ENDS OF METRIC MEASURE SPACES AND SOBOLEV INEQUALITIES

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Abstract. Generalizing work of Li and Wang, we prove sharp volume growth/decay rates for ends of metric measure spaces supporting a (p,p)-Sobolev inequality. A sharp result for (q,p)-Sobolev inequalities is also proved.

0. Introduction

As part of their study of complete manifolds whose spectrum of the Laplacian has a positive lower bound, Li and Wang ([LW, theorem 1.4]) prove the following result, where $\lambda_1(E)$ is the least eigenvalue of the Laplacian on E for the Dirichlet problem, $V(r) = \mu(E \cap B(o,r))$, and $V(\infty) = \mu(E)$. They also prove that the rates of volume decay and growth are sharp.

Theorem A. *Suppose that E is an end of a complete pointed manifold (X,d,μ,o) and that $\lambda = \lambda_1(E) > 0$. Then there exists a positive constant C_1, dependent only on E, such that either*

(1) *E is 2-parabolic and $V(\infty) - V(r) \le C_1 \exp(-2\lambda^{1/2} r)$ for all $r > 0$, or*
(2) *E is 2-nonparabolic and $V(r) \ge C_1^{-1} \exp(2\lambda^{1/2} r)$ for all sufficiently large r.*

The condition $\lambda_1(E) > 0$ is equivalent to the statement that E supports a $(2,2)$-Sobolev inequality. We prove the following generalization of Theorem A in the setting of proper metric measure spaces, where again the rates of volume decay and growth are sharp. We define $V(r)$ and $V(\infty)$ as before. For relevant definitions, see Section 1.

Theorem 0.1. *Suppose that E is an end of a proper pointed metric measure space (X,d,μ,o) and that E supports a $(p,p;\lambda)$-Sobolev inequality, $1 \le p < \infty$. Then there exist a positive constant C_1, dependent only on E and p, such that either*

(1) *E is p-parabolic and $V(\infty) - V(r) \le C_1 \exp(-p\lambda^{1/p} r)$ for all $r > 0$, or*
(2) *E is p-nonparabolic and $V(r) \ge C_1^{-1} \exp(p\lambda^{1/p} r)$ for all sufficiently large r.*

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It is well-known that a (q,q)-Sobolev inequality implies a (p,p)- Sobolev inequality when $p > q$. This holds with good estimates: from a $(q,q;\lambda)$-Sobolev inequality one deduces a $(p,p;\lambda')$-Sobolev inequality, where $\lambda' = (\frac{q}{p})^p \lambda^{p/q}$, see Section 1. Letting $p > q = 2$ and substituting λ' into Theorem 0.1, we recover the decay and growth rates given in Theorem A. Thus our result can be viewed as an extension of Theorem A.

It is also well known that least eigenvalues of the Laplacian and Sobolev-type inequalities are connected with volume growth of manifolds. See for instance chapter 3 of the lecture notes by Saloff-Coste [S], as well as the papers [B1] and [B2] of Brooks. Theorem 2 of [B1] is closely related to theorems A and 0.1.

The proof of Theorem A in [LW] depends on the theory of harmonic functions. Our assumptions are too weak to allow the use of the theory of partial differential equations in the proof of Theorem 0.1, but our proof is nevertheless rather simple.

We also consider (q,p)-Sobolev inequalities for $q > p$. In this case, there are again two very different possibilities. Compared with the $q = p$ case, the conclusion for the case of finite volume is stronger (no ends are possible), but that for the case of infinite volume is weaker.

Theorem 0.2. *Suppose that (X,d,μ,o) is a proper pointed metric measure space and that E is a component of $\{x \in X : d(x,o) > r_0\}$, for some $r_0 \geq 0$. If E supports a $(q,p;\lambda)$-Sobolev inequality for some $1 \leq p < q < \infty$, then there exists a constant C_1, dependent only on q and p, such that either*

(1) *E is bounded, and $\operatorname{diam}(E) \leq 2r_0 + C_1 \lambda^{-1/p} \mu(E)^{(q-p)/qp}$, or*
(2) *E is a p-nonparabolic end, and $V(r) \geq C_1^{-1} (\lambda r^p)^{q/(q-p)}$ for all sufficiently large r.*

Related ideas and results occur in the literature concerning the concentration of measure phenomenon and the Herbst argument; see Ledoux's work [Le].

