, σ_F and σ_X will denote the conformally distorted metrics of B, F and X with respect to the chosen base points and a fixed distortion function g .

Theorem 3.10. *Let B and F be pointed, proper length spaces, with B unbounded. Let $f : [0, \infty) \rightarrow [1, \infty)$ be a warping function satisfying the requirements of Lemma 3.1 and let $g : [0, \infty) \rightarrow (0, \infty)$ be a distortion function as in Lemma 3.9 satisfying also:*

$$(3.11) \quad g(a+b) \geq k_1 g(a)g(b) \text{ and } f(t)g(t) \geq k_2,$$

for some positive constants k_1, k_2 and all $a, b, t \geq 0$. Then the conformal boundary $\partial_g X$ of the warped product $X = B \times_{\tilde{f}} F$, where $\tilde{f} = f \circ |\cdot|_B$, is homeomorphic to the gluing of $\partial_g B \times \overline{F}$ onto $\partial_g F$ along $\partial_g B \times \partial_g F$, using the projection map. If $\partial_g F = \emptyset$, the gluing is simply $\partial_g B \times F$.

Proof. For each leaf the embedding $B \hookrightarrow B \times \{q\} \subseteq X$ is isometric, and $|(p, q)|_X \geq |p|_B$, hence $g(|(p, q)|_X) \leq k_0 g(|p|_B)$. Thus clearly any σ_B -Cauchy sequence $\{p_n\} \subset B$ is also a σ_X -Cauchy sequence in the leaf $B \times \{q\}$. Hence we have a map

$$\psi_B : \partial_g B \times F \rightarrow \partial_g X$$

Likewise the embedding of the standard fiber $F \hookrightarrow \{o_B\} \times F \subset X$ is isometric, with $|(o_B, q)|_X = |q|_F$, so we also have a map

$$\psi_F : \partial_g F \rightarrow \partial_g X$$

Now define a map $\psi : \partial_g B \times \overline{F} \rightarrow \partial_g X$ by

$$\psi((x, y)) = \begin{cases} \psi_B((x, y)) & (x, y) \in \partial_g B \times F \\ \psi_F(y) & (x, y) \in \partial_g B \times \partial_g F \end{cases}$$

Another way of getting the gluing of $\partial_g B \times \overline{F}$ and $\partial_g F$ is by considering the quotient $(\partial_g B \times \overline{F}) / \sim$, where $\partial_g B \times \{y\}$ is collapsed to a point for $y \in \partial_g F$. Since ψ is constant on equivalence classes we may consider it as a map $\psi : (\partial_g B \times \overline{F}) / \sim \rightarrow \partial_g X$.

Equip $\partial_g B \times \overline{F}$ with the product topology induced by the conformally distorted metrics σ_B, σ_F and give $(\partial_g B \times \overline{F}) / \sim$ the quotient topology. We will show that ψ is a homeomorphism.

By Lemma 3.9 the conformal distortion map $J_X : \partial_{RI} X \rightarrow \partial_g X$ is surjective. Using the description in Lemma 3.1, it follows that ψ is surjective. In fact, if $x \in \partial_g X$ and we choose a rough ray γ such that $J_X(\gamma) = x$, then either γ is asymptotic to a horizontal ray

(γ_B, q) and x is in the image $\psi_B(\partial_g B \times F)$, or γ is asymptotic to a vertical ray (o_B, γ_F) and x is in the image $\psi_F(\partial_g F)$.

To prove that ψ is injective, we first need some estimates. Let $\gamma = (\alpha, \beta) : [t_0, t_1] \rightarrow X$ be a path in X . By first moving in the fiber $\{o_B\} \times F$ and then horizontally in a leaf, we have a triangle with side lengths $|\beta(s)|_F, |\alpha(s)|_B, |(\alpha(s), \beta(s))|_X$, so by the triangle inequality:

$$|(\alpha(s), \beta(s))|_X \leq |\alpha(s)|_B + |\beta(s)|_F$$

so, using that g is quasidecreasing and condition (3.11), we have

$$(3.12) \quad g(|(\alpha(s), \beta(s))|_X) \geq k_0 g(|\alpha(s)|_B + |\beta(s)|_F) \geq k_0 k_1 g(|\beta(s)|_F) g(|\alpha(s)|_B)$$

