# Bordism of regularly defective maps

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**Abstract**

To a topological space $V$ we assign the bordism group $\mathfrak{N}_n^{\text{def}}(V)$ of regularly defective maps $f\colon M\circ\!\!\to V$ on closed $n$-dimensional manifolds $M$. These are triples $(M, \Delta, f)$ where $\Delta$ is a closed submanifold $\Delta \subset M$ and $f$ a continuous map $f\colon M \smallsetminus \Delta \to V$.
We briefly review the construction of the defect complex $DV$ given by M. Rost in [17] and show that $\mathfrak{N}_n^{\text{def}}(V)$ is isomorphic to ordinary bordism $\mathfrak{N}_n(DV)$. The bordism classes in $\mathfrak{N}_n^{\text{def}}(V) \cong \mathfrak{N}_n(DV)$ are detected by characteristic numbers twisted with cohomology classes of $DV$. Some of these numbers can be described without reference to the defect complex. As an example we treat the case of the circle $V = S^1$. We compute $\mathfrak{N}_n^{\text{def}}(S^1)$, construct a basis and a complete set of characteristic numbers.

## 1 Introduction

By a regularly defective map we mean a triple $(M, \Delta, f)$ consisting of a compact manifold $M$, a closed submanifold $\Delta \subset M$ and a continuous map $f\colon M \smallsetminus \Delta \to V$ into a topological space $V$. We additionally require that $\Delta$ be transverse to the boundary $\partial M$ of $M$. Usually the defect set $\Delta$ will be suppressed in the notation and we will write $f\colon M\circ\!\!\to V$.

Initially, interest in defective maps arose from the physics of ordered media, where $M$ is thought of as the coordinate space of a collection of particles, e.g. a domain in $\mathbb{R}^3$, cf. [11], [12] or [15]. A map $f\colon M \smallsetminus \Delta \to V$ encodes some additional piece of information like the orientation of the particles. Famous examples are axial or biaxial nematics, superfluid $^3$He, see [1], [7], [10], [13]. Physicists also have considered some invariants distinguishing topologically different defective maps $f$. Probably the simplest ones are obtained by considering the homotopy class of the restriction of $f$ to a tubular neighbourhood of the defect set, in particular the defect indices considered in [11], [14], and [1], [2].

We consider the natural notion of bordism on such maps: Two regularly defective maps $f\colon M \circ\!\!\to V$ and $f'\colon M' \circ\!\!\to V$ with defect sets $\Delta$, $\Delta'$ are bordant if there is a regulary defective map $F\colon W \circ\!\!\to V$ with defect set $\Gamma$ such that

1. $\partial W = M \dot{\cup} M'$,
2. $\partial \Gamma = \Delta \dot{\cup} \Delta'$ and
3. $F|_M = f$, $F|_{M'} = f'$.

Taking disjoint union defines an addition on the set of equivalence classes. We obtain the bordism group of regularly defective maps on $n$-dimensional manifolds which we denote by $\mathfrak{N}_n^{\mathrm{def}}(V)$. If $V = *$ consists of a point only we get bordism of pairs, which will be considered below.

Let $f\colon M \circ\!\!\to V$ be a regularly defective map with defect set $\Delta$ and fix a component $\Delta_0 \subset \Delta$. Consider the restriction of $f$ to the sphere bundle $SN$ of the normal bundle $N$ of $\Delta$. Choosing a fibre $SN_x$ of $SN|_{\Delta_0}$, $x \in \Delta_0$, and a homeomorphism $h\colon S^k \cong SN_x$ we get a map $f \circ h\colon S^k \to V$. On the set $[S^k, V]$ of homotopy classes of maps $S^k \to V$ we have an involution $\pm$ induced by reversing the orientation of $S^k$. The local defect index of $f$ at the component $\Delta_0$ is the class $\iota(f, \Delta_0) = [f \circ h] \in [S^k, V]/\pm$. It does not depend on the choice of $x$ and $h$. Up to sign it is the primary obstruction to extending $f$ over all of $\Delta_0$, cf. [3]. Regularly defective maps with $\iota(f, \Delta_0) \neq 0$ for each component $\Delta_0$ of the defect set are called topologically stable in the physics literature. In this case the defect can not be diminished by deformation, i.e. $f$ is not homotopic to a map extending to a superset of $M \setminus \Delta$.

We will also include the local defect index in the bordism groups. For a prescribed subset $\Lambda \subset \bigcup_k [S^k, V]/\pm$, a $\Lambda$-defective map $f\colon M \circ\!\!\to V$ is a regularly defective map all of whose local defect indices are contained in $\Lambda$. Requiring the maps $F$, $f'$ and $f$ in the above definition to be $\Lambda$-defective leads to the bordism groups $\mathfrak{N}_n^{\mathrm{def},\Lambda}(V)$.

In [17] M. Rost constructs the representing space $D_\Lambda V$ for the set $D_\Lambda(M, V)$ of concordance classes of $\Lambda$-defective maps $M \circ\!\!\to V$ by suitably enlarging $V$ such that each $\Lambda$-defective map $f\colon M \circ\!\!\to V$ induces a continuous map $F\colon M \to D_\Lambda V$, cf. section 2. He obtains a bijection $D_\Lambda(M, V) \to [M, D_\Lambda V]$. We do not need this result here but rely on the corresponding statement for bordism. Along the lines of [17] we obtain in section 2 a natural identification $\mathfrak{N}_n^{\mathrm{def},\Lambda}(V) = \mathfrak{N}_n(D_\Lambda V)$. Since $\mathfrak{N}_n(D_\Lambda V) \cong \bigoplus_{j=0}^n \mathfrak{N}_j(*) \otimes H_{n-j}(D_\Lambda V, \mathbb{Z}_2)$, cf. [4], the $\Lambda$-defective bordism groups can then be computed from the $\mathbb{Z}_2$-homology of the defect complex.

The bordism class of a regularly defective map $f\colon M \circ\!\!\to V$ is determined by the characteristic numbers

$$\langle w_I(M) \smile F^*\alpha, [M] \rangle \tag{1.1}$$

where $\alpha \in H^*(D_\Lambda V)$ and $F\colon M \to D_\Lambda V$ extends $f$. In section 3 we describe some of these geometrically, i.e. without reference to the defect complex. For fixed $\lambda \in \Lambda$ we denote by $\Delta^{(\lambda)}$ the union of those components of $\Delta$ with local defect index $\lambda$ and by $N^{(\lambda)}$ and $SN^{(\lambda)}$ the corresponding bundles over $\Delta^{(\lambda)}$.

We consider two types of characteristic numbers for regularly defective maps. First, omitting the map $f$ defines for each $\lambda \in \Lambda$ a natural map $\mathfrak{N}_n^{\mathrm{def},\Lambda}(V) \to \mathfrak{N}_n^{\mathrm{def}}(*) = \mathfrak{N}_n^{\mathrm{pair}}$, $[M, \Delta, f\colon M \setminus \Delta \to V] \mapsto [M, \Delta^{(\lambda)}]$ to bordism of pairs. By Theorem 1 in [19] this is completely described by the Stiefel-Whitney numbers $\langle w_I(TM), [M] \rangle$ of $M$ and the characteristic numbers

$$\mathfrak{Y}_{\lambda,I,J}(f) = \langle w_I(\Delta) \smile w_J(N), [\Delta^{(\lambda)}] \rangle \ .$$

Second we can restrict the map $f$ to the sphere bundle $\pi\colon SN \to \Delta$ of the normal bundle of the defect set. From the splitting $TSN^{(\lambda)} = \pi^* T\Delta^{(\lambda)} \oplus T_F SN^{(\lambda)}$ we construct the characteristic numbers

$$\mathfrak{Z}_{\lambda,\alpha,I,J}(f) = \langle w_I(\Delta) \smile w_J(N) \smile f^*\alpha, [SN^{(\lambda)}] \rangle$$

for $\alpha \in H^*(V)$.

