GEOMETRIC CHARACTERIZATIONS OF GROMOV HYPERBOLICITY

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Abstract. We prove the equivalence of three different geometric properties of metric-measure spaces with controlled geometry. The first property is the Gromov hyperbolicity of the quasihyperbolic metric. The second is a slice condition and the third is a combination of the Gehring-Hayman property and a separation condition.

0. Introduction.

The purpose of this paper is to show that three (apparently) different geometric properties of Euclidean domains (and more general metric spaces) are in fact equivalent. These properties have various far-reaching analytical consequences. The very different nature of these applications serves as motivation for proving this equivalence.

The first such property is the so-called Gromov hyperbolicity. This expresses the property of a general metric space to be "negatively curved" in the sense of coarse-geometry. The importance of Gromov hyperbolicity is widely appreciated. This notion was introduced by Gromov in the setting of geometric group theory [Gr1], [Gr2], [GhHa] but has played an increasing role in analysis on general metric spaces [BoHeKo], [BoSc] with applications to the Martin boundary, invariant metrics in several complex variables [BaBo1], [BaBo2] and extendability of Lipschitz mappings [Ln].

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The second property is related to a decomposition of a metric space in subdomains with nice geometry called "slices". The slicing condition was used in Euclidean domains for Sobolev imbedding results (see [BuKo1], [BuKo2], [BuOS], [BuSt1]) and for studying quasiconformal images of various classes of domains [BuSt2].

The third property has its origins in geometric function theory. It is a combination of the so-called Gehring-Hayman condition and a separation condition. The former property was proven in [GeHa] for the case of the hyperbolic metric on simply connected plane domains and more recently in [HeRo] for the case of the quasihyperbolic metric in Euclidean domains that are quasiconformally equivalent to uniform domains. This property is the expression of an interplay between the hyperbolic and Euclidean metric. It says that if the domain is nice enough then the hyperbolic geodesics are quasigeodesics for the Euclidean metric as well. This has numerous applications related to lengths of radii or boundary behavior of the conformal and quasiconformal mappings [BoKoRo],[BaBo3] [HeRo]. The Gehring-Hayman condition for more general metric spaces was recently considered by Bonk-Heinonen-Koskela in [BoHeKo] where it was indicated that Gromov hyperbolicity implies this condition and also a separation condition. The starting point of this paper was a question raised in [BoHeKo] about the converse of this implication. Our results include the positive answer to this question. The "slice condition" appears as an intermediate step in proving this implication. A side-benefit of the equivalence is the fact that the Gehring-Hayman and separation conditions are potentially much easier to verify for a given domain than the more complicated Gromov hyperbolicity and slice conditions. In view of [BuKo1], [BuKo2], [BuOS], [BuSt1], [BuSt2] our results show an unexpectedly rich analytic content of Gromov hyperbolicity in Euclidean domains.

To state our results precisely let us introduce some notation and terminology. Given a metric space (X, d), a subset S of X, and points $x, y \in X$, we write $\mathrm{len}_d(S)$ for the

length (meaning 1-dimensional Hausdorff measure $\mathcal{H}^1_d$) of S, and $\mathrm{dia}_d(S)$ for its diameter. Assuming additionally that $x, y \in S$, we use $\Gamma(x, y; S)$ to denote the class of all rectifiable paths $\gamma : [0, t] \to S$ for which $\gamma(0) = x$ and $\gamma(t) = y$. We also write $\Gamma(x, y) = \Gamma(x, y; X)$.

