# The local defect index up to finite ambiguity

Stefan Bechtluft-Sachs

Marco Hien

**Abstract:** The rational Hopf invariant of the primary obstruction to a nowhere vanishing section in a vectorbundle is computed in terms of characteristic classes. This leads to a local rigidity result on the local defect index of a map.

We consider a $q$-dimensional oriented vector bundle $E$ over a cell complex $\Delta$ and a section $\sigma$ of the sphere bundle $SE$ over the $p$-skeleton of $\Delta$. Let $c(\sigma) \in H^{p+1}(\Delta; \pi_p(S^{q-1}))$ denote the primary obstruction to extending $\sigma|_{\Delta^{p-1}}$ to the $p+1$ skeleton. It is well known that the coefficient group is infinite only if $p = q-1$ or $p = 4l-1$, $q = 2l+1$. In the first case the obstruction is the Euler class of $E$. In the second case, we compute the image of this obstruction under the rational Hopf invariant $h^{\mathbb{Q}}$, i.e. under the composite $h^{\mathbb{Q}} : \pi_{4l-1}(S^{2l}) \xrightarrow{h} \mathbb{Z} \hookrightarrow \mathbb{Q}$, where $h$ is the Hopf invariant.

**Theorem 1** *Let $E$ be an oriented real vector bundle dimension $2l+1$, $l \geq 1$ over a $4l$-dimensional cell complex $\Delta$. Let $\sigma : \Delta^{4l-1} \to SE$ be a section of the sphere bundle $SE$ over the $4l-2$ skeleton of $\Delta$ which extends to a section over the $(4l-1)$-skeleton. Then the image under the rational Hopf invariant of the primary obstruction $c(E,\sigma) \in H^{4l}(\Delta; \mathbb{Z})$ to extending $\sigma$ to $\Delta$ is given by*

$$p_l(E) - e(\sigma)^2 = 4\ (-1)^l\ h_*^{\mathbb{Q}} c(E,\sigma) \in H^{4l}(\Delta; \mathbb{Q})$$

*where $e(\sigma) \in H^{2l}(\Delta; \mathbb{Q})$ restricts to the Euler class of the the complementary bundle of $\sigma$ over $\Delta^{4l-2}$.*

Consider a regularly defect cross section $\sigma$ in a $q$-dimensional vector bundle $E$ over a manifold $M$ i.e. $\sigma$ is defined only outside some closed submanifold $\Delta \subset M$. The local defect index (at a point $x \in \Delta$) is $\iota(\sigma) : SN_x \to SE_x$ where $N$ is the normal bundle of $\Delta$ in $M$, see [3, 4, 2] for these notions in ordered media with defects. Let $p$ be the dimension of $N$ and assume $N$ and $E$ oriented over $\Delta$. Then we may identify $\iota(\sigma) \in \pi_{p-1}(S^{q-1})$ as the primary obstruction to extending the section $\sigma$ on $SN$ to $DN$. If $p = q$ we have that $e(E) = \iota(\sigma)e(N)$. In the case $p = 4l$, $q = 2l+1$ we can apply the previous theorem to the $4l$-skeleton of the relative cell complex $(DN, SN)$ and get

$$4\ (-1)^{4l}\ h^{\mathbb{Q}}(\iota(\sigma))e(N) = p_l(E) - e^2 \in H^{4l}(\Delta, \mathbb{Q})$$

where $e \in H^{2l}(\Delta) \cong H^{2l}(SN)$ is the Euler class of the complementary bundle of $\sigma$ on $SN$. Thus $\iota(\sigma)$ is determined up to finite ambiguity by $e$ (and the bundles $N$ and $E$) if the Euler class of $N$ is not torsion.

**Proof of the Theorem:** Consider the Postnikov decomposition of the map $BSO(2l) \to BSO(2l+1)$.

$$BSO(2l) \xrightarrow{q} B_{2l-1} \to \cdots \to B_2 \to B_1 \to B_0 = BSO(2l+1)$$

where the fibre of the map $B_{i+1} \to B_i$ is an Eilenberg MacLane space $K(\pi_{2l+i}(S^{2l}), 2l+i)$. Let $c \in H^{4l}(B_{2l-1}; \pi_{4l-1}(S^{2l}))$ be the primary obstruction to splitting $q$. The section $\sigma$ gives rise to a lift of the classifying map $\epsilon$ of $E$ to a map $\epsilon_\sigma : \Delta \to B_{2l-1}$ and we have $c(E,\sigma) = \epsilon_\sigma^* c$.