Section 4 deals with regularly defective bordism of the circle $V = S^1$. In Theorem 4.1.1 we calculate the (co)homology of $D_\Lambda(S^1)$ and thereby $\mathfrak{N}_*^{\mathrm{def},\Lambda}(S^1)$. A basis for $\mathfrak{N}_*^{\mathrm{def},\Lambda}(S^1)$ is given in section 4.2. For $V = S^1$ the $\mathfrak{Z}_{\lambda,\alpha,I,J}(f)$ are determined by the $\mathfrak{Y}_{\lambda,I,J}(f)$. Nonetheless we obtain a complete set of geometrically defined characteristic numbers for $\mathfrak{N}_*^{\mathrm{def},\Lambda}(S^1)$, $\Lambda \subset \mathbb{Z} \setminus 0$ in Theorem 4.3.1.

The bordism groups of normal coverings with Galois group $\mathbb{Z}$ are $\mathfrak{N}_*(S^1)$. Analogously $\mathfrak{N}_*^{\mathrm{def}}(S^1)$ may be identified with cobordism of regularly branched $\mathbb{Z}$-coverings. These are branched coverings $X \to M$ in the sense of [6] with the following additional properties: First they are required to have a submanifold

$\Delta$ of $M$ as branching, or "singular" set. Second they are to carry an action of the integers $\mathbb{Z}$ on $X$ which is transitive and free on the fibres over $M \setminus \Delta$. It is shown in [6] that, via completion, branched coverings $g : X \to M$ with singular set $\Delta$ biuniquely correspond to unbranched coverings over $M \setminus \Delta$. Taking the classifying map $f \colon M \setminus \Delta \to S^1 = B\mathbb{Z}$ relates these to defective maps to $S^1$.

A similiar calculation is performed in [16] for $V = \mathbb{RP}^\infty$, thus producing a branched analogue to the computation of line field cobordism by Koschorke, [9]. It turns out that $[f \colon M \circ\!\!\rightarrow \mathbb{RP}^\infty] \in \mathfrak{N}_*^{\mathrm{def},\Lambda}(\mathbb{RP}^\infty)$ is determined by the bordism class of $M$ and the $\mathfrak{Z}_{\lambda,\alpha,I,J}(f)$.

Finally in section 5 we compute the invariants $\mathfrak{Z}_{\lambda,\alpha,I,J}(f)$ for some examples showing that in general they give information neither contained in the local defect index nor in the characteristic numbers of bordism of pairs.

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## 2 The Defect Complex

We review the construction in [17] of the defect complex $D_\Lambda V$ of a topological space $V$. Let $\Lambda \subset \bigcup_{k=1}^\infty [S^k, V]/\pm$ which we sometimes view as a $\mathbb{Z}_2$-invariant subset $\Lambda \subset \bigcup_{k=1}^\infty [S^k, V]$. Let $EO(k) \to BO(k)$ denote the universal $O(k)$-bundle and $\gamma^k = EO(k) \times_{O(k)} \mathbb{R}^k$ the universal vector bundle. We endow the set $C(S^{k-1}, V)$ of continuous maps $S^{k-1} \to V$ with the compact-open topology and the $O(k)$-action $(g, f) \mapsto f \circ g^{-1}$ for $g \in O(k)$ and $f \in C(S^{k-1}, V)$.

Let $C_\Lambda(S^{k-1}, V)$ be the subspace of maps with homotopy class in $\Lambda$. For a $k$-dimensional $\mathbb{R}$-vector bundle $N \to \Delta$ we consider the associated $C_\Lambda(S^{k-1}, V)$-bundle

$$C_\Lambda(SN, V) = \bigcup_{x \in \Delta} C_\Lambda(SN_x, V) = P_{O(k)}(N) \times_{O(k)} C_\Lambda(S^{k-1}, V) \to \Delta \ ,$$

where $P_{O(k)}(N) \to \Delta$ is the orthonormal frame bundle of $N$. Let

$$\Delta_\Lambda^k = C_\Lambda(S\gamma^k, V) = EO(k) \times_{O(k)} C_\Lambda(S^{k-1}, V) \xrightarrow{\pi_\Lambda^k} BO(k)$$

denote the classifying $C_\Lambda(S^{k-1}, V)$-bundle. Then $C_\Lambda(SN, V) = \nu^* \Delta_\Lambda^k$ for a classifying map $\nu \colon \Delta \to BO(k)$.

Denote by $E^k_\Lambda = (\pi^k_\Lambda)^*\gamma^k$ the pull-back of $\gamma^k$ to $\Delta^k_\Lambda$ and let $DE^k_\Lambda$, $SE^k_\Lambda$ denote its disc respectively sphere bundle. The fibre of $SE^k_\Lambda$ over a point $q \in BO(k)$ is canonically $(S\gamma^k_\Lambda)_q \times C_\Lambda((S\gamma^k_\Lambda)_q, V)$. Hence we have the evaluation map $a^k_\Lambda\colon SE^k_\Lambda \to V$. We let $\Delta_\Lambda$, $E_\Lambda$, $DE_\Lambda$, $SE_\Lambda$, $a_\Lambda$ denote the union over all $k \geq 1$ of the corresponding objects and use $a_\Lambda$ to glue

$$D_\Lambda V := DE_\Lambda \cup_{a_\Lambda} V\ .$$

This set $D_\Lambda V$ is called the $\Lambda$-defect complex and $\Delta_\Lambda$ the universal defect set.

Two $\Lambda$-defective maps $f_i\colon M {\circ\!\!\to} V$, $i = 0, 1$ are concordant if there is a $\Lambda$-defective map $\widetilde{f}\colon M\times[0,1] {\circ\!\!\to} V$ extending $f_i\colon M\times\{i\} {\circ\!\!\to} V$. If $F\colon M \to D_\Lambda V$ is transverse to the universal defect set $\Delta$ (i.e. the induced section of $F^*E'_\Lambda$ is transverse to the zero section, $E'_\Lambda$ the pull-back of $E_\Lambda$ over itself), then $\Delta := F^{-1}(\Delta_\Lambda)$ is a submanifold of $M$. Viewing $\Delta_\Lambda \subset \mathring{D}E_\Lambda \subset D_\Lambda V$ as the 0-section we may define $R$ to be the obvious retraction $D_\Lambda V \smallsetminus \Delta_\Lambda \to V$. Then, $R \circ F\colon M {\circ\!\!\to} V$ is a $\Lambda$-defective map with defect set $\Delta$. It is shown in [17] that this construction induces a bijection $\mathfrak{R}\colon [M, D_\Lambda V] \overset{[F]\mapsto[R\circ F]}{\longrightarrow} D_\Lambda(M, V)$ of the set of homotopy classes of maps $M \to D_\Lambda V$ with the set $D_\Lambda(M, V)$ of concordance classes of $\Lambda$-defective maps $M {\circ\!\!\to} V$.

We rely on the following immediate consequence of this construction.

**Proposition 2.1** *For each $n$ there is a canonical isomorphism*

$$\begin{array}{ccc} \mathfrak{N}_n(D_\Lambda V) & \overset{\cong}{\longrightarrow} & \mathfrak{N}^{\mathrm{def},\Lambda}_n(V) \\ [F] & \longmapsto & [R \circ F]\ , \end{array}$$

*where we have chosen a representative $F$ transverse to the universal defect set $\Delta_\Lambda$.*

**Proof:** In [17], the inverse map $\mathfrak{L}\colon D_\Lambda(M, V) \to [M, D_\Lambda V]$ of $\mathfrak{R}$ is obtained by linear extension as follows. Let $f\colon M \smallsetminus \Delta \to V$ be a $\Lambda$-defective map and $\nu\colon \Delta \to BO(k)$, $\hat{\nu}\colon N \to \gamma^k$ be a classifying map for the normal bundle $N$ of $\Delta$. The map $f$ defines a section of the bundle $C(SN, V) = \nu^*\Delta^k_\Lambda$ defined above. Therefore we have a unique lift $\psi\colon \Delta \to \Delta^k_\Lambda$, $\hat{\psi}\colon N \to E^k_\Lambda$ of maps of vector bundles such that $f|_{SN} = a^k_\Lambda \circ \hat{\psi}|_{SN}$. Glueing $f|_{M\smallsetminus DN}$ with $\hat{\psi}|_{DN}$ along $SN$ yields a map $L(f)\colon M \to D_\Lambda V$. This map represents $\mathfrak{L}([f])$ and will be called a linear extension of $f$ in the sequel.

