An Integral Formula for the Maslov-Index of a Symplectic Path

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Abstract

The Maslov index of a not necessarily closed path M in the symplectic group $\mathrm{Sp}(2n)$ is expressed by an integral formula. We have an explicit formula for the integrand which is a rational 1-form on $\mathrm{Sp}(2n)$.

Keywords: Symplectic Group, Lagrangian Grassmannian, Maslov Index

1 Introduction

The homotopy class of a loop $\Lambda(t)$ in the Lagrangian Grassmannian $\mathbf{L}(2n)$ in standard symplectic space $\mathbb{R}^{2n}$ is detected by an integer, the *Maslov index*. Following Arnold, [1], this can be defined as an intersection number with a certain subset of $\mathbf{L}(2n)$, the Maslov cycle. In [6] this definition of the Maslov index was extended to not necessarily closed paths, and it is this index which enters in the Gutzwiller trace formula in semiclassical quantisation, [2], see also [4].

The group $\mathrm{Sp}(2n)$ of symplectic automorphisms of $\mathbb{R}^{2n}$ acts transitively on the set of Lagrangians and thus $\mathbf{L}(2n) = \mathrm{Sp}(2n)/\mathrm{H}(2n)$ is a homogeneous space. In applications the path $\Lambda(t)$ is frequently given in the form $M(t)\Lambda_0$ with a curve $M(t) \in \mathrm{Sp}(2n)$ of symplectic automorphisms and a fixed Lagrangian Λ_0. In view of calculations of the Maslov index as in [3] it seems desirable to compute the Maslov index $\mu(\Lambda) = \mu(M)$ directly in terms of M.

In the present note we derive such a formula. We show that the Maslov index is the line integral over a certain differential form $\chi_{\mathrm{Sp}(2n)} \in \Omega^1(\mathrm{Sp}(2n), \mathbb{R})$ plus end point terms which are also given by an explicit function Φ on $\mathrm{Sp}(2n)$.

For unitary paths, i.e. paths $M(t) \in \mathrm{U}(n) \subset \mathrm{Sp}(2n)$ in the unitary group the (complex) Trace $\mathrm{Tr}_{\mathbb{C}}$ on the Lie algebra $\mathfrak{u}(n)$ is an invariant for the adjoint representation, and therefore provides a bi-invariant 1-form on $\mathrm{U}(n)$. On the symplectic group $\mathrm{Sp}(2n)$ however a nontrivial biinvariant 1-form does not exist. The 1-form $\chi_{\mathrm{Sp}(2n)}$ and the function Φ we construct are invariant under the left action of Sp$(2n)$ and under the right action of the stabilizer subgroup H(n) which contains O(n).

We formulate our main result in the next section 2. Then, in section 3, we compute the 1-form $\chi_{\mathrm{Sp}(2n)}$ explicitely in terms of the entries of the symplectic matrix. Finally we prove our integral formula by verifying the axioms for the Maslov index in section 4.

2 A local formula for the Maslov index

We need to introduce some notation and recall some well known facts from symplectic geometry as may be found in [5], for instance. Let Λ_0 be a Lagrangian in $\mathbb{R}^{2n}$ and $\Lambda_0^\perp$ the orthogonal complement with respect to the standard scalar product $\langle\cdot|\cdot\rangle$. The symplectic structure ω on $\mathbb{R}^{2n}$ can be written as

$$\omega(x,y) = \langle x|\, Jy\rangle$$

where J is a complex structure, i.e. $J \in \mathrm{O}(2n)$ with $J^2 = -\mathbf{1}_{2n}$ and we have $J\Lambda_0 = \Lambda_0^\perp$.

Usually we choose coordinates such that $\Lambda_0 = \mathbb{R}^n \times \{0\}$, $\Lambda_0^\perp = \{0\} \times \mathbb{R}^n$ and so that the complex structure J becomes

$$J = \begin{pmatrix} 0 & -\mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix} \quad , \quad \mathbf{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{Mat}(n \times n, \mathbb{R}) \; . \tag{1}$$

With respect to the orthogonal decomposition

$$\mathbb{R}^{2n} = \Lambda_0 \oplus \Lambda_0^\perp \tag{2}$$

we can write a (symplectic) matrix M as

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3}$$

where

$$a \in \mathrm{Hom}(\Lambda_0, \Lambda_0),\; b \in \mathrm{Hom}(\Lambda_0^\perp, \Lambda_0),\; c \in \mathrm{Hom}(\Lambda_0, \Lambda_0^\perp),\; d \in \mathrm{Hom}(\Lambda_0^\perp, \Lambda_0^\perp) \; .$$

All these will be identified with $\mathrm{Mat}(n \times n; \mathbb{R})$.