We prove theorems 0.1 and 0.2 in Section 2, and prove that the decay and growth rates in those theorems are sharp in Section 3.

1. Preliminaries

Suppose (X,d,μ,o) is a *pointed metric measure space*, i.e. it is a metric space with distinguished point o and a Borel measure μ which assigns finite non-zero measure to every ball of positive radius. We say that this space is *proper* if closed balls (and so all closed bounded sets) are compact. We write $|x-a|$ for $d(x,a)$ and $|x|$ for $d(x,o)$. We also write $B(r) = \{x \in X : |x| < r\}$, $\overline{B}(r) = \{x \in X : |x| \leq r\}$ and $S(r) = \overline{B}(r) \setminus B(r)$. Note that the closure of $B(r)$ is contained in $\overline{B}(r)$, but in general the two sets may be different. An *end* is an unbounded component E of $X \setminus \overline{B}(r_0)$, $r_0 \geq 0$.

We denote the maximum and minimum of a pair of numbers s,t by $s \vee t$ and $s \wedge t$ respectively. Given $s,t \in [-\infty,\infty]$, $s \leq t$, we define the truncation $\operatorname{trunc}(f;\ s,t)$ of any real valued function f to be $(f \vee s) \wedge t$. We write $C = C(a,b,\dots)$ if C depends only on the parameters $a,b,\dots$.

Given $S \subset X$, we write $\mathrm{Lip}(S)$ for the set of Lipschitz functions on S and $\mathrm{Lip}_c(S)$ for the Lipschitz functions of compact support. For $\phi \in \mathrm{Lip}(X)$, we define

$$g_\phi(x) \equiv \begin{cases} \liminf\limits_{r \to 0^+} \sup\limits_{\{y\,:\,|x-y|=r\}} \dfrac{|\phi(y)-\phi(x)|}{|y-x|}, & \text{if } x \text{ is not an isolated point,} \\ 0, & \text{if } x \text{ is an isolated point,} \end{cases}$$

Then g_ϕ is an upper gradient of ϕ (in the sense of [HK] and [C]) and, if (X,d,μ) is a metric measure space satisfying some mild extra conditions[1] then, ignoring sets of μ-measure zero, the above $\liminf$ is actually a limit, and g_ϕ equals the minimal p-weak upper gradient of ϕ; see Proposition 1.11 and Corollaries 6.36 and 6.38 of [C]. Also, it is clear that if $\phi(x) = \psi(|x|)$, where ψ is a piecewise smooth Lipschitz function, then $g_\phi(x)$ is at most the larger of the absolute values of the two one-sided derivatives of ψ at $|x|$. We use this last fact repeatedly without comment.

Suppose $1 \le p < \infty$ is fixed, and that E is a component of $X \setminus \overline{B}(r_0)$ for some $r_0 > 0$. Then E is said to be *p-parabolic*, or simply *parabolic*, if for each $K \subset\subset X$ and $\epsilon > 0$ there exists a function $\phi \in \mathrm{Lip}_c(X)$, $\phi \ge 1$ on K, such that $\int_E g_\phi^p < \epsilon$. Otherwise, E is *p-nonparabolic*, or simply *nonparabolic*. We will only be concerned with proper metric measure spaces, so bounded components E are always parabolic (if $E \subset B(r)$, and $\phi = \mathrm{trunc}(r+1-|x|;\ 0,1)$, then $g_\phi|_E \equiv 0$).