Clearly $q^*c = 0$. For $i = 1 \ldots 2l-2$ the groups $\pi_{2l+i}(S^{2l})$ are finite and the Eilenberg MacLane spaces $K(\pi_{2l+i}(S^{2l}), 2l+i)$ are rationally acyclic. Hence the composition $B_{2l-1} \to \cdots \to B_2 \to B_1$ induces an isomorphism in rational cohomology. From the Serre spectral sequence and the fact that $H^*(K(\mathbb{Z}, 2l); \mathbb{Q}) = \mathbb{Q}[x]$, where $x \in H^{2l}(K(\mathbb{Z}, 2l); \mathbb{Q})$, we get that $H^{4l}(B_{2l-1}; \mathbb{Q})$ is spanned by monomials in the pull back of the universal Pontrijagin classes $p_i \in H(BSO(2l+1); \mathbb{Q})$ and an element $y \in H^{2l}(B_{2l-1}; \mathbb{Q})$ whose restriction to the fibre is $x$ and with $q^*y = e$, where $e \in H^{2l}(BSO(2l); \mathbb{Q})$ denotes the universal Euler class. Since the only relation among the characteristic classes in $H^*(BSO(2l); \mathbb{Q})$ is $e^2 = p_l$, we infer that the kernel of $q^*$ in $H^{4l}$ is spanned by $p_l - y^2$. Therefore we find $\alpha \in \mathbb{Q}$, such that $\alpha c = p_l - y^2$ and

$$\alpha c(E, \sigma) = \alpha \epsilon_\sigma^* c = \epsilon_\sigma^*(p_l - y^2) = p_l(E) - e^2 \ .$$

In order to determine the value of $\alpha$ it suffices to consider the special case of a real $(2l+1)$-dimensional vector bundle $E$ over $\Delta = S^{4l}$. Let $\eta : S^{4l-1} \to S^{2l}$ be the Whitehead product $[\iota_{2l}, \iota_{2l}]$, where $\iota_{2l}$ is the identity on $S^{2l}$. Its Hopf invariant is $h(\eta) = 2$ ([5]). For some integer $r$ the composition of the map $\rho : S^{4l-1} \to S^{4l-1}$ of degree $r$ with $\eta$ lifts to a map $S^{4l-1} \to SO(2l+1)$, which we take as the characteristic map for the vector bundle $E$. The primary obstruction $c(E)$ in the theorem may be identified with $[\eta \circ \rho] \in \pi_{4l-1}(S^{2l})$. The Pontrijagin class $p_l(E)$ can be interpreted similiarly: The complexified bundle $E \otimes \mathbb{C}$ admits a reduction to a $SU(2l)$ bundle $E'$ and $(-1)^l p_l(E) = c_{2l}(E') = e(E')$ is the same as the degree of the characteristic map of $E'$. Hence we have to compute the degree of the composite

$$S^{4l-1} \xrightarrow{E} SO(2l+1) \hookrightarrow SU(2l+1) \hookleftarrow SU(2l) \to S^{4l-1}$$

This is the same as

$$S^{4l-1} \xrightarrow{\epsilon} ST \overset{\zeta}{\hookrightarrow} L := SU(2l+1)/SU(2l-1) \hookleftarrow S^{4l-1}$$

where $T = TS^{2l}$ is the tangent bundle. The inclusion on the right hand is the inclusion of the fibre in the sphere bundle $L \to S^{4l+1}$ and induces an isomorphism in $H^{4l-1}$. We will show, that $\epsilon$ and $\zeta$ induce multiplication with $4r$ respectively 2 in $H^{4l-1}$. Then $(-1)^l p_l = 8r = 4h(\eta \circ \rho)$ and the theorem is proved.