By means of the complex structure (1) we can read a complex $(n\times n)$-matrix Y as a real $(2n \times 2n)$-matrix $\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$. The traces $\mathrm{Tr}_{\mathbb{C}}$ over $\mathbb{C}$ respectively $\mathrm{Tr} = \mathrm{Tr}_{\mathbb{R}}$ over $\mathbb{R}$ are related by

$$\mathrm{Tr}_{\mathbb{C}}(Y) = \frac{1}{2}\,\mathrm{Tr}(Y) - \frac{i}{2}\,\mathrm{Tr}(JY) = \mathrm{Tr}(\alpha) + i\,\mathrm{Tr}(\beta) \; . \tag{4}$$

The group $\mathrm{Sp}(2n)$ of symplectic automorphisms is

$$\mathrm{Sp}(2n) = \{M \in \mathrm{Gl}(2n) \mid M^t J M = J\}$$

and acts transitively on the Lagrangian Grassmannian $\mathbf{L}(2n)$. The stabilizer group, i.e. the isotropy group of this action at $\Lambda_0 = \mathbb{R}^n \times \{0\}$ is

$$\mathrm{H}(n) = \left\{ \begin{pmatrix} A & X \\ 0 & (A^t)^{-1} \end{pmatrix} \mid A \in \mathrm{Gl}(n) \ , \ AX \in \mathrm{Mat}(n \times n, \mathbb{R}) \text{ symmetric} \right\}$$

We will also need the subgroup $\mathrm{H}^+(n)$ consisting of those matrices in $\mathrm{H}(n)$ with A upper triangular and positive. The unitary group $\mathrm{U}(n)$ and the orthogonal group $\mathrm{O}(n)$ are naturally identified as subgroups of $\mathrm{Sp}(2n)$,

$$\begin{aligned} \mathrm{U}(n) &= \{M \in \mathrm{Sp}(2n) \mid MJ = JM\} = \mathrm{Sp}(2n) \cap \mathrm{O}(2n) \\ \mathrm{O}(n) &= \{M \in \mathrm{Sp}(2n) \mid MJ = JM \text{ and } M\Lambda_0 = \Lambda_0\} \\ &= \left\{ \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} \mid T \in \mathrm{O}(n) \right\} = \mathrm{H}(n) \cap \mathrm{O}(2n) \subset \mathrm{U}(n) \ . \end{aligned} \tag{5}$$

With these identifications we can represent the Lagrangian Grassmannian as homogeneous space,

$$\mathbf{L}(2n) = \mathrm{Sp}(2n)/\,\mathrm{H}(n) = \mathrm{U}(n)/\,\mathrm{O}(n) \ . \tag{6}$$

The square of the (complex) determinant

$$\mathbf{L}(2n) = \mathrm{U}(n)/\,\mathrm{O}(n) \to \mathrm{U}(1),\ F \mapsto \det_{\mathbb{C}}(F)^2 \tag{7}$$

induces an isomorphism of the fundamental groups

$$\pi_1(\mathbf{L}(2n)) \cong \pi_1(\mathrm{U}(1)) = \mathbb{Z} \ .$$

The Jordan-, or QR- decomposition, of the symplectic group is

$$\mathrm{Sp}(2n) = \mathrm{U}(n) \cdot \mathrm{H}^+(n)$$

and

$$\mathrm{H}^+(n) = \mathrm{D}^+(n) \ltimes \mathrm{Sym}(n)$$

is the semidirect product of the group of positive upper triangular matrices with the vector group of symmetric matrices. In particular, we have a diffeomorphism and a homotopy equivalence, both equivariant under the right action of $\mathrm{H}^+(n)$,

$$\mathcal{F}\colon\ \mathrm{Sp}(2n) \cong \mathrm{U}(n) \times \mathrm{H}^+(n) \xrightarrow{\simeq} \mathrm{U}(n) \ . \tag{8}$$

We will also need the map

$$\mathcal{H}\colon\ \mathrm{Sp}(2n) \cong \mathrm{U}(n) \times \mathrm{H}^+(n) \xrightarrow{\simeq} \mathrm{H}^+(n) \ . \tag{9}$$

These are defined by first decomposing a given $M \in \mathrm{Sp}(2n)$, $M = FH$ with $F \in \mathrm{U}(n)$, $H \in \mathrm{H}^+(n)$ uniquely determined by M. This can be done efficiently by any of the algorithms for the QR-decomposition. Then we set $\mathcal{F}(M) := F$ and $\mathcal{H}(M) := H$