Given $1 < p \le q < \infty$ and $\lambda > 0$, a subset S of X is said to *support a $(q,p;\lambda)$-Sobolev inequality* if

$$\lambda \left(\int_S \phi^q \, d\mu \right)^{p/q} \le \int_E g_\phi^p \, d\mu, \qquad \phi \in \mathrm{Lip}_c(S).$$

and λ is the *(q,p)-Sobolev constant* of S. A *(q,p)-Sobolev inequality* simply means a $(q,p;\lambda)$-Sobolev inequality for some $\lambda > 0$. Defining the p-Laplacian of a function u by $\Delta_p u \equiv \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, we note that if $p = q$ and S is a bounded Euclidean domain, then $-\Delta_p u = t|u|^{p-2}u$ has a solution $u \in C_0^1(S)$ for $t = \lambda$, but not for any $t < \lambda$; see [Li]. In the case $p = 2$, this says that λ is the least eigenvalue for the Laplacian Dirichlet problem.

We close this section by commenting on the dependence of the validity of a (p,p)-Sobolev inequality on p. First of all, given $q < p$ it is easy to give examples where this inequality holds, but the corresponding (q,q)-Sobolev inequality fails. On the other hand, when $q < p$, we claim that a $(q,q;\lambda)$-Sobolev inequality implies the validity of a $(p,p;\lambda')$-Sobolev inequality with $\lambda' = (q/p)^p \lambda^{p/q}$.

Indeed, given $u \in \mathrm{Lip}_c(S)$, we define $v(x) = |u|^{p/q}$. Then $v \in \mathrm{Lip}_c(S)$, and $g_v(x) \le \frac{p}{q} v(x)^{p/q-1} g_u(x)$ for all x. The desired estimate follows from the (q,q,λ)-Sobolev inequality applied to v via the use of Hölder's inequality.

[1]Specifically, these statements are true if the metric measure space is a doubling space that supports a $(1,p)$-Poincaré inequality, and $\phi, g_\phi \in L^q_{\mathrm{loc}}(X)$ for some $q > p$.

2. Proof of main results

Let us state some standing notation for the remainder of the paper. First, we assume that (X, d, μ, o) is a proper pointed metric measure space, and $|x|$, $B(r)$, $\overline{B}(r)$, and $S(r)$ are as defined in the previous section. E is a component of $X \setminus \overline{B}(r_0)$ for some fixed $r_0 \geq 0$; E may be bounded unless we specify that it is an end. We write $E(r) = E \cap B(r)$, $V(r) = \mu(E(r))$, for $r > r_0$, and $V(\infty) = \mu(E)$. We also define $E(r+) = E \cap \overline{B}(r)$ and $V(r+) = \mu(E(r+))$.

Proof of Theorem 0.1. Throughout this proof, C is a generic constant which can change from one instance to the next, but depends only on E and p.

Note first that $(X, \lambda^{-1/p}d, \mu, o)$ supports a $(p, p; 1)$-Sobolev inequality. It follows that it suffices to prove the theorem with $\lambda = 1$. Assuming that E is parabolic, we prove exponential decay of the volume. Let

$$\psi(x) = \begin{cases} \exp(r_0 + 1), & |x| \leq r_0 + 1, \\ \exp(|x|), & r_0 + 1 \leq |x| \leq r - 1, \\ \exp(r - 1), & r - 1 \leq |x|, \end{cases}$$

where $r \geq r_0 + 2$. Let $\eta_1(x) = \operatorname{trunc}(2(|x| - r_0 - \frac{1}{2});\ 0, 1)$. By parabolicity of E, we may choose $\eta_2 \in \operatorname{Lip}_c(X)$ such that $0 \leq \eta_2 \leq 1$, $\eta_2 \equiv 1$ on $B(r + 1)$, and $\int_E g_{\eta_2}^p \leq \epsilon \exp(-pr + p)$, where $\epsilon > 0$ is fixed but arbitrary. Finally, note that $\phi \equiv \eta_1 \eta_2 \psi \in \operatorname{Lip}_c(E)$.