Then

$$(3.13) \quad \int_{\gamma} g(|\gamma(s)|_X) \sqrt{v_{\alpha}^2 + f^2(|\alpha(s)|_B) v_{\beta}^2} ds \\ \geq \int_{\gamma} k_0 k_1 g(|\beta(s)|_F) g(|\alpha(s)|_B) f(|\alpha(s)|_B) v_{\beta} ds$$

By assumption there is a positive constant k_2 such that $g(t)f(t) \geq k_2$ for all $t > 0$. Hence

$$(3.14) \quad \mathfrak{L}_{\sigma_X}(\gamma) \geq k_0 k_1 k_2 \mathfrak{L}_{\sigma_F}(\beta),$$

so we conclude that

$$(3.15) \quad \sigma_X((p_1, q_1), (p_2, q_2)) \geq K_0 \sigma_F(q_1, q_2),$$

where $K_0 = k_0 k_1 k_2$. A similar crude analysis shows that

$$(3.16) \quad \mathfrak{L}_{\sigma_X}(\gamma) \geq K_1 \min_{t_0 \leq t \leq t_1} (g(|\beta(t)|_F)) \mathfrak{L}_{\sigma_B}(\alpha)$$

where $K_1 = k_0 k_1$, and we conclude from (3.14) that

$$(3.17) \quad \epsilon := \sigma_X((p_1, q_1), (p_2, q_2)) \\ \geq K_1 \min\{g(t) : t \in [|q_1|_F - \epsilon/K_0, |q_2|_F + \epsilon/K_0]\} \sigma_B(p_1, p_2),$$

assuming that $|q_1|_F \leq |q_2|_F$. By continuity (3.15) extends to $\partial_g X$, i.e. to Cauchy sequences of the form (x, y) , for $x \in \overline{B}, y \in \overline{F}$. The estimate (3.17) extends to Cauchy sequences of the form $(x, q), x \in \partial_g B, q \in F$, i.e. to the image $\psi(\partial_g \times F)$.

We then easily deduce that ψ is injective on $(\partial_g B \times \overline{F}) / \sim$. Because if $\psi((x_1, y_1)) = \psi((x_2, y_2))$ we must have $y_1 = y_2 := y \in \overline{F}$ by (3.15), and for $y \in F$ we see from (3.16) that $\psi((x_1, y)) = \psi((x_2, y))$ if and only if $x_1 = x_2$.

Using (3.15) and (3.17) we see that ψ^{-1} is continuous on the horizontal sequences $\psi(\partial_g \times F)$. Continuity of ψ^{-1} , in the quotient topology, at a vertical sequence $\psi_F(\partial_g F)$ follows directly from (3.15).

A similar analysis shows that ψ is continuous. However this also follows from the fact that the involved spaces are compact and Hausdorff. \square

4. EXOTIC BOUNDARIES OF CAT(0)-SPACES

In this section we apply the previous results to give examples of boundaries of CAT(0)-spaces. That the ideal and conformal boundaries typically differ in this category is illustrated by the simplest example: $\partial_I \mathbb{R}^n$ is homeomorphic to \mathbb{S}^{n-1} , but $\partial_g \mathbb{R}^n$ is always a single point for $n \geq 2$, and any quasidecreasing distortion function $g \in L^1([0, \infty))$.

Flat “sections” extending to infinity will always be collapsed in the conformal boundary. In this respect the conformal boundary is in a way dual to the so-called Tits boundary, c.f. [4]. However negatively curved pieces can also be collapsed, if one chooses a conformal scaling function with too fast decay.

The following theorem is the main result in [1]:

Theorem 4.1 (Alexander & Bishop). *If B and F are complete CAT(0) spaces and $f : B \rightarrow (0, \infty)$ is convex, then $B \times_f F$ is CAT(0).*

If we furthermore require that the function $f : [0, \infty) \rightarrow [1, \infty)$ in the construction of Theorem 3.10 is convex then, since distance functions are convex in CAT(0)-spaces [4, II.2.2], it follows easily that the composition $(f \circ |\cdot|)|_B$ is convex and thus, by the above theorem of Alexander and Bishop, that $B \times_f F$ is CAT(0) when B and F are CAT(0)-spaces.

4.1. Examples. We give a few examples of complete, Riemannian CAT(0)-spaces (i.e. Hadamard manifolds) with interesting boundary, calculated using Theorem 3.10. When B is an Hadamard manifold, we get a smooth convex warping function satisfying the requirements of Lemma 3.1 by choosing $f(t) = \cosh(t)$.