For the contribution of end points we need a function

$$\Phi\colon\ \mathrm{Sp}(2n) \to \mathbb{R}\ .$$

which is defined as follows: $\Phi(M)$ is computed from the unitary part

$$\mathcal{F}(M) = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \tag{10}$$

alone. Since $F^tF = \mathbf{1}_{2n} = FF^t$ we have

$$\alpha^t\alpha + \beta^t\beta = \mathbf{1}_n,\ \ \alpha^t\beta = \beta^t\alpha,$$

$$\alpha\alpha^t + \beta\beta^t = \mathbf{1}_n,\ \ \alpha\beta^t = \beta\alpha^t.$$

From these relations it follows that $\alpha^t\alpha$, $\beta^t\beta$ and $\alpha^t\beta$ commute and can therefore be simultaneously diagonalized. Since $\mathrm{Spec}(\beta^t\beta) \subset [0, 1]$ we may set

$$X := \arcsin\left((\beta^t\beta)^{1/2}\right) \in \mathrm{Mat}(n \times n; \mathbb{R})\ . \tag{11}$$

Note that X is nonnegative symmetric and $\mathrm{Spec}\, X \subset [0, \frac{\pi}{2}]$. Now define $T_\alpha, T_\beta \in \mathrm{Mat}(n \times n; \mathbb{R})$ as

$$T_\beta = \beta \sin(X)^{-1}\, \mathrm{pr}_{(\ker X)^\perp}\ ,\ T_\alpha = \alpha \cos(X)^{-1}\, \mathrm{pr}_{\ker\left(X - \frac{\pi}{2}\mathbf{1}_n\right)^\perp} \tag{12}$$

where pr denotes the orthogonal projection on the subspace indicated. We then have

$$\beta = T_\beta \sin(X),\ \alpha = T_\alpha \cos(X),\ T_\beta^t T_\beta = \mathrm{pr}_{(\ker X)^\perp},\ T_\alpha^t T_\alpha = \mathrm{pr}_{\ker\left(X - \frac{\pi}{2}\mathbf{1}_n\right)^\perp}$$

We define

$$\Phi(M) := \frac{1}{\pi}\, \mathrm{Tr}\left(T_\alpha^t T_\beta \left(X - \frac{\pi}{2}\mathbf{1}_n\right)\right)\ . \tag{13}$$

This is to arrange for $\mathbf{d}\,\Phi = \chi_{\mathrm{Sp}(2n)}$ on the complement of the Maslov cycle (see Lemma 21 and (23) in the proof of that Lemma).

We can now write down an integral formula for the Maslov index:

Theorem *Let $M\colon [0, 1] \to \mathrm{Sp}(2n)$ be a differentiable path in the symplectic group of $\mathbb{R}^{2n}$ and let $\Lambda_0 \subset \mathbb{R}^{2n}$ be a fixed Lagrangian subspace. Then the Maslov index $\mu(\Lambda) \in \frac{1}{2}\mathbb{Z}$ of the path $\Lambda\colon [0, 1] \to \mathbf{L}(2n)$, $\Lambda(t) = M(t)\Lambda_0$, in the Lagrangian-Grassmannian of $\mathbb{R}^{2n}$ is given by*

$$\mu(M) = \mu(\Lambda) = \int_M \chi_{\mathrm{Sp}(2n)} - \Phi(M(1)) + \Phi(M(0)) \tag{14}$$

where Φ *is as defined in* (13) *and*

$$\chi_{\mathrm{Sp}(2n)} = -\frac{1}{2\pi} \operatorname{Tr}\left(JM^t \,\mathbf{d}\, M\right) - \operatorname{Tr}\left(J\,\mathcal{H}(M)^t \,\mathbf{d}(\mathcal{H}(M))\right)$$

This form is explicitely computed in (17) *in terms of the coefficients of* M*.*

Recall that the stabilizer part $H = \mathcal{H}(M) \in \mathrm{H}^+(n) \subset \mathrm{Mat}(2n \times 2n)$ is uniquely determined from M by the following conditions (see (9)):

1. $M^t M = H^t H$),
2. $H = \begin{pmatrix} A & AX \\ 0 & (A^t)^{-1} \end{pmatrix}$ w.r.t. the splitting $\mathbb{R}^{2n} = \Lambda_0 \oplus \Lambda_0^\perp$,
3. $A \in \mathrm{Mat}(n \times n, \mathbb{R})$ is upper triangular with positive eigenvalues,
4. $X \in \mathrm{Mat}(n \times n, \mathbb{R})$ is symmetric.