Assuming X is rectifiably connected (i.e. each family $\Gamma(x, y)$ is non-empty), and $\rho : X \to (0, \infty)$ is a continuous density, then (X, d_ρ) is a metric space where d_ρ is defined by the equation

$$d_\rho(x, y) = \inf_{\gamma \in \Gamma(x,y)} \int_\gamma \rho(z)\, |dz|,$$

where $|dz|$ denotes the length element for the metric d. In particular, we denote d_ρ by l in the special case $\rho \equiv 1$; we call l the *inner metric associated with* d. Then (X, d) is said to be a *length space (in the sense of Gromov)* if $l = d$; in this case, d is said to be a *length metric* or *inner metric*. A metric space (X, d) is *geodesic* if every pair of points x, y can be joined by a *geodesic*, that is a path whose length equals the distance from x to y. Of course geodesic spaces are length spaces, but the converse is false (consider Euclidean space with the origin removed). If (X, d) is locally compact and the identity map $(X, d) \to (X, l)$ is continuous (and so a homeomorphism), it is shown in [BoHeKo, Lemma 2.6] that the length of a rectifiable path γ in (X, d_ρ) is given by the line integral

$$\mathrm{len}_\rho(\gamma) = \int_\gamma \rho(z)\, |dz|.$$

Thus every such d_ρ is a length metric.

Let us pause to mention two important special cases of metrics d_ρ. First, if (X, d) is Euclidean space, and $\rho(z) = 2/(1 + |z|^2)$, then d_ρ is the *spherical metric on* $\mathbb{R}^n$, and the *Riemann sphere* $\overline{\mathbb{R}^n}$ is the metric completion of $(\mathbb{R}^n, d_\rho)$ got by adding the single point ∞. Note that $\overline{\mathbb{R}^n}$ is a compact metric space of diameter π. Secondly, if d is the Euclidean (or spherical) metric on an open set in $\mathbb{R}^n$ (or $\overline{\mathbb{R}^n}$), then the associated metric l is called the *inner Euclidean* (or *inner spherical*) *metric*.

Suppose (X, d) is an incomplete rectifiably connected metric space whose metric completion is denoted $(\overline{X}, \overline{d})$. We define the *boundary distance* $d_X : X \to (0, \infty)$ by the equation $d_X(x) = \overline{d}(x, \partial X)$, where the *boundary of* X, ∂X, is simply $\overline{X} \setminus X$. We then define the associated *quasihyperbolic metric* to be d_ρ for $\rho(x) = 1/d_X(x)$. We sometimes write $d(x)$ in place of $d_X(x)$, and we normally denote the quasihyperbolic metric by k. We denote open balls in (X, d) by $B(x, r)$ or $B_d(x, r)$, and closed balls by $\overline{B}(x, r)$ or $\overline{B}_d(x, r)$.

For the rest of this section, we assume that (X, d) is incomplete and rectifiably connected, and that the associated space (X, k) is geodesic. These conditions are quite mild, and certainly true when d is the (inner) Euclidean or (inner) spherical metric on a Euclidean domain. More generally, the geodesic condition follows from the other two if additionally (X, d) is a locally compact length space, as proved in [BoHeKo, Proposition 2.8]. We denote by $[x, y]$ any quasihyperbolic geodesic from x to y; under the above conditions on d, these exist, but are not necessarily unique. It is convenient to use the same notation for paths and their images, allowing us to write for instance $w \in [x, y]$ to mean that w is a point on a geodesic path from x to y.

The following definitions come together with a parameter C which is also assumed to be at least 1; this lower bound is also implicitly assumed whenever these conditions are used later.

Our first definition is that of Gromov hyperbolicity. It says that large triangles in the quasihyperbolic metric are thin. More formally we say that (X, d) is *C-kG-hyperbolic* if the associated space (X, k) is C-Gromov hyperbolic, meaning that

$$\forall\, x, y, z \in X \;\; \forall\, [x, y],\, [x, z],\, [y, z] \;\; \forall\, w \in [x, y] : \quad k(w, [x, z] \cup [z, y]) \le C. \tag{Hyp}$$

Note that this condition, which says that an arbitrary point on one side of a geodesic triangle is at most a bounded distance away from some point on the other two sides, amounts to a type of negative curvature assumption for the space (X, k).