If an Hadamard manifold X^n has curvature bounded away from zero, $\kappa \leq \kappa_0 < 0$, and is thus Gromov hyperbolic, then the conformal boundary $\partial_g X$ is homeomorphic to the ideal boundary \mathbb{S}^{n-1} when we choose the distortion g to satisfy the so-called weak Floyd conditions

$$g(t) \leq Cg(s), \quad \text{whenever } s, t \geq 0, \quad s - 1 \leq t \leq 2s + 1, \\ \int_0^\infty g(t) dt \leq Cg(0),$$

for some fixed $C > 0$, together with the decay condition $g(t) \geq K \exp(-\epsilon_0 t)$ for some sufficiently small $\epsilon_0 = \epsilon_0(\kappa) > 0$; see Section 1.6 and Theorem 2.4 of [5]. Below we call any conformal distortion function g that satisfies these three conditions an ϵ_0 -moderate distortion function. For any given $\epsilon_0 > 0$, there always exist ϵ_0 -moderate distortion functions that fulfill the requirements of Theorem 3.10: for instance, it suffices to choose $g(t) = \exp(-\epsilon t)$ for sufficiently small $\epsilon \in (0, \epsilon_0]$.

With these choices of f and g we have:

- If $B = \mathbb{R}$ and $F = \mathbb{R}^{n-1}$, with $n > 2$, then X is CAT(0) and so $\partial_I X$ is homeomorphic to \mathbb{S}^{n-1} . However $\partial_g X$ is homeomorphic to $\mathbb{S}^{n-1} \sqcup_p \mathbb{S}^{n-1}$, a 1-point union of two $(n-1)$ -spheres. This follows since $\partial_g \mathbb{R}^{n-1}$ is a single point $\{p\}$ and $\overline{\mathbb{R}^{n-1}}$ is \mathbb{S}^{n-1} , while $\partial_g \mathbb{R}$ consists of two points.
- If $B = \mathbb{R}^2$ and $F = \mathbb{H}^{n-2}$ is hyperbolic n -space, $n > 2$, then X is CAT(0) and so $\partial_I X$ is homeomorphic to \mathbb{S}^{n-1} . However $\partial_g X = \mathbb{B}^{n-2}$, the closed unit ball.

Note that the above pair of examples already prove Theorem 1.1. It is also clear that if we iterate this procedure with some of the ingredients described above, it is possible

to construct disturbingly complicated conformal boundaries, even when $\partial_I X$ is a sphere of codimension 1.

We also note that while $X \cup \partial_I X$ is contractible, being homeomorphic to a closed unit ball in Euclidean space, it is easy to construct examples where \overline{X} is not contractible. This is the case in the first example above where $\partial_g X$ is homeomorphic to $\mathbb{S}^{n-1} \sqcup_p \mathbb{S}^{n-1}$.

Let us also give some non-Riemannian examples of CAT(0)-spaces. Here we can take e.g. $f(t) = \cosh(t)$ or $f(t) = \exp(t)$. Again g is some appropriate distortion function and $X = B \times_f F$.

- Let $B = \mathbb{R}$ and let F be a tree T . Then X is CAT(0), with $\partial_g X$ the doubling of T along $\partial_g T$. In this example the ideal and conformal boundaries are the same.
- Let $B = \mathbb{R}^2$ and let F be a tree T . Then X is CAT(0) and $\partial_g X$ is simply the closure of the tree $\overline{T} = T \cup \partial_g T$.
- Let B be a tree with finitely many branchings, so that $\partial_g B$ is finite. Then $\partial_g X$ is finitely many copies of \overline{F} glued along $\partial_g F$. If F is CAT(0), then so is X .

5. SIMPLY CONNECTED CONFORMAL BOUNDARIES

In this section we discuss further the topological properties of conformal boundaries of Hadamard manifolds. In particular we establish that the conformal boundary of an Hadamard manifold of dimension at least 3 is simply connected. Thus, even though we have seen that conformal boundaries of Hadamard manifolds can be quite complicated from a topological viewpoint, this provides a bound for the “level of complication”.