3 The 1-form on the symplectic group

We will now derive explicit formulas for the integrand in (14). The form $\chi = \frac{1}{2\pi i} z^{-1} \,\mathbf{d}\, z$ on $U(1) = S^1 \subset \mathbb{C}$ detects the homotopy type of a loop in U(1). To find the right 1-form for the Maslov index we first pull back χ to $\mathrm{U}(n)$ via the map $\det^2_{\mathbb{C}}$ in (7). Sine $\mathbf{d}(\det_{\mathbb{C}} F) = \det_{\mathbb{C}}(F) \operatorname{Tr}_{\mathbb{C}}(F^{-1} \,\mathbf{d}\, F)$ we get

$$\chi_{\mathrm{U}(n)} = \left(\det{}^2_{\mathbb{C}}\right)^* \chi = \frac{1}{\pi i} \operatorname{Tr}_{\mathbb{C}} F^{-1} \,\mathbf{d}\, F \in \Omega^1(\mathrm{U}(n)) \ .$$

This already gives a formula, similiar to that for the winding number, for the Maslov index of a *unitary* path $F(t)\Lambda_0$, $F\colon [0,1] \to \mathrm{U}(n)$,

$$\mu(F) = \frac{1}{\pi i} \int \operatorname{Tr}_{\mathbb{C}}(F^{-1} \,\mathbf{d}\, F) - \Phi(F(1)) + \Phi(F(0))$$

The 1-form on $\mathrm{Sp}(2n)$ we are looking for is the pull back of $\chi_{\mathrm{U}(n)}$ over the map $\mathcal{F}$ in (8). We compute this explicitely.

Lemma 15 *Denote by* $a, b, c, d \in \mathrm{Mat}(n \times n, \mathbb{R})$ *the components of a symplectic matrix* M*, i.e the matrices such that*

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{16}$$

with respect to the decomposition (2)*. Then*

$$\begin{aligned} \chi_{\mathrm{Sp}(2n)} = F^* \chi_{\mathrm{U}(n)} &= \frac{1}{\pi i} \operatorname{Tr}_{\mathbb{C}}\left(\mathcal{F}(M)^{-1} \,\mathbf{d}\, \mathcal{F}(M)\right) \\ &= \frac{-1}{2\pi} \operatorname{Tr}\left(JM^t \,\mathbf{d}\, M - JH^t \,\mathbf{d}\, H\right) \\ &= \frac{-1}{2\pi} \operatorname{Tr}\big[-b^t \,\mathbf{d}\, a - d^t \,\mathbf{d}\, c + a^t \,\mathbf{d}\, b + c^t \,\mathbf{d}\, d \\ &\qquad -(a^t a + c^t c) \,\mathbf{d}\left[(a^t a + c^t c)^{-1}(a^t b + c^t d)\right]\big] \end{aligned} \tag{17}$$

Proof: Let $M \in \mathrm{Sp}(2n)$ and $F = \mathcal{F}(M) = MH^{-1} \in \mathrm{U}(n)$ with $H \in \mathrm{H}^+(n)$. We get

$$F^{-1}\,\mathbf{d}\,F = HM^{-1}(\mathbf{d}\,M)H^{-1} + HM^{-1}M\,\mathbf{d}(H^{-1}) \in \Omega^1(\mathrm{Sp}(2n), \mathfrak{u}(n)) \ .$$

Because of (4) and since $H^tH = M^tM$,

$$\begin{aligned}
\mathrm{Tr}_{\mathbb{C}}\, F^{-1}\,\mathbf{d}\,F &= \frac{1}{2i}\,\mathrm{Tr}\, JF^{-1}dF = \frac{1}{2i}\,\mathrm{Tr}\left(JHM^{-1}(\mathbf{d}\,M)H^{-1} + JHM^{-1}M\,\mathbf{d}\left(H^{-1}\right)\right) \\
&= \frac{1}{2i}\,\mathrm{Tr}\left(H^{-1}JHM^{-1}\,\mathbf{d}\,M - JHH^{-1}(\mathbf{d}\,H)H^{-1}\right) \\
&= \frac{1}{2i}\,\mathrm{Tr}\left(JH^tHM^{-1}\,\mathbf{d}\,M - J(\mathbf{d}\,H)H^{-1}\right) \\
&= \frac{1}{2i}\,\mathrm{Tr}\left(JM^tMM^{-1}\,\mathbf{d}\,M + J(\mathbf{d}\,H)JH^tJ\right) \\
&= \frac{1}{2i}\,\mathrm{Tr}\left(JM^t\,\mathbf{d}\,M - JH^t\,\mathbf{d}\,H\right)
\end{aligned} \tag{18}$$

In view of (16) we can simplify this further. First

$$\begin{aligned}
\mathrm{Tr}(JM^t\,\mathbf{d}\,M) &= \mathrm{Tr}\begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}\begin{pmatrix} a^t & c^t \\ b^t & d^t \end{pmatrix}\begin{pmatrix} \mathbf{d}\,a & \mathbf{d}\,b \\ \mathbf{d}\,c & \mathbf{d}\,d \end{pmatrix} \\
&= \mathrm{Tr}(-b^t\,\mathbf{d}\,a - d^t\,\mathbf{d}\,c + a^t\,\mathbf{d}\,b + c^t\,\mathbf{d}\,d)
\end{aligned}$$