The notion of Gromov hyperbolicity, which makes sense in general metric spaces, was conceived in the setting of geometric group theory [Gr1], [Gr2], [GhHa]. For our case of the quasihyperbolic metric this property was extensively studied in [BoHeKo]. It turns out that the kG-hyperbolicity condition has nice consequences for the geometry of the metric itself. According to the results of this paper one such consequence, is the following "slice condition".

Given $x, y \in X$ and an arc $\gamma \in \Gamma(x, y)$, we write $(x, y; \gamma) \in \mathrm{slice}(C)$ if there exist pairwise disjoint open subsets $\{S_i\}_{i=0}^m$ of X, $m \geq 0$, with $d_i \equiv \mathrm{dia}_d(S_i) < \infty$ such that

$$\forall\ 0 < i < m,\ \ \forall\ \lambda \in \Gamma(x, y): \quad \mathrm{len}_d(\lambda \cap S_i) \geq d_i/C; \tag{Sli_1}$$

$$\forall\ 0 \leq i \leq m: \quad \mathrm{dia}_k(\gamma \cap S_i) \leq C; \tag{Sli_2}$$

$$\mathrm{len}_d\left(\gamma \setminus \bigcup_{i=0}^{m} S_i\right) = 0; \tag{Sli_3}$$

$$B_d(x, d(x)/C) \subset S_0, \quad B_d(y, d(y)/C) \subset S_m. \tag{Sli_4}$$

We say that (X, d) is a C-slice space, or that X satisfies (Sli), if $(x, y; \gamma) \in \mathrm{slice}(C)$ for all $x, y \in X$ satisfying $k(x, y) \geq \log 2$, and all quasihyperbolic geodesics $\gamma \in \Gamma(x, y)$.

The above slice condition is a variant of a condition introduced by Buckley and Koskela [BuKo2]; other variants of which were later used in [BuOS], [BuSt1] and [BuSt2]. In these papers, it was used in particular to obtain a variety of Sobolev- and Trudinger-type imbedding results, mostly in Euclidean domains. We shall prove in Section 4 that for domains on the Riemann sphere, kG-hyperbolicity implies the slice condition and all these other variant slice conditions.

The next property has its origin [GeHa] in the classical complex analysis. However the definition makes sense in general metric spaces. It says that geodesics in quasihyperbolic metric are quasi-minimizing the length in the original underlying metric. More formally, we

say that (X, d) satisfies the *C-Gehring-Hayman condition* if

$$\forall\, x, y \in X \;\; \forall\, [x, y] : \qquad \operatorname{len}_d([x, y]) \le C l(x, y). \tag{GH}$$

It turns out that the Gehring-Hayman condition by itself is not enough to encode all information contained in Gromov hyperbolicity or slice conditions. In addition we need the following separation type condition introduced in [BoHeKo].

We say that (X, d) satisfies a *C-ball-separation condition* if

$$\forall\, x, y \in X \;\; \forall\, [x, y] \;\; \forall\, w \in [x, y] \;\; \forall\, \lambda \in \Gamma(x, y) : \quad \lambda \cap B_d(w, Cd(w)) \neq \emptyset. \tag{BS}$$

The condition says that all paths from x to y are forced to intersect the ball $B_d(w, Cd(w))$. (We use the term "ball-separation" to distinguish this from a later condition where such paths are forced to intersect a set that in general might not be a ball, but has certain additional useful properties.)

We are now ready to state our main theorem for domains in $\overline{\mathbb{R}^n}$. Note that we could replace the inner spherical metric by the more familiar inner Euclidean metric if X is bounded; the constants would then also depend on $\operatorname{dia}(G)$.

Theorem 0.1. *If $X \subsetneq \overline{\mathbb{R}^n}$ is a domain, and d is the inner spherical metric, then the following conditions are equivalent.*

(1) *(X, d) is C_1-kG-hyperbolic.*

(2) *(X, d) is a C_2-slice space.*

(3) *(X, d) satisfies the C_3-Gehring-Hayman and the C_4-ball separation condition.*

The various constants C_i depend only on each other and on n.