Applying $R \circ -$ resp. $L(-)$ to bordisms one easily sees that $\mathfrak{R}$ and $\mathfrak{L}$ induce well defined maps $\mathfrak{R}'\colon \mathfrak{N}_n(D_\Lambda V) \to \mathfrak{N}^{\mathrm{def},\Lambda}_n(V)$, $[F] \mapsto [R \circ F]$ and $\mathfrak{L}'\colon \mathfrak{N}^{\mathrm{def},\Lambda}_n(V) \to \mathfrak{N}_n(D_\Lambda V)$.

Since $\mathfrak{R} \circ \mathfrak{L} = \mathrm{id}$ and $\mathfrak{L} \circ \mathfrak{R} = \mathrm{id}$ we obviously get $\mathfrak{R}' \circ \mathfrak{L}' = \mathrm{id}$ and $\mathfrak{L}' \circ \mathfrak{R}' = \mathrm{id}$. □

# 3 Characteristic numbers for $\mathfrak{N}_n^{\mathrm{def},\Lambda}(V)$

The bordism class of $f\colon M \setminus \Delta \to V$ is determined by the characteristic numbers $\langle w_I(M) \smile F^*\alpha\,,\,[M]\rangle$, where $F$ is a linear extension of $f$ as defined in the proof of proposition 2.1. In the following, we will investigate the relation between these numbers and the invariants $\mathfrak{Y}_{\lambda,I,J}(f)$ and $\mathfrak{Z}_{\lambda,\alpha,I,J}(f)$. For $q \geq 0$ and $\lambda \in \Lambda \cap [S^{k-1}, V]$ we define $\kappa_\lambda^q\colon H^q(\Delta_{\bar\lambda}) \to H^{q+k}(D_\Lambda V)$ as the composition

$$H^q(\Delta_{\bar\lambda}) \xrightarrow[\cong]{\Phi_{\bar\lambda}} H^{q+k}(DE_{\bar\lambda}, SE_{\bar\lambda}) \xrightarrow[\cong]{(\iota_{\bar\lambda}^*)^{-1}} H^{q+k}(D_\Lambda V, D_\Lambda V \setminus \mathring{D}E_{\bar\lambda}) \xrightarrow{\jmath_{\bar\lambda}^*} H^{q+k}(D_\Lambda V), \tag{3.1}$$

where $\Phi_{\bar\lambda}$ is the Thom isomorphism, $\iota_{\bar\lambda}\colon (DE_{\bar\lambda}, SE_{\bar\lambda}) \to (D_\Lambda V, D_\Lambda V \setminus \mathring{D}E_{\bar\lambda})$ the canonical map and $\jmath_{\bar\lambda}\colon (D_\Lambda V, \emptyset) \to (D_\Lambda V, D_\Lambda V \setminus \mathring{D}E_{\bar\lambda})$ the inclusion.

Additionally, we define $\mu_\lambda^q\colon H^q(SE_{\bar\lambda}) \to H^{q+1}(D_\Lambda V)$ as the composition

$$H^q(SE_{\bar\lambda}) \xrightarrow{\delta} H^{q+1}(DE_{\bar\lambda}, SE_{\bar\lambda}) \xrightarrow[\cong]{(\iota_{\bar\lambda}^*)^{-1}} H^{q+1}(D_\Lambda V, D_\Lambda V \setminus \mathring{D}E_{\bar\lambda}) \xrightarrow{\jmath_{\bar\lambda}^*} H^{q+1}(D_\Lambda V).$$

**Proposition 3.2** *Then we have*

$$\mathfrak{Y}_{\lambda,I,J}(f) = \langle w_I(M) \smile F^*\kappa_\lambda^*(w_J(E_{\bar\lambda}))\,,\,[M]\rangle.$$

**Proof:** Let $\Phi\colon H^q(\Delta^{(\lambda)}) \cong H^{q+k}(DN^{(\lambda)}, SN^{(\lambda)})$ be the Thom isomorphism and $\iota\colon (DN^{(\lambda)}, SN^{(\lambda)}) \hookrightarrow (M, M \setminus \mathring{D}N^{(\lambda)})$, $\jmath\colon (M, \emptyset) \hookrightarrow (M, M \setminus \mathring{D}N^{(\lambda)})$ the inclusions. Then we have

$$\begin{aligned}
\mathfrak{Y}_{\lambda,I,J}(f) &= \langle w_I(TM|_{\Delta^{(\lambda)}}) \smile w_J(N^{(\lambda)})\,,\,[\Delta^{(\lambda)}]\rangle \\
&= \langle w_I(TM|_{DN^{(\lambda)}}) \smile \Phi(w_J(N^{(\lambda)}))\,,\,[DN^{(\lambda)}, SN^{(\lambda)}]\rangle \\
&= \langle w_I(M) \smile (\iota^*)^{-1}\Phi(w_J(N^{(\lambda)}))\,,\,\iota_*[DN^{(\lambda)}, SN^{(\lambda)}]\rangle \\
&= \langle w_I(M) \smile \jmath^*(\iota^*)^{-1}\Phi(F|_{\Delta^{(\lambda)}})^* w_J(E_{\bar\lambda})\,,\,[M]\rangle\,.
\end{aligned}$$

Since $\jmath^*(\iota^*)^{-1}\Phi(F|_{\Delta^{(\lambda)}})^* = F^*\kappa_\lambda^*$ the proposition is proved. □

**Proposition 3.3** *Let $\bar{\pi}_{\bar{\lambda}}\colon SE_{\bar{\lambda}} \to \Delta_{\bar{\lambda}}$ denote the projection. Then we have*

$$\mathfrak{z}_{\lambda,\alpha,I,J}(f) = \langle w_I(M) \smile F^*\mu_\lambda^*(\bar{\pi}_{\bar{\lambda}}^* w_J(E_{\bar{\lambda}}) \smile a_{\bar{\lambda}}^*\alpha) \,,\, [M]\rangle.$$

**Proof:** Let $\iota_{SN^{(\lambda)}}\colon SN^{(\lambda)} \hookrightarrow DN^{(\lambda)}$, $\iota\colon (DN^{(\lambda)}, SN^{(\lambda)}) \hookrightarrow (M, M \smallsetminus \mathring{D}N^{(\lambda)})$ and $\jmath\colon (M, \emptyset) \hookrightarrow (M, M \smallsetminus \mathring{D}N^{(\lambda)})$ denote the inclusions. Then we have

$$\begin{aligned}
\mathfrak{z}_{\lambda,\alpha,I,J}(f) &= \langle (\iota_{SN^{(\lambda)}})^* w_I(DN^{(\lambda)}) \smile (\pi_{SN^{(\lambda)}})^* w_J(N^{(\lambda)}) \smile (f|_{SN^{(\lambda)}})^*\alpha \,,\, [SN^{(\lambda)}]\rangle \\
&= \langle w_I(DN^{(\lambda)}) \smile \delta((\pi_{SN^{(\lambda)}})^* w_J(N^{(\lambda)}) \smile (f|_{SN^{(\lambda)}})^*\alpha) \,,\, [DN^{(\lambda)}, SN^{(\lambda)}]\rangle \\
&= \langle w_I(M) \smile (\iota^*)^{-1}\delta((\pi_{SN^{(\lambda)}})^* w_J(N^{(\lambda)}) \smile (f|_{SN^{(\lambda)}})^*\alpha) \,,\, \iota_*[DN^{(\lambda)}, SN^{(\lambda)}]\rangle \\
&= \langle w_I(M) \smile \jmath^*(\iota^*)^{-1}\delta((\pi_{SN^{(\lambda)}})^* w_J(N^{(\lambda)}) \smile (f|_{SN^{(\lambda)}})^*\alpha) \,,\, [M]\rangle.
\end{aligned}$$

Let $\xi\colon N \to E_{\bar{\lambda}}$ denote the isometric bundle map, equal to $F$ on $DN$. Then

$$(\pi_{SN^{(\lambda)}})^* w_J(N^{(\lambda)}) \smile (f|_{SN^{(\lambda)}})^*\alpha = (\xi|_{SN^{(\lambda)}})^*(\bar{\pi}_{\bar{\lambda}}^* w_J(E_{\bar{\lambda}}) \smile a_{\bar{\lambda}}^*\alpha)$$

and using $\jmath^*(\iota^*)^{-1}\delta(\xi|_{SN^{(\lambda)}})^* = F^*\mu_\lambda^*$ we have proved proposition 3.3. □