For the second summand in (18) let $H = \begin{pmatrix} A & AX \\ 0 & (A^t)^{-1} \end{pmatrix} \in \mathrm{H}^+(n)$. We compute

$$\begin{aligned}
\mathrm{Tr}\left(JH^t\,\mathbf{d}\,H\right) &= \mathrm{Tr}\begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}\begin{pmatrix} A^t & 0 \\ X^tA^t & A^{-1} \end{pmatrix}\begin{pmatrix} \mathbf{d}\,A & \mathbf{d}(AX) \\ 0 & \mathbf{d}\left((A^t)^{-1}\right) \end{pmatrix} \\
&= \mathrm{Tr}(-X^tA^t\,\mathbf{d}\,A + A^t\,\mathbf{d}(AX)) \\
&= \mathrm{Tr}(A^tA\,\mathbf{d}\,X)
\end{aligned}$$

since X is symmetric. As $H^tH = M^tM$ we have $A^tA = a^ta + c^tc$ and $A^tAX = a^tb + c^td$ which yields

$$\mathrm{Tr}\left(JH^t\,\mathbf{d}\,H\right) = \mathrm{Tr}\left[(a^ta + c^tc)\ \mathbf{d}\left[(a^ta + c^tc)^{-1}(a^tb + c^td)\right]\right]$$

■

4 The axiomatic characterization of the Maslov index

For the proof of the Theorem, denote by $\widetilde{\mu}(M)$ the right hand side of (14) for a path $M(t) \in \mathrm{Sp}(2n)$. In order to show that $\widetilde{\mu}$ coincides with the Maslov index, we will check that $\widetilde{\mu}$ satisfies the five properties which were shown to characterize the Maslov index by Theorem 4.1 in [6].

We need two Lemmas. The first will permit us to move the path by paths in $\mathrm{O}(n)$.

Lemma 19 *The form* $\chi_{\mathrm{Sp}(2n)} \in \Omega^1(\mathrm{Sp}(2n))$ *from* (17) *and the function* Φ *of* (13) *are left-*$\mathrm{O}(n)$ *and right-*$\mathrm{H}(n)$*-invariant. Thus if* $M(t) \in \mathrm{Sp}(2n)$ *and* $T(t) \in \mathrm{O}(n)$, $H(t) \in \mathrm{H}(n)$ *are smooth paths, then*

$$\widetilde{\mu}(M) = \widetilde{\mu}(TMH) \ .$$

Proof: Let

$$M = FH^+ \quad \text{and} \quad H = T_1 H_1^+ \quad \text{with} \quad F \in \mathrm{U}(n),\ H^+, H_1^+ \in \mathrm{H}^+(n),\ T_1 \in \mathrm{O}(n)\ .$$

Then

$$\mathcal{F}(TMH) = \mathcal{F}(TFH^+T_1H_1^+) = \mathcal{F}(TFT_2H_2^+) = TFT_2$$

with some $T_2 = \mathcal{F}(H^+T_1H_1^+) \in \mathrm{O}(n)$ and some $H_2^+ \in \mathrm{H}^+(n)$, since

$$\mathrm{H}(n) = \mathrm{O}(n) \cdot \mathrm{H}^+(n) = \mathrm{H}^+(n) \cdot \mathrm{O}(n). \tag{20}$$

Now

$$\begin{aligned}
\mathrm{Tr}_{\mathbb{C}}\left((TFT_2)^{-1}\,\mathbf{d}(TFT_2)\right) &= \mathrm{Tr}_{\mathbb{C}}\left((TFT_2)^{-1}TF(\mathbf{d}\,T_2)\right) \\
&\quad + \mathrm{Tr}_{\mathbb{C}}\left((TFT_2)^{-1}T(\mathbf{d}\,F)T_2\right) \\
&\quad + \mathrm{Tr}_{\mathbb{C}}\left((TFT_2)^{-1}(\mathbf{d}\,T)FT_2\right) \\
&= \mathrm{Tr}_{\mathbb{C}}(F^{-1}\,\mathbf{d}\,F)
\end{aligned}$$

since $T^{-1}\,\mathbf{d}\,T$ lies in the Lie algebra of $\mathrm{O}(n)$ and therefore $\mathrm{Tr}_{\mathbb{C}}(T^{-1}\,\mathbf{d}\,T) = 0$.