For the assertion about $\zeta$ observe that the restriction of the bundle $L \to S^{4l+1} \subset \mathbb{C}^{2l+1}$ to $S^{2l} \subset \mathbb{R}^{2l+1}$ is the sphere bundle $S(T \oplus T)$, where $T$ is the tangent bundle of $S^{2l}$ and $\oplus$ denotes the fibre product over $S^{2l}$. The fibre inclusions $S^{4l-1} \hookrightarrow S(T \oplus T)$ and $S^{4l-1} \hookrightarrow L$ induce isomorphisms in $H^{4l-1}$. Thus we have to compute the map $H^{4l-1}(S(T \oplus T)) \to H^{4l-1}(ST)$. By the multiplicativity of Thom classes, we may decompose this map as

$$H^{4l-1}(S(T \oplus T)) \underset{\cong}{\xrightarrow{\partial}} H^0(S^{2l}) \xrightarrow{\cup e(T)} H^{2l}(S^{2l}) \underset{\cong}{\xleftarrow{\partial}} H^{4l-1}(ST)$$

where $\partial$ denotes the connecting homomorphism in the Gysin sequences of $T \oplus T$ and $T$ respectively. Since $e(T) = 2$, $\zeta$ induces multiplication by 2.

For the statement about $\epsilon$ let $r$ be the order of the finite group $\pi_{4l-2}(S^{2l-1})$. Fix a unit vector $p \in S^{2l-1} \subset D^{2l} \subset \mathbb{R}^{2l}$ and denote by $q : S^{2l-1} \to SO(2l)$ the characteristic map of the tangent bundle of $S^{2l}$, thus

$$ST = D^{2l} \times S^{2l-1}/\sim \;\text{ where } (x, v) \sim (p, q(x)v) \text{ for } x \in \partial D^{2l} \tag{1}$$

The map $x \mapsto q(x)p$ on $S^{2l-1}$ has degree 2 ([1]). We represent the Whitehead product $\eta = [\iota_{2l}, \iota_{2l}]$ by the map

$$S^{4l-1} = D^{2l} \times S^{2l-1} \cup S^{2l-1} \times S^{2l-1} \times I \cup S^{2l-1} \times D^{2l} \xrightarrow{pr_1 \cup * \cup pr_2} S^{2l} \tag{2}$$

where $pr_i$ denotes the projection on the $i$th factor and $*$ is the constant map. In order to lift $\eta \circ \rho$ we need to lift the $r$ fold connected sum of (2), which is obtained by glueing (2) in the interior of the cylinders $S^{2l-1} \times S^{2l-1} \times I$. We can lift $\eta \circ \rho$ on the handles $D^{2l} \times S^{2l-1}$ and $S^{2l-1} \times D^{2l}$ by mapping $D^{2l} \times S^{2l-1} \ni (x, y) \mapsto (x, q(y)p) \in ST$ and $S^{2l-1} \times D^{2l} \ni (x, y) \mapsto (y, q(x)p) \in ST$. By the identification (1) this determines a map $\lambda_0 : S^{2l-1} \times S^{2l-1} \times \{0, 1\} \to S^{2l-1}$ mapping $(x, y, 0) \mapsto q(x)q(y)p$ and $(x, y, 1) \mapsto q(y)q(x)p$. Since the bidegree of $\lambda_0$ on both components is $(2, 2)$, we can extend it to $(S^{2l-1} \vee S^{2l-1}) \times I$. Furthermore $S^{2l-1} \times S^{2l-1} \times I$ arises from $S^{2l-1} \times S^{2l-1} \times \{0, 1\} \cup S^{2l-1} \vee S^{2l-1} \times I$ by attaching a cell of dimension $4l - 1$. Its attaching map composed with $\lambda_0$ yields a map $\lambda : S^{4l-2} \to S^{2l-1}$. By assumption, $r\lambda$ is nullhomotopic. Hence we get an extension on the $r$-fold connected sum of the cylinders.

Clearly the lift so constructed has degree $4r$. Eventually substituting $r$ by a multiple we can further lift $\epsilon$ to $SO(2l + 1)$ to realize it by a vector bundle $E$.

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# References


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