# 4 Regularly defective bordism of the circle

## 4.1 Homology of $D_\Lambda(S^1)$

In the sequel (co)homology is always understood with $\mathbb{Z}_2$-coefficients. In this section we think of the set $\Lambda$ of admitted defect indices as a symmetric subset $\Lambda \subset \pi_1(S^1) = \mathbb{Z}$, $\Lambda = \Lambda_+ \cup -\Lambda_+$ with $\Lambda_+ \subset \mathbb{N}_0$

**Theorem 4.1.1** *Let $\Lambda_+^{ev} = \Lambda_+ \cap 2\mathbb{Z}$, $\phi\colon \bigoplus_{\lambda\in\Lambda_+} \mathbb{Z}_2 \to \mathbb{Z}_2$, $(a_\lambda)_\lambda \mapsto \sum_\lambda \lambda a_\lambda$ and assume $0 \neq \Lambda$. Then*

$$H_k(D_\Lambda S^1) \cong \begin{cases} \mathbb{Z}_2 & \text{for } k = 0 \\ \mathbb{Z}_2/\mathrm{im}(\phi) & \text{for } k = 1 \\ \ker(\phi) \subset \bigoplus_{\lambda\in\Lambda_+} \mathbb{Z}_2 & \text{for } k = 2 \\ \bigoplus_{\lambda\in\Lambda_+^{ev}} \mathbb{Z}_2 & \text{for } k \geq 3 \end{cases}$$

*and*

$$H^k(D_\Lambda S^1) = \mathrm{Hom}(H_k(D_\Lambda S^1), \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & \text{for } k = 0 \\ \ker(\psi) & \text{for } k = 1 \\ \left(\prod_{\lambda\in\Lambda_+} \mathbb{Z}_2\right)/\mathrm{im}(\psi) & \text{for } k = 2 \\ \prod_{\lambda\in\Lambda_+^{ev}} \mathbb{Z}_2 & \text{for } k \geq 3 \end{cases}$$

*where* $\psi\colon \mathbb{Z}_2 \to \prod_{\lambda\in\Lambda_+}\mathbb{Z}_2$, $1 \mapsto (\lambda \bmod 2)_{\lambda\in\Lambda_+}$. *If* $0\in\Lambda$ *then* $H_k(D_\Lambda S^1) \cong H_k(D_{\Lambda\smallsetminus 0}S^1)\oplus \mathbb{Z}_2^{k-1}$ *and* $H^k(D_\Lambda S^1) \cong H^k(D_{\Lambda\smallsetminus 0}S^1)\times \mathbb{Z}_2^{k-1}$ *(reading* $\mathbb{Z}_2^0 = \mathbb{Z}_2^{-1} = 0$*).*

For the proof of the theorem consider the subspaces

$$C_\Lambda(S^1,S^1) := \{f\colon S^1\to S^1 \mid \deg(f)\in\Lambda\},$$
$$C^{\mathrm{nor}}_\Lambda(S^1,S^1) := \{f\colon S^1\to S^1 \mid \underset{\lambda\in\Lambda}{\exists}\ \underset{z_0\in S^1}{\exists}\ \underset{z\in S^1}{\forall}\ f(z)=z_0z^\lambda\}$$

of $C(S^1,S^1)$. Let $\lambda>0$ and let $\bar\lambda := \{\lambda,-\lambda\}$. Obviously, $C^{\mathrm{nor}}_\lambda(S^1,S^1)$ is a strong deformation retract of $C_\lambda(S^1,S^1)$. The deformation of the identity into a retraction can be chosen to be compatible with the $SO(2)$-action on $C_\lambda(S^1,S^1)$. Therefore, $\Delta^{\mathrm{nor}}_\lambda := EO(2)\times_{SO(2)} C^{\mathrm{nor}}_\lambda(S^1,S^1)$ is a strong deformation retract of

$$\Delta_{\bar\lambda} = EO(2)\times_{O(2)} C_{\bar\lambda}(S^1,S^1) = EO(2)\times_{SO(2)} C_\lambda(S^1,S^1).$$

We identify $SO(2)=S^1$, and consider the $S^1$-action $S^1\times S^1\to S^1$, $(w,z)\mapsto \alpha(w)z$ on $S^1$, where $\alpha\colon S^1\to S^1$, $w\mapsto w^{-\lambda}$. Then the homeomorphism $C^{\mathrm{nor}}_\lambda(S^1,S^1)\to S^1$, $f\mapsto f(1)$ is compatible with the $S^1$-actions and we get

$$\Delta^{\mathrm{nor}}_\lambda = EO(2)\times_\alpha S^1 =: \alpha_* EO(2).$$

Consider the vector bundle $\xi_\lambda := \alpha_*EO(2)\times_{S^1}\mathbb{C}\to BSO(2)$. We have $c_1(\xi_\lambda) = -\lambda c_1$, where $c_1\in H^2(BSO(2),\mathbb{Z})$ denotes the universal first Chern class. Reducing modulo 2 we get $w_2(\xi_\lambda)=\lambda w_2$. Since $H^*(BSO(2)) = \mathbb{Z}_2[w_2]$ and thus $H^n(BSO(2))=0$ for $n$ odd, the Gysin sequence of $p_\lambda\colon \Delta^{\mathrm{nor}}_\lambda = S\xi_\lambda\to BSO(2)$ yields an exact sequence

$$0\to H^{n-1}(\Delta^{\mathrm{nor}}_\lambda)\xrightarrow{\phi_{n-1}} H^{n-2}(BSO(2))\xrightarrow{\smile\lambda w_2} H^n(BSO(2))\xrightarrow{p_\lambda^*} H^n(\Delta^{\mathrm{nor}}_\lambda)\to 0$$

for each even $n\ge 2$. If $\lambda$ is odd, then $\smile\lambda w_2\colon H^{n-2}(BSO(2))\to H^n(BSO(2))$ is an isomorphism and consequently $H^k(\Delta^{\mathrm{nor}}_\lambda)=0$ for all $k\ge 1$.

If $\lambda$ is even, then $\smile\lambda w_2\colon H^0(BSO(2))\to H^2(BSO(2))$ is zero and it follows that there exists a class $\alpha\in H^1(\Delta^{\mathrm{nor}}_\lambda)$ with $\phi_1(\alpha)\neq 0$. Then $\delta(\alpha)\in H^2(D\xi_\lambda, S\xi_\lambda)$ is the Thom class of the vector bundle $\xi_\lambda\to BSO(2)$. Therefore the restriction of $\alpha$ to each fibre generates the first $\mathbb{Z}_2$-cohomology of the fibre. Recalling that $\Delta_{\bar\lambda}\simeq\Delta^{\mathrm{nor}}_\lambda$ we obtain from the Leray-Hirsch Theorem:

**Proposition 4.1.2** *Let $\lambda > 0$. If $\lambda$ is odd then $H^k(\Delta_{\bar{\lambda}}) = 0$ for $k \geq 1$. If $\lambda$ is even then there is a nontrivial class $\alpha \in H^1(\Delta_{\bar{\lambda}})$ and $H^*(\Delta_{\bar{\lambda}})$ is a free module over $H^*(BSO(2))$ with basis $\{1, \alpha\}$.*

Applying the Thom isomorphism theorem, we get

**Proposition 4.1.3** *Let $\Lambda \subset \mathbb{Z} \setminus 0$ be a $\mathbb{Z}_2$-invariant subset, $\Lambda_+ := \Lambda \cap \mathbb{N}$ and $\Lambda_+^{ev} := \Lambda_+ \cap 2\mathbb{Z}$. Then*

$$H_k(D_\Lambda S^1, S^1) \cong \begin{cases} 0 & \text{for } k = 0, 1 \\ \bigoplus_{\lambda \in \Lambda_+} \mathbb{Z}_2 & \text{for } k = 2 \\ \bigoplus_{\lambda \in \Lambda_+^{ev}} \mathbb{Z}_2 & \text{for } k \geq 3 \end{cases}$$

*and*

$$H^k(D_\Lambda S^1, S^1) \cong \begin{cases} 0 & \text{for } k = 0, 1 \\ \prod_{\lambda \in \Lambda_+} \mathbb{Z}_2 & \text{for } k = 2 \\ \prod_{\lambda \in \Lambda_+^{ev}} \mathbb{Z}_2 & \text{for } k \geq 3. \end{cases}$$

Thus, in order to prove theorem 4.1.1 for $0 \notin \Lambda$, it remains to show

**Proposition 4.1.4** *The boundary*

$$\partial \colon H_2(D_\Lambda S^1, S^1) \cong \bigoplus_{\lambda \in \Lambda_+} \mathbb{Z}_2 \to H_1(S^1) \cong \mathbb{Z}_2$$

*is given by $(a_\lambda)_{\lambda \in \Lambda_+} \mapsto \sum_{\lambda \in \Lambda_+} \lambda a_\lambda$ and the coboundary $\delta \colon H^1(S^1) \cong \mathbb{Z}_2 \to H^2(D_\Lambda S^1, S^1) \cong \prod_{\lambda \in \Lambda_+} \mathbb{Z}_2$ by $1 \mapsto (\lambda \bmod 2)_{\lambda \in \Lambda_+}$.*

**Proof:** Obviously, it suffices to show that $\delta \colon H^1(S^1) \to H^2(D_{\bar{\lambda}} S^1, S^1)$ is zero if and only if $\lambda$ is even.