We will need the concept of a quotient (pseudo)metric, see [4] p. 65, so let us discuss this. Given a metric space (X, d) and an equivalence relation \sim on X , there is a pseudometric d'' on X given by the formula:

$$d''(x, y) = \inf \left\{ \sum_{i=1}^m d(x_{i-1}, x_i) \mid x_0 = x, x_m = y \right\}, \quad x, y \in X.$$

where

$$d'(u, v) = \begin{cases} 0, & u \sim v, \\ d(u, v), & \text{otherwise,} \end{cases}$$

and the infimum is taken over all finite chains of points in X from x to y . Since

$$d''(x, y) = d''(x', y'), \quad \text{whenever } x \sim x', y \sim y',$$

d'' induces a pseudometric d_\sim on X/\sim , which we call the *quotient pseudometric*.

We call \sim a *metric equivalence relation* if d_\sim is a metric on X/\sim . In this section we let

$$q : X \rightarrow X/\sim$$

denote the quotient map.

For completeness we include the following useful characterization of metric equivalence relations on compact spaces.

Lemma 5.1. *Suppose (X, d) is a compact metric space and \sim an equivalence relation on X . Then the following are equivalent:*

- (a) *the relation \sim is a metric equivalence relation;*
- (b) *the quotient topology on X_\sim is Hausdorff;*

(c) *all equivalence classes in X are closed and the quotient topology on X/\sim equals the d_\sim topology.*

Proof. In this proof, let τ_1 be the quotient topology, and τ_2 be the d_\sim topology on X/\sim . Since q is a contraction, τ_2 is always coarser than τ_1 .

It is easy to see that (a) implies (b) since if d_\sim is a metric, then τ_2 is Hausdorff and so τ_1 is also Hausdorff.

We next prove that (b) implies (c), so suppose that τ_1 is Hausdorff. Thus singletons in X/\sim are closed and so equivalence classes in X are closed. Every continuous bijection from a compact space to a Hausdorff space is a homeomorphism so, since $(X/\sim, \tau_1)$ is compact and $(X/\sim, \tau_2)$ is Hausdorff, it follows that $\tau_1 = \tau_2$.

Lastly we prove that (c) implies (a). The closure of equivalence classes in X is equivalent to τ_1 being T_1 . Assuming that τ_1 is T_1 and that $\tau_1 = \tau_2$, it follows that d_\sim must be a metric since otherwise there would be distinct points $u, v \in X/\sim$ such that $d_\sim(u, v) = 0$, and every τ_2 -neighborhood of u would include v . \square

We note that there are equivalence relations on compact sets whose equivalence classes are closed but which fail to be metric equivalence relations. For instance, if X is the Euclidean interval $[-1, 1]$, and we define an equivalence relation whose equivalence classes are $\{x, -x\}$, $0 < x < 1$, together with the singleton sets $\{0\}$, $\{-1\}$, and $\{1\}$, then $d_\sim([-1, 1]) = 0$ even though -1 and 1 are inequivalent.

Note also that compactness is needed in the lemma above. Consider e.g. the complex plane, with the relation $0 \sim ti$ for all $t \geq 0$ and $1 \sim ti + 1/(t+1)$ for all $t \geq 0$ and no other nontrivial equivalence. Then $d_\sim([0, 1]) = 0$ but $[0] \neq [1]$, while the quotient topology is Hausdorff.

We will need the following result, which seems quite fundamental but which the authors were unable to find in the literature.

Proposition 5.2. *If X is a compact length space and \sim a metric equivalence relation on X with connected equivalence classes, then X/\sim with the quotient metric is a compact length space and the natural map*

$$q_* : \pi_1(X) \rightarrow \pi_1(X/\sim)$$

is surjective.

Proof. That in general X/\sim is a compact length space, when X is a compact length space and \sim is a metric equivalence relation follows from Lemma 5.20 in [4], p.65.

In the following we assume that all curves are parametrized on the unit interval $I = [0, 1]$ and use standard notation from homotopy theory e.g. brackets $[\gamma]$ are used to denote homotopy classes.

We need to prove that q_* is surjective, which amounts to showing that any homotopy class $[\gamma]$ in $\pi_1(X/\sim)$ can be represented by a lifted curve, $[\gamma] = [q \circ \omega]$, where $q : X \rightarrow X/\sim$ is the quotient map. Since X/\sim is a compact length space, it follows that any sufficiently short curve is homotopically trivial, c.f. [6] p. 215. We thus assume that any curve of length less than $2K > 0$ in X/\sim is homotopically trivial.