For the invariance of the function Φ of (13) it suffices to prove left and right invariance under $\mathrm{O}(n)$, because of (20) as before. We also may assume $M = F$ unitary since right-$\mathrm{H}^+(n)$-invariance of Φ is clear from its definition. So let

$$F = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

be unitary with real $(n \times n)$-matrices α and β. We then have

$$TF = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} = \begin{pmatrix} T\alpha & -T\beta \\ T\beta & T\alpha \end{pmatrix}$$

From (12) and (11) we see that this replaces the matrices T_α, T_β used to define Φ by TT_α, TT_β while leaving X unchanged. By definition (13) $\Phi(TF) = \Phi(F)$. Similiarly, multiplying T from the right, replaces α, β by αT, βT. Thus, by (12) and (11), T_α, T_β, X are replaced by $T_\alpha T$, $T_\beta T$ and $T^{-1}XT$, since $\sin(T^{-1}XT) = T^{-1}\sin(X)T$. By (13) and the conjugation invariance of the trace, Φ again is unchanged. ∎

By the next Lemma $\widetilde{\mu}$ vanishes on a path $M(t)$ such that $M(t)\Lambda_0$ stays away from the Maslov cycle $\{\Lambda \in \mathbf{L}(2n) \mid \Lambda \cap \Lambda_0 \neq 0\}$. Thus, away from the Maslov cycle, the function Φ is an integral of $\chi_{\mathrm{Sp}(2n)}$.

Lemma 21 *For $M \in \mathrm{Sp}(2n)$ let $\beta(M), \alpha(M) \in \mathrm{Mat}(n \times n; \mathbb{R})$ be the matrices in the unitary part*

$$\mathcal{F}(M) = \begin{pmatrix} \alpha(M) & -\beta(M) \\ \beta(M) & \alpha(M) \end{pmatrix}$$

of M as in (10). *Then on the set $\mathrm{Sp}^\times(2n) = \{M \mid \beta(M)$ invertible $\}$ we have $\mathbf{d}\,\Phi = \chi_{Sp(2n)}$.*

In particular $\widetilde{\mu}$ vanishes on paths staying in $\mathrm{Sp}^\times(2n)$.

The set $Sp^\times(2n)$ is the set of all M such that $M(t)\Lambda_0 \cap \Lambda_0 = 0$. Thus $Sp^\times(2n)$ is the preimage of the complement of the Maslov cycle under the projection $\mathrm{Sp}(2n) \to \mathbf{L}(2n)$.

Proof: It suffices to prove the Lemma on $\mathrm{U}^\times(n) = \mathrm{U}(n) \cap \mathrm{Sp}^\times(2n)$ since both the form $\chi_{\mathrm{Sp}(2n)}$ and Φ are right-$\mathrm{H}^+(n)$ invariant. We will now show that $\int_F \chi_{(2n)} = [\Phi(F(t)]_{t=0}^{t=1}$ for arbitrary paths $F\colon [0,1] \to \mathrm{U}(n)$.

Let $M = F \in \mathrm{U}^\times(n)$. Then the matrices $X(F)$ and $T_\beta(F)$ defined as in (11) and (12) are invertible and depend smoothly on F. Let

$$X_s := (1-s)\frac{\pi}{2}\mathbf{1}_n + sX(F) \quad \text{for} \quad s \in [0,1] \quad \text{and}$$

$$\alpha_s = T_{\alpha(F)} \cos(X_s) \quad , \quad \beta_s = T_{\beta(F)} \sin(X_s) \ .$$

The path

$$F_s = \begin{pmatrix} \alpha_s & -\beta_s \\ \beta_s & \alpha_s \end{pmatrix} \tag{22}$$

has $F_1 = F = M$ and

$$F_0 = \begin{pmatrix} 0 & -T_{\beta(M)} \\ T_{\beta(F)} & 0 \end{pmatrix} = J \begin{pmatrix} T_{\beta(F)} & 0 \\ 0 & T_{\beta(F)} \end{pmatrix} = JT_{\beta(F)} \in J\,\mathrm{O}(n) \ .$$

Now by definition Φ vanishes on $J\,\mathrm{O}(n)$, so $\Phi(F_0) = 0$. We integrate

$$\begin{aligned}
\int_{F_s} \chi_{Sp(2n)} &= \frac{1}{\pi}\int_{F_s} \mathrm{Tr}\left(\alpha_s^t\,\mathbf{d}\,\beta_s - \beta_s^t\,\mathbf{d}\,\alpha_s\right) \\
&= \frac{1}{\pi}\int_{F_s} \mathrm{Tr}\left(\cos(X_s)T^t_{\alpha(F)}T_{\beta(F)}\cos(X_s)\,\mathbf{d}\,X_s + \sin(X_s)T^t_{\beta(F)}T_{\alpha(F)}\sin(X_s)\,\mathbf{d}\,X_s\right) \\
&= \frac{1}{\pi}\int_{F_s} \mathrm{Tr}\left(T^t_{\alpha(F)}T_{\beta(F)}\,\mathbf{d}\,X_s\right) \\
&= \frac{1}{\pi}\,\mathrm{Tr}\left(T^t_{\alpha(F)}T_{\beta(F)}(X_1 - X_0)\right) \\
&= \frac{1}{\pi}\,\mathrm{Tr}\left(T^t_{\alpha(F)}T_{\beta(F)}\left(X(F) - \frac{\pi}{2}\mathbf{1}_n\right)\right) \\
&= \Phi(F_1) = \Phi(F)
\end{aligned} \tag{23}$$