Let $\iota \colon D^2 \to DE_{\bar{\lambda}}$ denote the inclusion of a fibre of the disc bundle $DE_{\bar{\lambda}}$ and $\jmath \colon (DE_{\bar{\lambda}}, SE_{\bar{\lambda}}) \to (D_{\bar{\lambda}} S^1, S^1)$ the canonical map. Then we have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1(S^1) & \xrightarrow[\cong]{\delta} & H^2(D^2, S^1) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle (\iota|_{S^1})^*} & & \uparrow{\scriptstyle \cong\ \iota^*} & & \\
\cdots & \longrightarrow & H^1(SE_{\bar{\lambda}}) & \xrightarrow{\delta} & H^2(DE_{\bar{\lambda}}, SE_{\bar{\lambda}}) & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle (\jmath|SE_{\bar{\lambda}})^*} & & \uparrow{\scriptstyle \cong\ \jmath^*} & & \\
\cdots & \longrightarrow & H^1(S^1) & \xrightarrow{\delta} & H^2(D_{\bar{\lambda}} S^1, S^1) & \longrightarrow & \cdots
\end{array}$$

Since $(\jmath \circ \iota)|_{S^1} \colon S^1 \to S^1$ has degree $\pm\lambda$, it follows that $(\iota|_{S^1})^* \circ (\jmath|_{SE_{\bar{\lambda}}})^* = 0$ if and only if $\lambda$ is even. $\square$

For the case $0 \in \Lambda$, observe that $\Delta_0^{\mathrm{nor}} = BO(2) \times C_0^{\mathrm{nor}}(S^1, S^1)$.

### 4.2 A Basis for $\mathfrak{N}_*^{\mathrm{def},\Lambda}(S^1)$

Let $\{B_i^k \,|\, k \geq 0, i \in I(k)\}$ be a set of closed differentiable manifolds with $\dim B_i^k = k$ such that $\{[B_i^k] \,|\, i \in I(k)\}$ forms a basis of $\mathfrak{N}_k$ for each $k \geq 0$. It is well known that one can explicitly specify such a set using products of real projective spaces and Milnor manifolds, cf. [18], [4].

Let $\{F_j^l \colon M_j^l \to X \,|\, l \geq 0, j \in J(l)\}$ be a set of singular manifolds in a topological space $X$ such that $\{(F_j^l)_*[M_j^l] \,|\, j \in J(l)\}$ is a basis of $H_l(X)$. Then the singular manifolds

$$F_j^l \circ \mathrm{pr}_2 \colon B_i^k \times M_j^l \to X,$$

for $k, l \geq 0$, $i \in I(k)$, $j \in J(l)$ represent a basis of $\mathfrak{N}_k(X)$, cf. [5]. Here $[M_j^l]$ denotes the fundamental class of $M_j^l$ over $\mathbb{Z}_2$.

Using this fact and the identification $\mathfrak{N}_*^{\mathrm{def},\Lambda}(V) \cong \mathfrak{N}_*(D_\Lambda V)$ of Proposition 2.1 we immediately get

**Proposition 4.2.1** *Let $f_j^l \colon M_j^l \circ\!\!\rightarrow V, l \geq 0, j \in J(l)$ be $\Lambda$-defective maps with defect sets $\Delta_j^l$ and let $F_j^l \colon M_j^l \to D_\Lambda V$ be linear extensions. If $\{(F_j^l)_*[M_j^l] \,|\, j \in J(l)\}$ is a basis for $H_l(D_\Lambda V)$ for each $l$ then the $\Lambda$ defective maps*

$$f_j^l \circ \mathrm{pr}_2 \colon (B_i^k \times M_j^l) \smallsetminus (B_i^k \times \Delta_j^l) \to V,$$

*$k, l \geq 0, i \in I(k), j \in J(l)$ represent a basis of $\mathfrak{N}_*^{\mathrm{def},\Lambda}(V)$.*

In the following we explicitly give such a set of $\Lambda$-defective maps $f_j^l$, $l \geq 0$, $j \in J(l)$ for the case $V = S^1$.

Let $\Lambda \subset \mathbb{Z} \smallsetminus 0$ be a symmetric subset and $\Lambda_+ := \Lambda \cap \mathbb{N}$, $\Lambda_+^{ev} := \Lambda_+ \cap 2\mathbb{Z}$. With the techniques of section 4.1 it is straightforward to show that the following $\Lambda$-defective maps fulfil the assumptions of proposition 4.2.1. For simplicity, we omit the case $0 \in \Lambda$.

**Dimension 0:** We take $J(0) := \{0\}$, $M_0^0 := \{*\}$ and choose a constant map $f_0^0 \colon M_0^0 \to S^1$.

**Dimension 1:** If $\Lambda$ contains odd indices then $H_1(D_\Lambda S^1) = 0$ and consequently $J(1) = \emptyset$. Else $H_1(D_\Lambda S^1) \cong \mathbb{Z}_2$ and we take $J(1) := \{1\}$, $M_1^1 := S^1$ and $f_1^1 := \mathrm{id}_{S^1}$.

**Dimension 2:** Let $\lambda_1 < \lambda_2 < \dots$ be the sequence of the odd indices in $\Lambda_+$ and let $n \leq \infty$ be the number of such indices. Let

$$J(2) := \Lambda_+^{ev} \cup \{(\lambda_i, \lambda_{i+1}) \,|\, 1 \leq i < n\}.$$

Let $D^2 \subset \mathbb{C}$ denote the unit disc. For each index $\lambda \in \Lambda_+^{ev}$ we define $g_\lambda \colon D^2 \smallsetminus 0 \to S^1$, $z \mapsto z^\lambda/|z^\lambda|$. Since $g_\lambda(z) = g_\lambda(-z)$, we get a well defined $\Lambda$-defective map $f_\lambda^2$ on $M_\lambda^2 := \mathbb{RP}^2$ by identifying antipodal points in $S^1 \subset D^2$.

For $1 \leq i < n$ let $M^2_{(\lambda_i,\lambda_{i+1})} := S^2$ and define $f^2_{(\lambda_i,\lambda_{i+1})} \colon S^2 \circ\!\!\to S^1$ to be a map with $\lambda_{i+1}$ point defects of index $\lambda_i$ and $\lambda_i$ point defects of index $-\lambda_{i+1}$.