Note that all conditions (1)–(3) are all rather general even in the Euclidean context. In particular, we claim that they all hold for all bounded simply-connected planar domains and,

more generally, whenever X is a quasiconformal image of an inner uniform domain. This fact gives a large class of examples for Theorem 0.1 and was one of the initial motivations for our investigations. Let us briefly recollect the results that imply this claim: for (1) it follows by results in [BoHeKo] (specifically Theorem 1.11 and the remarks following it). In the case of (2), the claim for a slightly different slice condition follows from [BuSt2, Theorem 3.1]; for our slice condition, see Section 4. Finally, the claim for (3) follows from the claim for (1) together with the fact that (1) implies (3), which is proven in [BoHeKo, Section 7].

Moreover, Theorem 0.1 is merely an easy-to-state special case of what we actually prove in the remainder of the paper. We establish the individual implications mentioned above for varying classes of metric spaces, which include both the inner spherical case in each instance. In fact, we shall see that (1)–(3) are equivalent whenever (X, d) is an incomplete locally compact length space space satisfying the following two conditions:

(a) (X, d) can be equipped with a Borel measure μ which makes it into an upper regular, locally regular, locally Loewner space;

(b) The associated space (X, k) is roughly starlike.

The first condition is always true in the inner spherical or inner Euclidean cases, with μ being Lebesgue measure. The second holds for inner spherical domains if the quasihyperbolic metric is Gromov hyperbolic. In Section 4, we shall discuss more general situations in which conditions (a) and (b) hold. In particular we prove that rough starlikeness of (X, k) holds in the presence of Gromov hyperbolicity in most instances.

Our plan of attack is as follows. After some preliminary material in the next section, we show that (2) implies (3) in Section 2, and also that (3) implies a variant condition, which we call (3a), consisting of Gehring-Hayman and a more useful type of separation condition than ball-separation. We discuss rough starlikeness of (X, k) in Section 3, and prove that

(1) implies (2) in Section 4. In sections 5 and 6 we show that (3a) implies (1). Finally, we develop some fundamental results related to Loewner spaces in the appendix.

Before ending this section, let us comment briefly on the various implications in Theorem 0.1. One of the motivations for this research is the paper of Bonk, Heinonen, and Koskela [BoHeKo], where it is shown that (1) implies (3), and conjectured that (a formally stronger, but actually equivalent, variant of) condition (3) implies (1). Our theorem includes a different proof of the first implication, as well as a proof of this conjecture. Finally, we remark that (3) is potentially much easier to verify than the seemingly more complicated conditions (1) and (2).

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1. Notation and terminology

We gather here a list of some additional notation and terminology that is used throughout the rest of the paper.

If a metric space (X, d) is incomplete, rectifiably connected, and locally compact, and if the identity map $(X, d) \to (X, l)$ is continuous (as is certainly true if (X, d) is a length space), then we say that (X, d) is *minimally nice*. The significance of minimal niceness is that it guarantees that (X, k) is complete, proper, geodesic, and homeomorphic to (X, d); see [BoHeKo, Proposition 2.8]. Recall that a proper metric space is one in which all closed balls are compact.

Many of the results in this paper require at least that (X, d) is minimally nice, but let us merely assume for the remainder of this section that (X, d) is an incomplete and rectifiably connected, and that the associated space (X, k) is geodesic.

If $\gamma_1 \in \Gamma(x, y; X)$ and $\gamma_2 \in \Gamma(y, z; X)$, for some $x, y, z \in X$, then $\gamma_1 + \gamma_2$ denotes the joined path which first traverses γ_1, and then γ_2. More generally, $\sum_{i=0}^{j} \gamma_j$ is a joined path which traverses the paths γ_i in their natural order. Similarly, $-\gamma$ is the reverse parametrization of γ, and $\gamma_1 - \gamma_2$ is $\gamma_1 + (-\gamma_2)$.