It is easy to see that any connected subset $C \subseteq X$ has the following property: Given any $x, y \in C$ and any $\epsilon > 0$ there exists finitely many paths $\omega_i : I \rightarrow X, i \in \{1, \dots, n\}$, such that $\omega_1(0) = x, \omega_n(1) = y, \omega_i(1) = \omega_{i+1}(0) \in C$ and $\mathfrak{L}(\omega_i) \leq \epsilon$.

Thus $\omega = \omega_1 * \cdots * \omega_n \in \Gamma(x, y)$ and if C is an equivalence class of \sim then $q \circ \omega$ consists of finitely many loops with $C \in X/\sim$ as basepoint. If $\epsilon < 2K$ then since q is contracting (1-Lipschitz), $q \circ \omega$ will be homotopically trivial.

Let now a closed curve $\gamma : I \rightarrow X/\sim$ represent a homotopy class in $\pi_1(X/\sim)$. We may assume that γ is rectifiable, since any free nontrivial homotopy class has a minimizing geodesic. Choose an $\epsilon \in (0, K)$ and parameter values $0 = t_1 < t_2 < \cdots < t_n = 1$ in I , such that the length of each segment $\gamma([t_i, t_{i+1}])$ is less than $K - \epsilon$, hence $d_\sim(\gamma(t_i), \gamma(t_{i+1})) < K - \epsilon$.

Consider a fixed $i \in \{1, \dots, n-1\}$. By definition of d_\sim we can find a finite chain of points $\{x_1, y_1, x_2, y_2, \dots, x_m, y_m\} \subset X$ such that $d_\sim(\gamma(t_i), \gamma(t_{i+1})) + \epsilon > \sum_{j=1}^m d(x_j, y_j)$, with $x_1 \sim \gamma(t_i)$, $y_m \sim \gamma(t_{i+1})$ and $y_j \sim x_{j+1}$. Since equivalence classes are connected we have by the discussion above for each $j < m$ a path $\beta_j \in \Gamma(y_j, x_{j+1})$ such that $q \circ \beta_j$ is homotopically trivial in X/\sim .

Consider then a continuous curve ω_i consisting of geodesic segments connecting each x_j and y_j as well as curve segments β_j connecting y_j to x_{j+1} as above. That is, if $\alpha_j \in \Gamma(x_j, y_j)$ is a geodesic and $\beta_j \in \Gamma(y_j, x_{j+1})$ is as above, we define

$$\omega_i = \alpha_1 * \beta_1 * \alpha_2 * \beta_2 * \cdots * \beta_{m-1} * \alpha_m.$$

Since the push forward of each β_j curve is homotopically trivial and since q is contracting it is now clear, that the push forward curve $q \circ \omega_i$ is homotopic to a curve $\tilde{\omega}_i$ connecting $\gamma(t_i)$ to $\gamma(t_{i+1})$ with length

$$\mathfrak{L}(\tilde{\omega}_i) \leq \sum_{j=1}^m d(x_j, y_j) < d_\sim(\gamma(t_i), \gamma(t_{i+1})) + \epsilon.$$

Thus the closed curve $\tilde{\omega}_i * \gamma_{|[t_i, t_{i+1}]|}^{-1}$ has length less than $2K$ and is thus homotopically trivial. It follows that $\tilde{\omega}_i$, and thus $q \circ \omega_i$, is homotopic to $\gamma_{|[t_i, t_{i+1}]|}$.

For $i \in \{1, \dots, n-1\}$ we get curves ω_i in X as above, which we paste together to a continuous closed curve using curves $\zeta_i \in \Gamma(\omega_i(1), \omega_{i+1}(0))$ (with indices counted modulo $n-1$) such that $q \circ \zeta_i$ is homotopically trivial. Then γ is homotopic to the curve built up of these segments,

$$[\gamma] = [q_*(\omega_1 * \zeta_1 * \omega_2 * \zeta_2 * \cdots * \omega_{n-1} * \zeta_{n-1})].$$

□

Note that in the generality of the proposition above, it is easy to construct examples, where some continuous curves in X/\sim cannot be lifted to X ; in contrary to the “usual” setting, where $q : X \rightarrow X/\sim$ is a fibration.

We are now ready to provide the main result of this section. Here we will allow more general conformal boundaries defined with respect to a continuous distortion $\rho : X \rightarrow (0, \infty)$ such that

$$(5.3) \quad \rho \in C_0(X) \text{ and } \rho \circ \gamma \in L^1([0, \infty))$$

for any geodesic ray $\gamma \in \partial_I X$; here $C_0(X)$ denotes the continuous functions converging to zero at infinity. As we see from the proof of Lemma 3.9 this class of deformations includes the previously considered distance deformations $\rho = g(d(\cdot, o))$ with g quasidecreasing.