**Dimension $2k+3, k \geq 0$:** Let $J(2k{+}3) := \Lambda_+^{ev}$ and fix some $\lambda \in J(2k{+}3)$. Consider the canonical bundle $\gamma_k \to \mathbb{CP}^k$. In view of the previous section we use the lens space

$$\Delta_\lambda(2k+3) := S^{2k+1}/\lambda = S(\gamma_k^{\otimes_{\mathbb{C}}\lambda}) \cong P_{SO(2)}(\gamma_k) \times_{SO(2)} C^{\mathrm{nor}}_{-\lambda}(S^1, S^1)$$

as a finite dimensional approximation of the universal defect set and $D\pi^*\gamma_k \cup_{a_\lambda} S^1$ for the defect complex. Here $\pi$ denotes the projection $S^{2k+1}/\lambda \to \mathbb{CP}^k$ and $a_\lambda \colon S\pi^*\gamma_k \to S^1$ maps $([x], v) \mapsto z^\lambda$ if $v = zx$, $x \in S^{2k+1}$, $v \in S^{2k+1}$, $z \in S^1$. Let

$$M_\lambda^{2k+3} := D\pi^*\gamma_k/\pm$$

be obtained by identifying antipodal points in the circle bundle $S\pi^*\gamma_k$. We have a fibre bundle $M_\lambda^{2k+3} \to S^{2k+1}/\lambda$ with fibre $\mathbb{RP}^2$. Since $\lambda$ is even there is a map $f_{\lambda,2k+3} \colon M_\lambda^{2k+3} \circ\!\!\to S^1$ with defect set $S^{2k+1}/\lambda$ and local defect index $\lambda$ induced by $a_\lambda$. By the discussion in the previous section the linear extension of $f_{\lambda,2k+3}$ maps $[M_\lambda^{2k+3}]$ to the generator of $H_{2k+3}(D_\lambda S^1)$.

**Dimension $2k+2, k \geq 1$:** For $\lambda \in J(2k+2) := \Lambda_+^{ev}$ let $g_{\lambda,2k+2}$ be the composition

$$\Delta_\lambda(2k+2) := \mathbb{RP}^{2k} \hookrightarrow \mathbb{RP}^{2k+1} = S^{2k+1}/2 \to S^{2k+1}/\lambda\ .$$

We define $M_\lambda^{2k+2}$ to be the pull-back of $M_\lambda^{2k+3}$ to $\Delta_\lambda(2k{+}2)$ with $g_{\lambda,2k+2}$ and let $f_{\lambda,2k+2}$ be the composition $M_\lambda^{2k+2} \to M_\lambda^{2k+3} \circ\!\!\to S^1$. Since $g_{\lambda,2k+2}$ induces an isomorphism in $H_{2k}$ we get that the linear extension of $f_{\lambda,2k+2}$ maps $[M_\lambda^{2k+2}]$ to the generator of $H_{2k+2}(D_\lambda S^1)$.

### 4.3 Characteristic numbers for $\mathfrak{N}_n^{\mathrm{def},\Lambda}(S^1)$

Let $\Lambda \subset \mathbb{Z} \setminus 0$ be a $\mathbb{Z}_2$-invariant subset. In this section we prove the following:

**Theorem 4.3.1** *Let $f\colon M \circ\!\!\rightarrow S^1$ be a $\Lambda$-defective map. There is a unique class $\alpha \in H^1(M)$ with $\alpha|_{M\setminus\Delta} = f^*\varphi_{S^1}$, where $\varphi_{S^1}$ denotes the generator of $H^1(S^1)$. For each $\lambda \in \Lambda_+^{ev}$ there is a unique class $\beta_\lambda \in H^1(\Delta^{(\lambda)}) \subset H^1(\Delta)$ with the following property: If $\iota\colon S^1 \to \Delta^{(\lambda)}$ is any continuous map, $\hat\iota\colon S(\iota^* N^{(\lambda)}) \to SN^{(\lambda)}$ the canonical map over $\iota$ and $\sigma\colon S^1 \to S(\iota^*N^{(\lambda)})$ an arbitrary cross-section, then*

$$\langle \beta_\lambda,\, \iota_*[S^1]\rangle = \deg(f|_{SN^{(\lambda)}} \circ \hat\iota \circ \sigma) \bmod 2.$$

*The bordism class of $f\colon M \setminus \Delta \to S^1$ is determined by the characteristic numbers*

$$\begin{aligned}
&\langle w_I(M) \smile \alpha\,,\, [M]\rangle,\\
&\langle w_I(TM|_\Delta) \smile w_2(N)^{q-1}, [\Delta^{(\lambda)}]\rangle = \mathfrak{Y}_{\lambda,I,(0,q-1)}(f) \quad \text{with } \lambda \in \Lambda_+, \qquad (4.3.2)\\
&\langle w_I(TM|_\Delta) \smile w_2(N)^{q-1} \smile \beta_\lambda, [\Delta^{(\lambda)}]\rangle \quad \text{with } \lambda \in \Lambda_+^{ev}
\end{aligned}$$

*together with the bordism class of $M$.*

Let $F\colon M \to D_\Lambda S^1$ be a linear extension of $f$. Throughout this section let $\kappa_\lambda^q\colon H^q(\Delta_{\bar\lambda}) \to H^{q+2}(D_\Lambda S^1)$ denote the homomorphism (3.1) in the case $V = S^1$. Theorem 4.3.1 is an immediate consequence of the following propositions.

First we assume that $\Lambda \subset 2\mathbb{Z}$. Then $H^1(D_\Lambda V) \cong \mathbb{Z}_2$. Recall that we have $H^1(D_\Lambda V) = 0$ if $\Lambda \not\subset 2\mathbb{Z}$. Let $\eta$ be the nontrivial element in $H^1(D_\Lambda V)$.

**Proposition 4.3.3** *The restriction $H^1(M) \to H^1(M \setminus \Delta)$ is injective and we have $(F^*\eta)|_{M\setminus\Delta} = f^*\varphi_{S^1}$, where $\varphi_{S^1} \in H^1(S^1)$ denotes the generator.*

**Proof:** Since $\Delta$ has codimension 2, we have $H^1(M, M\setminus\Delta) = 0$ and the long exact sequence yields the injectivity of the restriction. Let $j\colon S^1 \to D_\Lambda S^1$ denote the inclusion. Since $H^1(D_\Lambda S^1, S^1)$ is zero, $j^*\colon H^1(D_\Lambda S^1) \to H^1(S^1)$ is bijective. Consequently, we have $\varphi_{S^1} = j^*\eta$. As $F|_{M\setminus\Delta}$ is homotopic to $j\circ f$, it follows that $(F^*\eta)|_{M\setminus\Delta} = f^*j^*\eta = f^*\varphi_{S^1}$. □

Now, let $\Lambda$ be an arbitrary $\mathbb{Z}_2$-invariant subset of $\mathbb{Z}\setminus 0$. We have $H^2(D_\Lambda S^1) \cong \big(\prod_{\lambda\in\Lambda_+} \mathbb{Z}_2\big)/\operatorname{im}(\psi)$, where $\psi$ is the homomorphism defined in theorem 4.1.1.

**Proposition 4.3.4** *Let $\eta \in H^2(D_\Lambda S^1)$ and let $(a_\lambda)_{\lambda\in\Lambda_+} \in \prod_{\lambda\in\Lambda_+}\mathbb{Z}_2$ be an element representing $\eta$ under the isomorphism of Theorem 4.1.1. Then*

$$\langle w_I(M) \smile F^*\eta\,,\,[M]\rangle = \sum_{\lambda\in\Lambda_+} a_\lambda\,\langle w_I(TM|_\Delta)\,,\,[\Delta^{(\lambda)}]\rangle = \sum_{\lambda\in\Lambda_+} a_\lambda\,\mathfrak{Y}_{\lambda,I,(0)}(f)\,.$$

**Proof:** Since $\Delta$ is compact, we may assume that $\Lambda$ is finite. Moreover, it suffices to consider $\lambda \in \Lambda_+$ with $a_\lambda = 1$ and $a_\mu = 0$ for $\mu \in \Lambda_+ \smallsetminus \{\lambda\}$, hence $\eta = \kappa^0_\lambda(1)$. Proposition 3.2 yields

$$\langle w_I(M) \smile F^*\eta\,,\,[M]\rangle = \mathfrak{Y}_{\lambda,I,(0)}(f) = \langle w_I(TM|_\Delta)\,,\,[\Delta^{(\lambda)}]\rangle.$$

□

For the even dimensions $\geq 4$ we have

**Proposition 4.3.5** *Let $q \geq 2$ and $\eta = (a_\lambda)_{\lambda\in\Lambda^{ev}_+} \in \prod_{\lambda\in\Lambda^{ev}_+}\mathbb{Z}_2 \cong H^{2q}(D_\Lambda S^1)$. Then*

$$\begin{aligned}\langle w_I(M) \smile F^*\eta, [M]\rangle &= \sum_{\lambda\in\Lambda^{ev}_+} a_\lambda\,\langle w_I(TM|_\Delta) \smile w_2(N)^{q-1}, [\Delta^{(\lambda)}]\rangle \\ &= \sum_{\lambda\in\Lambda^{ev}_+} a_\lambda\,\mathfrak{Y}_{\lambda,I,(0,q-1)}(f)\,.\end{aligned}$$