by definition (13) of Φ. Here we have used that X_s, $\mathbf{d}\,X_s$, $T^t_{\alpha(F)}T_{\beta(F)}$ commute, together with the identity $\cos^2(X_s) + \sin^2(X_s) = \mathbf{1}_n$.

Since F_s defined by (22) depends smoothly on F, we have a deformation retraction of $\mathrm{U}^\times(n)$ (and $\mathrm{Sp}^\times(2n)$) on $J\,\mathrm{O}(n) \cong \mathrm{O}(n)$. Therefore any path $F(t) \in \mathrm{U}^\times(n)$ is homotopic relative end points to the catenation $a * T * b$ of two paths a, b of type (22) and a path T in $J\,\mathrm{O}(n)$. Since $\chi_{Sp(2n)}$ is the pull back of the closed form χ on U(1) it is closed as well and therefore line integrals over $\chi_{Sp(2n)}$ depend on the homotopy class relative end points only. For an arbitrary path F in $\mathrm{U}^\times(n)$ we can therefore compute

$$\int_F \chi_{Sp(2n)} = \int_{a*T*b} \chi_{Sp(2n)} = \int_a \chi_{Sp(2n)} + \int_T \chi_{Sp(2n)} + \int_b \chi_{Sp(2n)} \ .$$

The integral over T vanishes since $(JT)^{-1}\,\mathbf{d}(JT) = T^{-1}\,\mathbf{d}T$ has trace 0. The integrals over a and b have been evaluated in (23). Thus we get that

$$\int_F \chi_{\mathrm{Sp}(2n)} = \Phi(F(1)) - \Phi(F(0))$$

for any path $F(t)$ in $\mathrm{U}^\times(n)$ which proves the Lemma. ∎

We now proceed to verify the axioms from [6] for the Maslov index for $\widetilde{\mu}$.

1. Homotopy: *Two paths in* $\mathrm{Sp}(2n)$ *are homotopic relative end points if and only if the respective values of* $\widetilde{\mu}$ *coincide.*

Since $\chi_{\mathrm{Sp}(2n)} = \left((\det^2_{\mathbb{C}}) \circ F\right)^* \chi$ is closed we have that $\widetilde{\mu}$ is homotopy invariant relative end points. The map $(\det^2_{\mathbb{C}}) \circ F$ induces an injective map $\pi_1(\mathrm{Sp}(2n)) \to \pi_1(\mathrm{U}(1))$ and the isomorphism $\pi_1(U(1)) \cong \mathbb{Z}$ is given by integrating the form χ. For closed loops the end point terms (14) cancel and the claim follows.

2. Catenation: *Let* $M_i(t)$, $i = 0, 1$ *be paths in* $\mathrm{Sp}(2n)$ *with* $M_0(1) = M_1(0)$ *and* $M = M_0 * M_1$ *denote the path with* $M(t) = M_0(2t)$ *for* $t \leq 1/2$ *and* $M_1(2t-1)$ *for* $t \geq 1/2$. *Then* $\widetilde{\mu}(M) = \widetilde{\mu}(M_0) + \widetilde{\mu}(M_1)$.

The contributions of the end points $M_0(1)$ and $M_1(0)$ cancel. The claim thus follows from the additivity of the integral.

3. Product: *If* $M(t) = \begin{pmatrix} M_1(t) & 0 \\ 0 & M_2(t) \end{pmatrix}$ *with* $M_i(t) \in \mathrm{Sp}(2n_i)$, $n_1 + n_2 = n$, *then* $\widetilde{\mu}(M_1) + \widetilde{\mu}(M_2) = \widetilde{\mu}(M)$.

This follows from the analogous property of the trace.

4. Zero: *Let* $M(t)$ *be a path such that* $\dim(M(t)\Lambda_0 \cap \Lambda_0) = k > 0$ *is constant for all* t. *Then* $\widetilde{\mu}(M) = 0$.