When we say that a point x lies on a path γ, we have in mind that $x = \gamma(s)$ for some specific value of s, which is assumed to be the same wherever x is used. This allows us to use $\gamma[x, y]$ to denote the segment $\gamma|_{[s,t]}$ of the (possibly non-injective path γ, whenever $x = \gamma(s)$ and $y = \gamma(t)$ for some $s < t$; recall that $[x, y]$ by itself denotes an unspecified quasihyperbolic geodesic segment. We use other standard interval notation analogously: for instance, $(x, y) = [x, y] \setminus \{x, y\}$. Within a proof, all instances of $[x, y]$ refer to the same quasihyperbolic geodesic path. Furthermore, whenever we select points $u, v \in [x, y]$, then $[u, v]$ always denotes the obvious subgeodesics of $[x, y]$ or $[y, x]$ (rather than an arbitrary geodesic between these points). We write $[a, b] \subset [c, d]$ if $[a, b]$ is a subgeodesic of $[c, d]$ *that preserves orientation.* Given $C \geq 1$, we say that a path γ is a (C, C')*-rough quasigeodesic* if $\mathrm{len}_k(\gamma[x, y]) \leq Ck(x, y) + C'$ whenever x, y lie on γ. If we assume that $C' = 0$, we speak simply of a C*-quasigeodesic.* For the rest of this paragraph, let us use $*$*-geodesic* to indicate any one of "geodesic", "quasigeodesic", and "rough quasigeodesic". Suppose γ is a $*$-geodesic, and that α is a reparametrization of γ with the property that $\mathrm{len}_k(\alpha[s, t]) = t - s$ for any numbers $s < t$, $s, t \in I$, where I denotes the domain of α. Then γ is $*$*-geodesic segment* if I is of finite length, γ is a $*$*-geodesic ray* if I has the form $[t, \infty)$ for some $t \in \mathbb{R}$, and γ is a $*$*-geodesic line* if $I = \mathbb{R}$. Note that, although we sometimes use the qualifier "quasihyperbolic" for emphasis, terms of the form "$*$-geodesic" refer by default to the quasihyperbolic metric in the rest of this paper; if more than one such metric is being used, we use notation such as a $(C; k)$*-geodesic* to specify which one we mean.

We denote the maximum and minimum of a pair of numbers s, t by $s \vee t$ and $s \wedge t$

respectively. Finally we note that we frequently drop constants and other parameters in many pieces of notation whenever they are unimportant or understood. For instance we may simply talk about kG-hyperbolic and slice spaces.

2. From Slice to Gehring-Hayman and Separation

In this section, we prove that the slice condition (Sli) implies the Gehring-Hayman condition (GH) and ball-separation condition (BS). We also prove that (GH) and (BS) together imply another more useful separation condition (Sep). We begin with some necessary definitions.

Given $K \geq 1$, we say that a metric space (X, d) lies in the class $QCX(K)$ if it satisfies the following *K-quasiconvexity* condition:

$$\forall\, x \in X,\ \ \forall\, 0 < r,\ \ \forall\, a, b \in \overline{B}_d(x, r/2): \quad l(a, b) < Kr. \tag{2.1}$$

Here as usual, l denotes the arclength metric associated with d. More generally, we say that (X, d) lies in the class $QCX_{\mathrm{loc}}(K)$ if it satisfies the *local K-quasiconvexity* condition

$$\forall\, x \in X,\ \ \forall\, 0 < r < d(x)/K,\ \ \forall\, a, b \in \overline{B}_d(x, r/2): \quad l(a, b) < Kr. \tag{2.2}$$

Of course length spaces are trivially 1-quasiconvex, and proper subdomains of $\mathbb{R}^n$ or $\overline{\mathbb{R}^n}$ with respect to the inner Euclidean or inner spherical metrics, respectively, are trivially in $QCX(1)$. Similarly proper subdomains of $\mathbb{R}^n$ or $\overline{\mathbb{R}^n}$ with respect to the Euclidean or spherical metrics, respectively, lie in $QCX_{\mathrm{loc}}(1)$. The last example is a special case of the easily proven fact that if (X, d) is a minimally nice subspace of a length space (Y, d'), and X is open in Y, then (X, d) lies in $QCX_{\mathrm{loc}}(1)$.