Then

$$\int_{E \setminus E(r+1)} (g_\phi^p - \phi^p)\, d\mu \leq \int_{E \setminus E(r+1)} \psi^p g_{\eta_2}^p\, d\mu \leq \epsilon, \tag{2.1}$$

and

$$\begin{aligned} \int_{E(r+1) \setminus E(r-1+)} (g_\phi^p - \phi^p)\, d\mu &= -\int_{E(r+1) \setminus E(r-1+)} \psi^p\, d\mu \\ &= -e^{p(r-1)}(V(r + 1) - V(r - 1+)). \end{aligned} \tag{2.2}$$

Also $g_\phi(x) \leq \phi(x)$ when $r_0 + 1 < |x| \leq r - 1$, and so

$$\int_{E(r-1+) \setminus E(r_0+1+)} (g_\phi^p - \phi^p)\, d\mu \leq 0 \tag{2.3}$$

Finally,

$$\int_{E(r_0+1+)} (g_\phi^p - \phi^p)\, d\mu \leq \int_{E(r_0+1+)} (2\psi)^p\, d\mu = 2^p e^{p(r_0+1)} V(r_0 + 1+). \tag{2.4}$$

Adding together (2.1), (2.2), (2.3), and (2.4), the Sobolev inequality yields

$$e^{p(r-1)}(V(r + 1) - V(r - 1+)) \leq 2^p e^{p(r_0+1)} V(r_0 + 1+) + \epsilon,$$

and so

$$V(r+) - V(r - 1+) \leq Ce^{-pr}.$$

Writing this last inequality for $r = R + j - 1$, $j \in \mathbb{N}$, and adding up the resulting sequence of inequalities, it follows that $V(\infty) - V(R) \le Ce^{-pR}$, as required.

Suppose instead that E is nonparabolic. We first prove by contradiction that $\mu(E) = \infty$. Suppose $\mu(E) < \infty$. If $K \subset\subset X$, and $\epsilon > 0$, then $K \subset B(r)$ for some $r \ge r_0$, and we can choose $r' > r$ such that $V(\infty) - V(r') < \epsilon$. Letting $\phi(x) = \operatorname{trunc}(r' + 1 - |x|;\ 0, 1)$, we see that $\int_E g_\phi^p \le V(r' + 1+) - V(r') < \epsilon$, contradicting nonparabolicity of E. Thus $\mu(E) = \infty$.

It remains to prove that volume grows at the desired exponential rate. Let

$$\psi(x) = \begin{cases} \exp(-r_1), & |x| \le r_1, \\ \exp(-|x|), & r_1 \le |x| \le r - 1, \\ \exp(-r + 1), & r - 1 \le |x|, \end{cases}$$

where $r \ge r_1 + 1$ and $r_1 = r_1(E)$ is chosen so large that $V(r_1) > 2^{p+1}V(r_0 + 1+) + 1$. Let $\eta_1(x) = \operatorname{trunc}(2(|x| - r_0 - \frac{1}{2});\ 0, 1)$, let $\eta_2(x) = \operatorname{trunc}(r - |x|;\ 0, 1)$, and let $\phi = \eta_1\eta_2\psi$.

Now, $g_{\eta_2} \le 1$, and so

$$\int_{E \setminus E(r-1+)} (g_\phi^p - \phi^p)\, d\mu \le \int_{E(r+) \setminus E(r-1+)} \psi^p\, d\mu = \exp(p - pr)(V(r+) - V(r - 1+)). \tag{2.5}$$

Also $g_\phi(x) \le \phi(x)$ when $r_1 \le |x| \le r - 1$ (note that both endpoints require special attention), and so

$$\int_{E(r-1+) \setminus E(r_1)} (g_\phi^p - \phi^p)\, d\mu \le 0. \tag{2.6}$$

Since ϕ is constant on $E(r_1) \setminus E(r_0 + 1+)$, we have

$$\int_{E(r_1) \setminus E(r_0+1+)} (g_\phi^p - \phi^p)\, d\mu = -\exp(-pr_1)(V(r_1) - V(r_0 + 1+)) \tag{2.7}$$

and finally

$$\int_{E(r_0+1+)} (g_\phi^p - \phi^p)\, d\mu \le \int_{E(r_0+1+)} (2\psi)^p\, d\mu \le 2^p \exp(-pr_1)V(r_0 + 1+) \tag{2.8}$$

Adding together (2.5), (2.6), (2.7), and (2.8), the Sobolev inequality yields

$$\exp(-pr_1)(V(r_1) - 2^p V(r_0 + 1+)) \le \exp(p - pr)(V(r+) - V(r - 1+))$$

Since $V(r_1) > 2^{p+1}V(r_0 + 1+) + 1$, and $r_1 = r_1(E)$, we see that $V(r+) \ge C_1 \exp(pr)$, for some $C_1 = C_1(E, p)$. □