**Proof:** We may again assume that we have a $\lambda \in \Lambda^{ev}_+$ with $a_\lambda = 1$ and $a_\mu = 0$ for $\mu \in \Lambda^{ev}_+ \smallsetminus \{\lambda\}$. Then $\eta = \kappa^{2q-2}_\lambda(w_2(E_{\bar\lambda})^{q-1})$. Proposition 3.2 yields

$$\langle w_I(M) \smile F^*\eta\,,\,[M]\rangle = \mathfrak{Y}_{\lambda,I,(0,q-1)}(f) = \langle w_I(TM|_\Delta) \smile w_2(N)^{q-1}\,,\,[\Delta^{(\lambda)}]\rangle.$$

□

Now, let $q \geq 1$ and $\eta = (a_\lambda)_{\lambda\in\Lambda^{ev}_+} \in \prod_{\lambda\in\Lambda^{ev}_+}\mathbb{Z}_2 \cong H^{2q+1}(D_\Lambda S^1)$. For $\lambda \in \Lambda_+$ let $F^{(\lambda)}\colon \Delta^{(\lambda)} \to \Delta_{\bar\lambda}$ be the restriction of $F$ to $\Delta^{(\lambda)}$.

**Proposition 4.3.6** *For $\lambda \in \Lambda^{ev}_+$ let $\beta_\lambda$ be the generator of $H^1(\Delta_{\bar\lambda})$ as in Proposition 4.1.2 . Then*

$$\begin{aligned}&\langle w_I(M) \smile F^*\eta\,,\,[M]\rangle = \\ &\qquad = \sum_{\lambda\in\Lambda^{ev}_+} a_\lambda\,\langle w_I(TM|_{\Delta^{(\lambda)}}) \smile w_2(N|_{\Delta^{(\lambda)}})^{q-1} \smile F^{(\lambda)*}\beta_\lambda\,,\,[\Delta^{(\lambda)}]\rangle\,.\end{aligned}$$

**Proof:** Assume again that $a_\lambda = 1$ and $a_\mu = 0$ for $\mu \in \Lambda_+^{ev} \setminus \{\lambda\}$. Then we have $\eta = \kappa_\lambda^{2q-1}(w_2(E_{\bar\lambda})^{q-1} \smile \beta_\lambda)$. With the Thom isomorphism $\Phi$ and the inclusions $\iota\colon (DN^{(\lambda)}, SN^{(\lambda)}) \hookrightarrow (M, M \setminus \mathring{D}N^{(\lambda)})$ and $\jmath\colon (M, \emptyset) \hookrightarrow (M, M \setminus \mathring{D}N^{(\lambda)})$ we get

$$\begin{aligned}
&\langle w_I(TM|_{\Delta^{(\lambda)}}) \smile w_2(N^{(\lambda)})^{q-1} \smile F^{(\lambda)*}\beta_\lambda\,,\, [\Delta^{(\lambda)}]\rangle \\
&\qquad = \langle w_I(TM|_{DN^{(\lambda)}}) \smile \Phi F^{(\lambda)*}(w_2(E_{\bar\lambda})^{q-1} \smile \beta_\lambda)\,,\, [DN^{(\lambda)}, SN^{(\lambda)}]\rangle \\
&\qquad = \langle w_I(M) \smile (\iota^*)^{-1}\Phi F^{(\lambda)*}(w_2(E_{\bar\lambda})^{q-1} \smile \beta_\lambda)\,,\, \iota_*[DN^{(\lambda)}, SN^{(\lambda)}]\rangle \\
&\qquad = \langle w_I(M) \smile \jmath^*(\iota^*)^{-1}\Phi F^{(\lambda)*}(w_2(E_{\bar\lambda})^{q-1} \smile \beta_\lambda)\,,\, [M]\rangle\,.
\end{aligned}$$

Using $\jmath^*(\iota^*)^{-1}\Phi F^{(\lambda)*} = F^*\kappa_\lambda^*$ we obtain

$$\langle w_I(TM|_{\Delta^{(\lambda)}}) \smile w_2(N^{(\lambda)})^{q-1} \smile F^{(\lambda)*}\beta_\lambda\,,\, [\Delta^{(\lambda)}]\rangle = \langle w_I(M) \smile F^*\eta\,,\, [M]\rangle.$$

□

Thus, it remains to describe the classes $F^{(\lambda)*}\beta_\lambda \in H^1(\Delta^{(\lambda)})$ for $\lambda \in \Lambda_+^{ev}$.

**Proposition 4.3.7** *Let $\lambda \in \Lambda_+^{ev}$, let $\iota\colon S^1 \to \Delta^{(\lambda)}$ be a continuous mapping and let $\hat\iota\colon S(\iota^*N^{(\lambda)}) \to SN^{(\lambda)}$ denote the canonical map over $\iota$. For an arbitrary cross-section $\sigma\colon S^1 \to S(\iota^*N^{(\lambda)})$ we then have:*

$$\langle F^{(\lambda)*}\beta_\lambda\,,\, \iota_*[S^1]\rangle = \deg(f|_{SN^{(\lambda)}} \circ \hat\iota \circ \sigma) \bmod 2. \tag{4.3.8}$$

**Proof:** Let $\sigma_1$ and $\sigma_2$ be two cross-sections in $S(\iota^*N^{(\lambda)}) \to S^1$. Then obviously $\deg(f|_{SN^{(\lambda)}} \circ \hat\iota \circ \sigma_2) - \deg(f|_{SN^{(\lambda)}} \circ \hat\iota \circ \sigma_1)$ is a multiple of $\lambda$ and consequently

$$\deg(f|_{SN^{(\lambda)}} \circ \hat\iota \circ \sigma_2) \equiv \deg(f|_{SN^{(\lambda)}} \circ \hat\iota \circ \sigma_1) \mod 2.$$

Therefore, is suffices to show the existence of a cross-section $\sigma$ which fulfils (4.3.8). Let $\widetilde\pi\colon \Delta_{\bar\lambda} \to BSO(2)$ denote the projection map. As $\pi_1(BSO(2)) = 0$, $\widetilde\pi_\lambda \circ F^{(\lambda)} \circ \iota$ is null homotopic and we can assume that $\widetilde\pi_\lambda \circ F^{(\lambda)} \circ \iota \equiv x \in BSO(2)$. Let $\widetilde\gamma^2$ denote the universal vector bundle over $BSO(2)$. An arbitrary element $v \in S\widetilde\gamma_x^2$ yields a cross-section $\widetilde\sigma\colon (\Delta_{\bar\lambda})_x \to SE_{\bar\lambda}|_{(\Delta_{\bar\lambda})_x}$. Let $\sigma\colon S^1 \to S(\iota^*N^{(\lambda)})$ be the cross-section induced by $\widetilde\sigma$.

The map $a_{\bar\lambda} \circ \widetilde\sigma$ is equal to the evaluation map

$$(\Delta_{\bar\lambda})_x = C_\lambda(S\widetilde\gamma_x^2, S^1) \longrightarrow S^1, \quad g \longmapsto g(v)$$

and therefore is a homotopy equivalence. Thus, $(a_{\bar\lambda}\circ\widetilde{\sigma})^*\varphi_{S^1}$ is the generator of $H^1((\Delta_{\bar\lambda})_x)=\mathbb{Z}_2$, i.e. $(a_{\bar\lambda}\circ\widetilde{\sigma})^*\varphi_{S^1}=\beta_\lambda|_{(\Delta_{\bar\lambda})_x}$. We obtain

$$\iota^*F^{(\lambda)*}\beta_\lambda=\iota^*F^{(\lambda)*}(a_{\bar\lambda}\circ\widetilde{\sigma})^*\varphi_{S^1}=(f|_{SN^{(\lambda)}}\circ\hat\iota\circ\sigma)^*\varphi_{S^1}$$

and the proposition is proved. $\square$

Thus the characteristic numbers (4.3.2) together with the bordism class of $M$ determine all the numbers (1.1).

# 5 Further Examples

We end with some examples of nonvanishing invariants $\mathfrak{z}_{\lambda,\alpha,I,J}(f)$ distinguishing bordism of regularly defective maps from bordism of pairs.