We now state the first main theorem of this section.

Theorem 2.3. *Suppose that $(X,d) \in QCX_{\mathrm{loc}}(K)$ is minimally nice and satisfies a C-slice condition. Then (X,d) satisfies a C'-Gehring-Hayman condition and a C'-ball separation condition, for some $C' = C'(C,K)$.*

When we prove that Gehring-Hayman plus separation implies Gromov-hyperbolicity, it suits us to use a different type of separation condition than ball-separation. We now define this other separation condition and later show that it is implied by Gehring-Hayman plus ball-separation.

Suppose $C \geq 1$. We say that (X,d) satisfies a *C-separation condition*, or simply that it satisfies condition (Sep), if for all $x, y \in X$ with $k(x,y) \geq \log 2$, every quasihyperbolic geodesic $[x,y]$, and every $w \in [x,y]$, there exists an open set $S_w \equiv S_w^{[x,y]} \subset X$ such that:

$$w \in S_w \subset B_d(w, Cd(w)); \tag{Sep$_1$}$$

$$\forall\, \lambda \in \Gamma(x,y): \quad \mathrm{len}_d(\lambda \cap S_w) \geq d(w)/C; \tag{Sep$_2$}$$

$$\mathrm{dia}_k([x,y] \cap S_w) \leq \log(C+1). \tag{Sep$_3$}$$

We also define $S_w = B(x, d(x)/2)$ in the case $k(x,y) < \log 2$. In this case, (Sep$_1$) and (Sep$_3$) are still valid with $C = 1$, and $w \in S_w$, but of course (Sep$_2$) may fail.

Theorem 2.4. *Suppose that a minimally nice metric space (X,d) satisfies a C-ball separation condition and that (X,d) satisfies a C'-Gehring-Hayman condition. Then (X,d) also satisfies a C_1-separation condition, for some $C_1 = C_1(C,C')$.*

Our first aim is to prove the Gehring-Hayman part of Theorem 2.3. The following lemma is crucial in the proof and it will also be used later on.

Lemma 2.5. *Suppose that (X,d) is an incomplete rectifiably connected space. Let u, v be points on a quasihyperbolic geodesic $[z,w]$ between distinct points $z, w \in X$, let S be a positive*

length subset of $[z,w]$*, and let* $c \equiv k(z,w)$*. Then*

$$v \in \overline{B}_d(u,(e^c-1)d(u)); \tag{2.6}$$

$$(1-e^{-c})d(u) \le \operatorname{len}_d([z,w]) \le (e^c-1)d(u); \tag{2.7}$$

$$e^{-c}d(u) \le d(v) \le e^c d(u); \tag{2.8}$$

$$e^{-c}d(u) \le \frac{\operatorname{len}_d(S)}{\operatorname{len}_k(S)} \le e^c d(u). \tag{2.9}$$

Proof. Let $\lambda : [0,L] \to X$ be the parametrization of $[z,w]$ by d-arclength. Then $d(\lambda(t)) \le d(z)+t$, and so

$$\operatorname{len}_k(\lambda) = c \ge \int_0^L \frac{dt}{d(z)+t} = \log(1+L/d(z)),$$

thus giving the second inequality in (2.7) in the case $u = z$. The first inequality in (2.7) for $u = z$ follows in a similar manner as the second by instead using the estimate $d(\lambda(t)) \ge d(z)-t$. These inequalities for general u follow by similar estimates applied separately to $[u,w]$ and $[u,z]$, as the reader can readily verify. The containment (2.6) is an immediate corollary of the second inequality in (2.7), as is the second inequality in (2.8); the first inequality then follows by symmetry. Finally, (2.8) implies (2.9). □

Theorem 2.10. *Suppose that* $(X,d) \in QCX_{\mathrm{loc}}(K)$ *is minimally nice and satisfies a* C*-slice condition. Then* (X,d) *satisfies a* C'*-Gehring-Hayman condition for some* $C' = C'(C,K)$*.*