Remark 2.9. The careful reader may have noticed that if we replace E by X, assuming that X has infinite diameter, the proof is still valid but no longer gives the full story, since we can now rule out the parabolic case. This can be seen most simply by taking $K = \overline{B}(1)$ in the definition of parabolicity. If $\phi \ge 1$ on K, the Sobolev inequality implies that $\int_X g_\phi^p \ge \lambda\mu(K) > 0$, thus ruling out parabolicity.

Proof of Theorem 0.2. As in the proof of Theorem 0.1, we may normalize so that $\lambda = 1$. Suppose first that E is parabolic. For given radii $R > r > r' > r_0$, and numbers $0 < \delta < (r - r')/2$, $0 < \epsilon$, we let $\phi = \psi\eta$, where

$$\psi(x) = \operatorname{trunc}\left(\frac{|x| - r' - \delta}{r - r' - 2\delta};\ 0, 1\right),$$

and $\eta \in \operatorname{Lip}_c(X)$ is such that $\eta \equiv 1$ on $\overline{B}(R)$ and $\int_E g_\eta^p \, d\mu < \epsilon$. Then

$$\int_E \phi^q \, d\mu \geq V(R+) - V(r)$$

and

$$\begin{aligned}\int_E g_\phi^p \, d\mu &\leq \int_{E \setminus E(R)} g_\eta^p \, d\mu + \int_{E(r-\delta+) \setminus E(r'+\delta)} g_\psi^p \, d\mu \\ &< \epsilon + (r - r' - 2\delta)^{-p}(V(r - \delta+) - V(r' + \delta)).\end{aligned}$$

Consequently,

$$(V(R+) - V(r))^{p/q} < \epsilon + (r - r' - 2\delta)^{-p}(V(r - \delta+) - V(r' + \delta)).$$

Now $\epsilon, \delta > 0$ are arbitrary (subject to $\delta < r - r'$), so we may take a limit as they both tend to zero to obtain

$$(V(R+) - V(r))^{p/q} \leq (r - r')^{-p}(V(r) - V(r'+)). \tag{2.10}$$

Letting R tend to infinity, we deduce that E is of finite measure.

Finishing the proof in the parabolic case is easier if we assume that $V(r) = V(r+)$ for all $r > 0$ (so that V is continuous). Let us temporarily add this assumption, allowing us to find an increasing sequence of radii r_j, $j \in \mathbb{N}$, such that $V(\infty) - V(r_j) = 2^{-j}V(\infty)$. Applying (2.10) with R, r, r' replaced by r_{j+1}, r_j, r_{j-1}, we see that

$$\begin{aligned}\operatorname{diam}(E) &\leq 2r_0 + 2\sum_{j=1}^{\infty} |r_j - r_{j-1}| \\ &\leq 2r_0 + 2^{1+1/q} \sum_{j=1}^{\infty} (2^{-j}\mu(E))^{1/p-1/q} \\ &\leq 2r_0 + C\mu(E)^{(q-p)/qp},\end{aligned} \tag{2.11}$$

where $C = C(p, q)$.

Let us now modify the above argument for the case where we allow $V(r+)$ to be larger than $V(r)$. Inductively, we pick r_j, $j \in \mathbb{N}$, to be the least number $r \geq r_{j-1}$ for which

$$V(\infty) - V(r+) \leq 2^{-1}(V(\infty) - V(r_{j-1}+)) \leq V(\infty) - V(r).$$

Using first the left-hand inequality and then the right-hand one, we get

$$\begin{aligned}2(V(r_{j+1}+) - V(r_j)) &= 2(V(\infty) - V(r_j)) - 2(V(\infty) - V(r_{j+1}+)) \\ &\geq V(\infty) - V(r_j) \\ &\geq (V(\infty) - V(r_{j-1}+)) - (V(\infty) - V(r_j)) \\ &= V(r_j) - V(r_{j-1}+). \end{aligned} \tag{2.12}$$

Also

$$V(\infty) - V(r_j+) \leq 2^{-j}\mu(E),$$

so (2.10) tells us that

$$|r_j - r_{j-1}| \leq 2^{1/q}(V(r_j) - V(r_{j-1}+))^{1/p-1/q} \leq 2^{1/q}\left(2^{-j}\mu(E)\right)^{1/p-1/q}.$$

We can now deduce (2.11) as before.