By lemma 19 we may replace $M(t)$ by $M(t)H(t)$ with $H(t) \in \mathrm{H}(n)$ without changing $\widetilde{\mu}$. We can thus achieve that that

$$M(t) = \mathcal{F}(M(t)) = \begin{pmatrix} \alpha(t) & -\beta(t) \\ \beta(t) & \alpha(t) \end{pmatrix} \in \mathrm{U}(n)$$

is unitary. We also have $\ker\beta(t) = M(t)\Lambda_0 \cap \Lambda_0$ which is of constant dimension k by our assumption. Also $\alpha(t)$ maps $\ker\beta(t)$ isometrically into Λ_0. Therefore replacing $M(t)$ by $T_1(t)M(t)T_2(t)$ with suitable paths $T_1(t), T_2(t) \in \mathrm{O}(n)$, we may assume that $V = \ker \beta(t)$ is independent of t and that $\alpha(t)$ is the identity on V. Again by Lemma 19, we have $\widetilde{\mu}(M) = \widetilde{\mu}(T_1 M T_2)$.

Let W be the orthogonal complement of V in Λ_0. We then can split

$$\mathbb{R}^{2n} = (V \oplus JV) \oplus (W \oplus JW)$$

as the orthogonal sum of two symplectic subspaces. With respect to this splitting we now have

$$M(t) = \begin{pmatrix} \mathbf{1}_{V\oplus JV} & 0 \\ 0 & Q(t) \end{pmatrix} .$$

The path $Q(t) \in \mathrm{Sp}^\times(W \oplus JW) = \mathrm{Sp}^\times(2(n-k))$ avoids the Maslov cycle. By Lemma 21, $\widetilde{\mu}(M) = \widetilde{\mu}(\mathbf{1}) + \widetilde{\mu}(Q) = 0$.

5. Normalization: *Let* $Y(t) \in \mathrm{Sym}(n)$ *be a path of symmetric matrices and* $M(t) = \begin{pmatrix} \mathbf{1} & 0 \\ Y(t) & \mathbf{1} \end{pmatrix}$ *the corresponding path of symplectic shears. Then*

$$\widetilde{\mu}(M) = [\mathrm{sign}(Y(1)) - \mathrm{sign}(Y(0))]/2 ,$$

where the signature $\mathrm{sign}(Y)$ *is the number of positive eigenvalues of* Y *minus the number of negative eigenvalues.*

To see this, let $T(t) \in \mathrm{O}(n)$ be such that

$$T(t)^{-1} Y(t) T(t) = \begin{pmatrix} \lambda_1(t) & & \\ & \ddots & \\ & & \lambda_n(t) \end{pmatrix}$$

is diagonal. Then

$$T^{-1}(t)M(t)T(t) = \begin{pmatrix} T(t)^{-1} & 0 \\ 0 & T(t)^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ Y(t) & \mathbf{1} \end{pmatrix} \begin{pmatrix} T(t) & 0 \\ 0 & T(t) \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{1} & 0 \\ T(t)^{-1}Y(t)T(t) & \mathbf{1} \end{pmatrix} .$$

By Lemma 19 and the product property,

$$\widetilde{\mu}(M) = \widetilde{\mu}(T^{-1}MT) = \sum_{k=0}^{n} \widetilde{\mu} \begin{pmatrix} 1 & 0 \\ \lambda_k(t) & 1 \end{pmatrix} .$$

It suffices therefore to verify the normalization property in the case $n = 1$, $Y(t) = \lambda(t)$ and $M(t) = \begin{pmatrix} 1 & 0 \\ \lambda(t) & 1 \end{pmatrix}$. The integral in (14), i.e. the integral over the form (17), becomes

$$\int_M \chi_{\mathrm{Sp}(2)} = \frac{-1}{2\pi} \int_M -\mathbf{d}\,\lambda - (1+\lambda^2)\,\mathbf{d}\left((1+\lambda^2)^{-1}\lambda\right) = \frac{[\arctan(\lambda(t))]_{t=0}^{t=1}}{\pi}$$

For the end point term in (14) we compute

$$F = \mathcal{F}\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} = \frac{1}{\sqrt{1+\lambda^2}}\begin{pmatrix} 1 & -\lambda \\ \lambda & 1 \end{pmatrix}$$

which gives

$$X = \arcsin\left(\frac{|\lambda|}{\sqrt{1+\lambda^2}}\right) = \arctan(|\lambda|) \quad , \quad T_\beta = \mathrm{sign}(\lambda) \quad , \quad T_\alpha = 1$$

in the notation of (11) and (12). From (13) we get

$$\Phi(M) = \begin{cases} \frac{1}{\pi}\arctan(\lambda) - \frac{1}{2} & \text{if} \quad \lambda > 0 \\ 0 & \text{if} \quad \lambda = 0 \\ \frac{1}{\pi}\arctan(\lambda) + \frac{1}{2} & \text{if} \quad \lambda < 0 \end{cases} .$$

This yields $\widetilde{\mu}(M) = [\mathrm{sign}(\lambda(1)) - \mathrm{sign}(\lambda(0))]/2$ as required.

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