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ADMISSIBLE MEASURES IN ONE DIMENSION

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ABSTRACT. In this note we show that p-admissible measures in one dimension (i.e. doubling measures admitting a p-Poincaré inequality) are precisely the Muckenhoupt A_p-weights.

In the last two decades it has been observed that much of the theory for p-harmonic functions can be extended to the situation when the Lebesgue measure on $\mathbf{R}^n$ is replaced by another measure satisfying certain conditions; see e.g. Fabes–Kenig–Serapioni [2] and Heinonen–Kilpeläinen–Martio [4]. More precisely, Theorem 2 in Hajłasz–Koskela [3] and Theorem 5.2 in Heinonen–Koskela [5] show that the following two conditions are exactly what is needed for the theory to go through.

Definition 1. A measure μ on $\mathbf{R}^n$ is called *p-admissible* with $p \geq 1$ if it satisfies the following two conditions:

- It is *doubling*, i.e. there is a constant $C > 0$ such that
$$\mu(2B) < C\mu(B)$$
for all balls $B \subset \mathbf{R}^n$, where $2B$ denotes the ball concentric with B and with twice the radius.
- It admits the *weak p-Poincaré inequality*, i.e. there exist $C > 0$ and $\lambda \geq 1$ such that
$$\frac{1}{\mu(B)} \int_B |u - u_B| \, d\mu \leq Cr \left(\frac{1}{\mu(\lambda B)} \int_{\lambda B} |\nabla u|^p \, d\mu \right)^{1/p}$$
holds whenever B is a ball with radius r and u is, say, a locally Lipschitz function on λB. Here and in what follows, $u_B = \mu(B)^{-1} \int_B u \, d\mu$.

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The Hölder inequality implies that every p-admissible measure is also p'-admissible for all $p' > p$. Conversely, by a recent result due to Keith–Zhong [6], every p-admissible measure with $p > 1$ is also p'-admissible for some $p' < p$.

Unfortunately, in many situations, the Poincaré inequality is rather difficult to verify. In this note we give a more straightforward characterization of admissible measures in one dimension, namely we prove the following result.

Theorem 2. *Let μ be a measure on $\mathbf{R}$ and let $p \geq 1$. Then μ is p-admissible in $\mathbf{R}$ if and only if $d\mu = w\,dx$ and w is a Muckenhoupt A_p-weight.*

Definition 3. A nonnegative function w on $\mathbf{R}^n$ is a *Muckenhoupt A_p-weight* with $p \geq 1$, if for some $C > 0$ and all balls $B \subset \mathbf{R}^n$,

$$\frac{1}{|B|}\int_B w\,dx < \begin{cases} C\biggl(\dfrac{1}{|B|}\displaystyle\int_B w^{1/(1-p)}\,dx\biggr)^{1-p} & \text{for } p > 1, \\ C \operatorname*{ess\,inf}_B w & \text{for } p = 1, \end{cases}$$

where $|B|$ denotes the Lebesgue measure of B.

Remark 4. Note that Theorem 2 fails in $\mathbf{R}^n$ if $n \geq 2$. By e.g. Corollary 15.35 in Heinonen–Kilpeläinen–Martio [4], the measures $d\mu = |x|^\alpha\,dx$ with $\alpha > 0$ are p-admissible in $\mathbf{R}^n$, $n \geq 2$, for all $p > 1$, but belong to A_p if and only if $p > 1 + n\alpha$.

To prove Theorem 2, we use the following lemma. For a proof, see the corollary on p. 200 in Stein [7].

Lemma 5. *Let μ be a nonnegative Borel measure on $\mathbf{R}^n$ and assume that there exists $C > 0$ such that*

$$\frac{1}{|B|}\int_B f(x)\,dx \leq C\biggl(\frac{1}{\mu(B)}\int_B f^p\,d\mu\biggr)^{1/p}$$

for all balls $B \subset \mathbf{R}^n$ and all nonnegative measurable functions f on B. Then μ is absolutely continuous with respect to the Lebesgue measure, $d\mu = w\,dx$ and w is a Muckenhoupt A_p-weight.

In the rest of this note, $C > 0$ denotes a constant whose value may vary with each usage but depends only on the doubling constant of μ and on the constants in the Poincaré inequality.

Proof of Theorem 2. The "if" part of the theorem is proved e.g. in Theorem 15.21 in Heinonen–Kilpeläinen–Martio [4]. To prove the "only if" part, let $f \geq 0$ be a measurable function supported on an interval $I \subset \mathbf{R}$. For $k \in \mathbf{N}$, let $f_k = \min\{f, k\}$ and

$$u_k(x) = \int_{-\infty}^x f_k(t)\chi_I(t)\,dt.$$

Then u_k is Lipschitz and we can test the weak p-Poincaré inequality with it on the concentric double $2I$ of I. On the right-hand side we have

$$C|I|\biggl(\frac{1}{\mu(2\lambda I)}\int_{2\lambda I}(u_k')^p\,d\mu\biggr)^{1/p} \leq C|I|\biggl(\frac{1}{\mu(I)}\int_I f^p\,d\mu\biggr)^{1/p}.$$

To estimate the left-hand side in the Poincaré inequality, let I_- and I_+ denote the parts of $2I \setminus I$ lying to the left and to the right of I, respectively. Then $u_k = 0$

on I_- and

$$u_k = \int_I f_k(x)\,dx$$

on I_+. Using the doubling property of μ, the left-hand side in the Poincaré inequality can be estimated as

$$\begin{aligned}\frac{1}{\mu(2I)}\int_{2I}|u_k-(u_k)_{2I}|\,d\mu &\ge \frac{1}{\mu(2I)}\biggl(\int_{I_-}(u_k)_{2I}\,d\mu + \int_{I_+}\biggl(\int_I f_k(x)\,dx-(u_k)_{2I}\biggr)\,d\mu\biggr)\\ &\ge C\int_I f_k(x)\,dx.\end{aligned}$$

Inserting both estimates into the weak p-Poincaré inequality, together with the monotone convergence theorem, shows that the assumptions in Lemma 5 are satisfied and hence $d\mu = w\,dx$ with w a Muckenhoupt A_p-weight. □

Remark 6. If we knew a priori that μ is absolutely continuous with respect to the Lebesgue measure, then Theorem 2 could also be obtained after some calculation from Theorem 1.4 in Chua–Wheeden [1].

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