We next consider the nonparabolic case. As in the proof of Theorem 0.1, it follows that $\mu(E) = \infty$. Let $0 < \epsilon < 1$ be arbitrary but fixed and let $\eta(x) = \operatorname{trunc}(2(|x| - r_0 - \frac{1}{2});\ 0, 1)$. For fixed but arbitrary $r > r_0 + 1$ and $0 < \delta < r$, let $\psi(x) = \operatorname{trunc}((2r - |x|)/(r - \delta);\ 0, 1)$ and let $\phi = \psi\eta$. Then

$$\int_E \phi^q \, d\mu \geq V(r+) - V(r_0 + 1)$$

and

$$\int_{E \setminus E(r)} g_\phi^p \, d\mu \leq (r - \delta)^{-p}(V(2r+) - V(r + \delta)), \tag{2.13}$$

$$\int_{E(r)} g_\phi^p \, d\mu \leq 2^p V(r_0 + 1+). \tag{2.14}$$

Thus

$$(V(r+) - V(r_0 + 1))^{p/q} - 2^p V(r_0 + 1+) \leq (r - \delta)^{-p}(V(2r+) - V(r + \delta))$$

Assuming r is large enough that $(V(r) - V(r_0+1))^{q/p} > 2^{p+1}V(r_0+1+)$ and letting δ tend to zero, we deduce that

$$V(2r+) - V(r+) \geq \frac{r^p}{2}(V(r+) - V(r_0 + 1))^{p/q}, \qquad r > r_1, \tag{2.15}$$

where $r_1 = r_1(E, q, p) \geq r_0 + 2$.

Writing $f(r) = V(r+) - V(r_0 + 1)$, we next show that

$$f(r) \geq c r^{pq/(q-p)}, \tag{2.16}$$

for all $r \geq r_0 + 2$ and some $c = c(E, p, q) > 0$. Trivially such an estimate holds for $r_0 + 2 \leq r \leq r_1$, and it suffices to prove it for $r = r_j$, $j \in \mathbb{N}$, where $r_j = 2^{j-1}r_1$ for all $j > 1$.

To get the inductive process off the ground, we assume that $c > 0$ has been chosen so that (2.16) holds for $j = 1$. Without loss of generality, we assume that $c \le c_0 \equiv 2^{-pq^2/(q-p)^2 - q/(q-p)}$. This last inequality is assumed to ensure that the following useful inequality holds:

$$2^{-1-pq/(q-p)} c^{p/q} \ge c. \tag{2.17}$$

Assuming that (2.16) holds for $r = r_k$, we wish to prove that (2.16) holds for $r = r_{k+1} = 2r_k$. Applying (2.15), we see that

$$f(2r_k) - f(r_k) \ge \frac{r_k^p}{2} f(r_k)^{p/q}$$

and so using (2.17), we deduce that

$$\begin{aligned} f(2r_k) &\ge c r_k^{pq/(q-p)} + \frac{r_k^p \cdot c^{p/q} r_k^{p^2/(q-p)}}{2} \\ &\ge (2r_k)^{pq/(q-p)} \cdot 2^{-pq/(q-p)} \left(c + 2^{-1} c^{p/q} \right) \\ &> c(2r_k)^{pq/(q-p)}. \end{aligned}$$