Once again we will utilize the map $J_X : \partial_I X \rightarrow \partial_\rho X$, defined by taking sequences along rays.

Lemma 5.4. *Let X be a proper CAT(0)-space and $\rho : X \rightarrow (0, \infty)$ satisfy (5.3), then $J_X : \partial_I X \rightarrow \partial_\rho X$ is well-defined, surjective and continuous.*

Proof. The proof of well-definedness and surjectivity is similar to the proof of Lemma 3.9, while continuity follows directly from the definition of the cone topology [4, II.8] and the fact that $d(o, x_n) \rightarrow \infty \implies \rho(x_n) \rightarrow 0$ for any point $o \in X$, since $\rho \in C_0(X)$. \square

Corollary 5.5. *The conformal boundary $\partial_\rho M$ of an Hadamard n -manifold, $n \geq 3$, is simply connected.*

Proof. Since the map $J_M : \partial_I M \rightarrow \partial_\rho M$ is surjective, continuous and the considered spaces are compact, $\partial_\rho M$ has the quotient topology on $\partial_I M^n \cong \mathbb{S}^{n-1}$ induced by J_M .

We can extend J_M to a map $M \cup \partial_I M \rightarrow \overline{M}$ by defining $J_M(x) = x$ for $x \in M$. It is a routine matter to check that this extension is continuous, with the cone topology on $M \cup \partial_I M$ which is homeomorphic to the closed unit ball in \mathbb{R}^n . We will then pull back the conformally distorted metric σ_ρ to $M \cup \partial_I M$ using J_M , yielding the pseudometric $\tilde{\sigma}(x, y) = \sigma_\rho(J_M(x), J_M(y))$, for $x, y \in M \cup \partial_I M$.

We only need to show that the fibers $J_M^{-1}(x)$ for $x \in \partial_\rho M$ are connected, allowing us to apply the previous proposition, since if we consider $\partial_I M$ as \mathbb{S}^{n-1} equipped with its standard geodesic metric $d_{\mathbb{S}}$, then because the topology on $\partial_\rho M$ is Hausdorff, the quotient metric on $\partial_\rho M$ associated to this choice of metric on \mathbb{S}^{n-1} becomes a length space metric¹. If some fiber $F = J_M^{-1}(x) \subset M \cup \partial_I M$ is not connected, then since the fibers are closed F can be decomposed into two disjoint closed sets $F = F_1 \cup F_2$. Thus since $M \cup \partial_I M$ is compact, there is an open set U such that $F_1 \subset U$ and $\overline{U} \cap F_2 = \emptyset$.

Thus any continuous curve joining a point $y_1 \in F_1$ to a point $y_2 \in F_2$ must intersect the compact set $\partial U \subset M \cup \partial_I M$. That y_1 and y_2 belong to the same equivalence class means that there exist curves $\gamma : [0, 1] \rightarrow M \cup \partial_I M$ with $\gamma(0) = y_1, \gamma(1) = y_2$ and $\gamma((0, 1)) \in M$ of arbitrarily short $\tilde{\sigma}$ -length. Note that $\tilde{\sigma} = \sigma_\rho$ is a metric on M .

By compactness of ∂U we have $\inf\{\tilde{\sigma}(y_1, y)|y \in \partial U\} = \delta > 0$ and it follows that any curve γ joining y_1 and y_2 as above will have $\tilde{\sigma}$ -length at least δ . Thus y_1 cannot be equivalent to y_2 , which is a contradiction. Hence $J_M^{-1}(x)$ must be connected. \square

Notice that what we actually prove is: Suppose that we can put a topology on $X \cup \partial_I X$ such that $\partial_I X$ is homeomorphic to a length space, $X \cup \partial_I X$ is compact and $J_X : X \cup \partial_I X \rightarrow \overline{X}$ is continuous, then $J_{X*} : \pi_1(\partial_I X) \rightarrow \pi_1(\partial_\rho X)$ is surjective.

This statement of the result should apply to other spaces besides Hadamard manifolds and CAT(0)-spaces.

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¹Note that although this quotient metric will induce the same topology on $\partial_\rho M$ as the conformal metric σ , these metrics can be quite different in other respects!

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