**Example:** The unit tangent bundle of $\mathbb{RP}^{2k}$ is explicitly given as $ST\mathbb{RP}^{2k}=\{(x,y)\in S^{2k}\times S^{2k}\mid x\perp y\}/\sim$, with the antipodal identification $(x,y)\sim(-x,-y)$. For independent $x,y\in\mathbb{R}^{2k+1}$, let $\langle x,y\rangle\in G_2^+(\mathbb{R}^{2k+1})$ denote the oriented subspace spanned by these vectors and define a map $\hat f\colon ST\mathbb{RP}^{2k}\to V=G_2^+(\mathbb{R}^{2k+1})$ by $[x,y]\mapsto\langle x,y\rangle$. Mapping $[x,y]\mapsto(\langle x,y\rangle,[x])$ defines a homeomorphism of $ST\mathbb{RP}^{2k}$ with the projective bundle of the canonical bundle over $G_2^+(\mathbb{R}^{2k+1})$. This is the circle bundle of a 2-dimensional vector bundle $L$ over $G_2^+(\mathbb{R}^{2k+1})$ and under the above identifications, the map $\hat f$ extends to the bundle projection of $L$. Glueing the disc bundle $DL$ of $L$ with the obvious regularly defective extension of $\hat f$ to the disc bundle $DT\mathbb{RP}^{2k}$ we obtain a regularly defective map

$$f\colon M=DL\cup_{ST\mathbb{RP}^{2k}}DT\mathbb{RP}^{2k}\circ\!\!\rightarrow V=G_2^+(\mathbb{R}^{2k+1})$$

with defect set $\mathbb{RP}^{2k}$. If $k\geq 2$ its local defect index $\lambda$ is a generator of $\mathbb{Z}=\pi_{2k-1}(V)$. We compute the $\mathfrak{z}_{\lambda,\alpha,I,J}(\hat f)$.

From the Leray-Hirsch Theorem we infer that $\hat f^*$ is injective and that $H^*(ST\mathbb{RP}^{2k})$ is a free $H^*(V)$-module with base $\{1,y\}$ for some $y\in H^1(ST\mathbb{RP}^{2k})$. The Gysin-sequence shows that $H^*(ST\mathbb{RP}^{2k})\cong\mathbb{Z}_2[b,y]/(b^2,y^{2k})$ as graded $\mathbb{Z}_2$-algebras with $\deg(b)=2k$ and $y$ the generator of $H^1(ST\mathbb{RP}^{2k})\cong H^1(\mathbb{RP}^{2k})$. Hence $H^*(V)\cong\mathbb{Z}_2[b,y^2]/(b^2,y^{2k})$. One can now easily compute the $\mathfrak{z}_{\lambda,\alpha,I,J}(f)$. For instance taking $\alpha=y^{2k-2}b$, $I=(1,0,\dots,0)$ and $J=(0)$ we get $\mathfrak{z}_{\lambda,\alpha,I,J}(f)=1$.

For the second set of examples we need the following.

**Lemma 5.1** *For multiindices $L, I$ and $l \in \mathbb{N}$ there are universal polynomials $p_{L,I,l} \in \mathbb{Z}_2[T_1, ...T_l]$ with the following property: Let $\pi\colon N \to \Delta$ be an $l$-dimensional vector bundle, $U_N$ its Thom class and $\Phi$ the Thom isomorphism. Then $P_L\colon H^*(\Delta) \to H^*(\Delta)$ mapping*

$$x \longmapsto \sum_{I \in \mathbb{N}_0^t, t \in \mathbb{N}} p_{L,I,l}(w_1, \ldots, w_l)\, \mathrm{Sq}^I(x)$$

*fulfils* $\mathrm{Sq}^L(\Phi(x)) = \Phi(P_L(x))$.

**Proof:** We use induction on the length of $L$. For $L \in \mathbb{N}$ we compute

$$\mathrm{Sq}^L(\Phi(x)) = \sum_{r+s=L} \pi^*(\mathrm{Sq}^r(x)) \underbrace{\mathrm{Sq}^s(U_N)}_{=\Phi(w_s)} = \Phi\left(\sum_{r+s=L} w_s\, \mathrm{Sq}^r(x))\right).$$

The assertion follows by induction using the formulae of Wu and Cartan. Obviously the polynomials do not depend on the bundle $\pi : N \to \Delta$. □

**Proposition 5.2** *Let $M$ be a compact $n$-dimensional manifold and $\Delta$ a closed $k$-dimensional connected submanifold. Let $\pi : N \to \Delta$ be the normal bundle. Let $I, \tilde{I}, J, \tilde{J}, L, \tilde{L}$ be multiindices and $y \in H^{s-n+k+1}(\Delta)$, $y \neq 0$ with $s > n - k - 1$ such that*

1. $w_{n-k}(N) = 0$,
2. $H^{n-k}(M, \Delta) = H^s(M, \Delta) = 0$,
3. $\langle P_L(1) \smile w_I(\Delta) \smile w_J(N)\,,\, [\Delta]\rangle \neq 0$,
4. $\langle P_{\tilde{L}}(y) \smile w_{\tilde{I}}(\Delta) \smile w_{\tilde{J}}(N)\,,\, [\Delta]\rangle \neq 0$.

*Then there are maps $f_{1,2} : M \setminus \Delta \to V := K(\mathbb{Z}_2, n-k-1) \times K(\mathbb{Z}_2, s)$ with the same nontrivial local defect index $\lambda$ representing different nontrivial elements in $\mathfrak{N}_n^{\mathrm{def},\lambda}(V)$.*

For an explicit example choose $M = S^n$ and $\Delta$ any submanifold diffeomorphic to $\mathbb{RP}^k$ with $k$ even, $n > 2k + 1$, $n - k < s < n$, $I = (1, 0, \ldots, 0)$, $J = \tilde{J} = (0, \ldots, 0)$, $L = 1$, $\tilde{I} = (n - s - 1, 0 \ldots, 0)$, $\tilde{L} = 0$, $y = z^{s-n+k+1}$ with $z$ the generator of $H^*(\mathbb{RP}^k)$.

**Proof:** The $f_i$ will be distinguished by suitable $\mathfrak{Z}_{\lambda,\alpha,I,J}$. Let $u$ and $v$ be the characteristic elements of $K(\mathbb{Z}_2, n-k-1)$ and $K(\mathbb{Z}_2, s)$. Let $b \in H^{n-k-1}(SN)$ with $\rho(b) = 1$ where $\rho$ is the connecting homomorphism in the Gysin sequence. Let $i\colon SN \to M \setminus \Delta$ be the inclusion. The long exact sequence of $(M \setminus \mathring{D}N, SN)$ shows that there is an $x \in H^{n-k-1}(M \setminus \Delta)$ such that $i^*(x) = b$. There is a unique map $f : M \setminus \Delta \to K(\mathbb{Z}_2, n-k-1)$ such that $f^*(u) = x$. Analogously there is a $\tilde{x} \in H^s(M \setminus \Delta)$ with $i^*(\tilde{x}) = yb$ and a map $g : M \setminus \Delta \to K(\mathbb{Z}_2, s)$ with $g^*(v) = \tilde{x}$. Define $f_1 := f \times \text{const.}$ and $f_2 := f \times g$.

Now we show that both maps have nontrivial local defect indices. Since $\delta i^* f^*(u)$ is the Thom class of $N$ we know that its restriction on any fibre of $SN$ is not zero. This shows that the local defect index of $f$ is not zero for any $p \in \Delta$.

By assumption 3, $\mathfrak{Z}_{\lambda,\mathrm{Sq}^L(u)\times 1,I,J}(f_i) = \mathfrak{Z}_{\lambda,\mathrm{Sq}^L(u),I,J}(f) \neq 0$. Hence the $f_i$ are not null bordant. But on the other hand $f_1$ and $f_2$ are not bordant since $\mathfrak{Z}_{\lambda,1\times\mathrm{Sq}^{\tilde{L}}(v),\tilde{I},\tilde{J}}(f_1) = 0$ and $\mathfrak{Z}_{\lambda,1\times\mathrm{Sq}^{\tilde{L}}(v),\tilde{I},\tilde{J}}(f_2) \neq 0$. □

## References


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