This finishes the proof of (2.16) for all $r \ge r_0 + 2$. However, note that we have allowed c to depend on E, which is not allowed in the statement of the theorem. Let us examine the above induction argument carefully. Assume that (2.16) holds for $r = r_k$ with $c = c_k \equiv \epsilon_k c_0$, where $0 < \epsilon_k < 1$. We replace (2.17) by the exact equation

$$2^{-1-pq/(q-p)} c_k^{p/q} = \epsilon_k^{p/q-1} c_k \tag{2.18}$$

allowing us to improve the inductive estimate to get

$$f(2r_k) > \epsilon_k^{p/q} c_0 (2r_k)^{pq/(q-p)}.$$

Thus we get (2.16) for $r = r_{k+1}$ with $c = c_{k+1} = \epsilon_k^{p/q} c_0$. Thus we can take $\epsilon_{k+1} = \epsilon_k^{p/q}$. It follows that $\epsilon_j \ge 1/2$ for $j \ge j_0$, where j_0 depends only on E, p, and q. This implies the desired end-independent rate of volume growth. □

3. Sharpness and Examples

In this section, we prove sharpness of the growth/decay estimates in the previous section. First we give two examples that prove sharpness for Theorem 0.1. By the usual normalization, it suffices to prove sharpness for $\lambda = 1$.

Example 3.1. Let (X, d, μ) consist of the real line equipped with Euclidean distance and the measure $d\mu(x) = \exp(px)\, dx$. Let $o = 0$, $r_0 = 0$, and $E = (0, \infty)$.

Suppose $\phi \in \mathrm{Lip}_c(E)$. Using the Fundamental Theorem of Calculus, Hölder's inequality, and Fubini's theorem, we see that

$$\begin{aligned}
\int_0^\infty \phi^p(x) e^{px}\,dx &\le \int_0^\infty \left(\int_x^\infty g_\phi(y)\,dy\right)^p e^{px}\,dx \\
&\le \int_0^\infty \left(\int_x^\infty g_\phi^p(y) e^{(p-1)y}\,dy\right)\cdot\left(\int_x^\infty e^{-y}\,dy\right)^{p-1} e^{px}\,dx \\
&= \int_0^\infty \left(\int_x^\infty g_\phi^p(y) e^{(p-1)y}\,dy\right) e^{x}\,dx \\
&\le \int_0^\infty g_\phi^p(y) e^{(p-1)y}\left(\int_0^y e^{x}\,dx\right) dy \\
&\le \int_0^\infty g_\phi^p(y) e^{py}\,dy.
\end{aligned}$$

Thus E supports a $(p,p;1)$ Sobolev inequality, and it clearly implies sharpness of the volume growth rate in Theorem 0.1. The full space X also satisfies a $(p,p;1)$-Sobolev inequality, as can be seen by making a few minor modifications to the above estimates.

Example 3.2. Let (X,d,μ) be as before. Let $o = 0$, $r_0 = 0$, and $E = (-\infty, 0)$. By a similar calculation to that of Example 3.1, we see that E supports a $(p,p;1)$-Sobolev inequality. It clearly implies sharpness of the volume decay rate in Theorem 0.1.

Example 3.3. If we take X to be the real interval $(-\infty, 0]$ with the Euclidean metric and measure $d\mu(x) = \exp(px)\,dx$, then as above $E = (-\infty, 0)$ is an end satisfying a $(p,p;1)$-Sobolev inequality. However, Remark 2.9 tells us that X itself does not satisfy a (p,p)-Sobolev inequality. This is also easy to see directly—simply take the test function $\phi_R(x) = \mathrm{trunc}(R - |x|;\ 0, 1)$, and let R tend to infinity.

Example 3.4. For the reader who is more interested in Riemannian manifolds, the previous examples can readily be modified to give examples of that type. For instance the space in Example 3.1 is closely associated with the Riemannian manifold $M = \mathbb{R} \times N$, where N is any compact Riemannian manifold and the metric on M is the warped product metric

$$ds^2 = dt^2 + \exp(2pt) ds_N^2$$

Let us choose $o = (0, o_N) \in M$, $r_0 > \mathrm{diam}(N)$, and pick E to be the infinite volume end of $M \setminus \overline{B}(r_0)$. Then E satisfies a $(p,p;1)$-Sobolev inequality since, if

$\phi \in \mathrm{Lip}_c(M)$ is zero outside E, then

$$\begin{aligned}
\int_M \phi^p &= \int_0^\infty \int_N \phi^p(t,n)\, dA_N(n)\, e^{pt}\, dt \\
&\le \int_0^\infty \int_N \left(\int_t^\infty g_\phi(u,n)\, du \right)^p dA_N(n)\, e^{pt}\, dt \\
&\le \int_0^\infty \int_N \left(\int_t^\infty g_\phi^p(u,n) e^{(p-1)u}\, du \right) \cdot \left(\int_t^\infty e^{-u}\, du \right)^{p-1} dA_N(n)\, e^{pt}\, dt \\
&\le \int_0^\infty \int_N \left(\int_x^\infty g_\phi^p(u,n) e^{(p-1)u}\, du \right) dA_N(n)\, e^{t}\, dt \\
&\le \int_0^\infty \int_N g_\phi^p(u,n) e^{(p-1)u} \left(\int_0^u e^t\, dt \right) dA_N(n)\, du \\
&\le \int_0^\infty \int_N g_\phi^p(u,n) e^{pu}\, dA_N(n)\, du = \int_M g_\phi^p,
\end{aligned}$$

where dA_N denotes area measure on N. Note that if N is the unit circle, then E has sharp volume growth.

Example 3.5. The growth exponent $qp/(q-p)$ for the volume in Theorem 0.2 is optimal. This can be seen whenever $n = pq/(q-p)$ happens to be an integer by considering $\mathbb{R}^n$ with the Euclidean metric and Lebesgue measure. For non-integer values of n, the exponent is still optimal, as can be seen by considering an Ahlfors n-regular metric space that supports (a $(1,p)$-Poincaré inequality and) a $(pn/(n-p),p)$-Sobolev inequality when $p < n$. Such spaces exist for all $n \ge 1$; see [La]. By considering suitable open, connected subsets of such spaces one can also see that the exponent $(q-p)/qp$ for the diameter of bounded components is optimal.

We mentioned in the introduction that we do not require the metric measure space to be doubling or to satisfy a Poincaré inequality. While that is formally obvious, let us give an example of a space that fails to satisfy these last two conditions but which satisfies a (p,p)-Sobolev inequality.

Example 3.6. Consider a "binary tree with a tail". Specifically, the space X consists of a tree all of whose nodes are connected with their offspring via line segments of length 1. The root node is a single node o at level -1, which is connected with a single node at level 0. For $j \in \mathbb{Z}$, $j \ge 0$, there are 2^j nodes at level j, and each has two offspring nodes. We equip X with the length metric and the measure $d\mu(x) = e^{p|x|}dx$, where dx denotes the length measure (Hausdorff 1-measure) on X, and $|x| = |x - o|$ as always.

Each node x of T that is at a non-negative level j has a unique mother node $M(x)$ at level $j-1$. Whenever x is a node, we define the branch rooted at x, $T(x)$, to be the set consisting of x, all of its descendents, and all connected line segments. We denote by $[x,y]$ the closed line segment from a node x to a daughter node y.

Let E be either all of X or the end $X \setminus \{o\}$. Then E is not doubling since $V(r+1) > e^p V(r)$ whenever $r \in \mathbb{N}$, $r > 0$. The fact that E supports a (p,p)-Sobolev inequality can be verified as in the previous examples; we leave the details to the reader. However, the following weak $(1, p)$-Poincaré inequality is false:

$$\inf_{a\in\mathbb{R}} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |\phi - a| \, d\mu \le C_0 r \left(\frac{1}{\mu(B(x,r))} \int_{B(x,2r)} g_\phi^p \, d\mu \right)^{1/p}, \qquad \phi \in \mathrm{Lip}(B(x,r)),\ B(x,2r) \subset E. \tag{3.7}$$

In fact, it is easy to verify that the following sequence of test functions $(\phi_j)_{j=2}^\infty$ disprove (3.7):

$$\phi_j(x) = \begin{cases} 1, & x \in T(x_j'), \\ -1, & x \in T(x_j''), \\ |x - x_j|, & x \in [x_j, x_j'], \\ -|x - x_j|, & x \in [x_j, x_j''], \\ 0, & \text{otherwise}, \end{cases}$$

where x_j is any fixed node at level $2j$, and x_j', x_j'' are its daughters.

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