Proof. Let us fix points $x \ne y \in X$, and a quasihyperbolic geodesic $[x,y]$ and write $L = \operatorname{len}_d([x,y])$. Without loss of generality, we assume that $d(y) \le d(x)$. Suppose first that $k(x,y) < \log 2$. Using the elementary estimate $e^t - 1 \le 2t$ for all $0 \le t \le \log 2$, and the second inequality in (2.7), we see that $L \le 2k(x,y)d(y)$. Suppose $\lambda \in \Gamma(x,y)$ is a competing path, with $L' = \operatorname{len}_d(\lambda)$. If $L' \ge d(y)/2$, we get a $(4\log 2)$-Gehring-Hayman inequality for x,y. If $L' < d(y)/2$, then $\lambda \subset B_d(y,d(y)/2)$, and so $L' \ge \operatorname{len}_k(\lambda)d(x)/2 \ge k(x,y)d(x)/2$.

Putting this together with the upper bound for L, we get a 4-Gehring-Hayman inequality for x, y. We may therefore suppose that $k(x,y) \geq \log 2$.

Suppose that $d(x,y) < d(x)/4K$, and so $l(x,y) < d(x)/4$. Taking $\lambda \in \Gamma(x,y)$ so that $\mathrm{len}_d(\lambda) < d(x)/4$, we see that $d(u) > 3d(x)/4$ for all $u \in \lambda$, and so $\mathrm{len}_k(\lambda) \leq 4l(x,y)/3d(x) < 1/3$. This contradicts the fact that $k(x,y) \geq \log 2 > 1/3$. Thus $d(x,y) \geq d(x)/4K$.

Let $\{S_i, d_i\}_{i=0}^m$ be the slices and slice diameters associated with $x, y, [x,y]$. Writing $L_i = \mathrm{len}_d([x,y] \cap \overline{S}_i)$, we get upper bounds for the numbers L_i in the case of *small* and *large* slices by separate arguments. We first consider small slices, meaning slices S_i for which $d_i < d(z_i)/3K$ for some point $z_i \in [x,y] \cap S_i$. Fixing one such small slice S_i, we let a, b be the first and last points of intersection of $[x,y]$ with $\overline{S}_i$, and let $\lambda \in \Gamma(a,b)$ be such that $\mathrm{len}_d(\lambda) < Kd(a,b)$. Thus $\mathrm{len}_d(\lambda) < d(z_i)/3$, and so $d(w) > d(z_i)[1-1/(3K)-1/3] \geq d(z_i)/3$ for all points w on λ. Thus

$$\mathrm{len}_k[a,b] < \frac{3\,\mathrm{len}_d(\lambda)}{d(z_i)} < \frac{3Kd_i}{d(z_i)}. \tag{2.11}$$

The rightmost quantity in (2.11) is less than 1, so (2.8) implies that $d(w) < ed(z_i)$ for all $w \in [a,b] \cap \overline{S}_i$. Thus

$$L_i \leq \mathrm{len}_d[a,b] \leq ed(z_i)\,\mathrm{len}_k[a,b] \leq 3eKd_i. \tag{2.12}$$

We next turn to the big slices, i.e., slices S_i such that $d_i \geq d(z)/3K$ for all $z \in [x,y] \cap S_i$. Consequently, from (Sli_2) we see that for all $0 < i < m$,

$$L_i \leq 3Kd_i\,\mathrm{len}_k(\gamma \cap S_i) \leq 3CKd_i. \tag{2.13}$$

As for the case $i = 0$, the quasihyperbolic length of the segment of $[x,y]$ from x to its last intersection point with $\overline{S_0}$ is at most C according to (Sli_2). Using (2.7) from Lemma 2.5,

we deduce that $L_0 \le (\exp(C)-1)d(x)$. Similarly $L_m \le (\exp(C)-1)d(y)$. Combining these last two estimates with (2.12), (2.13), and (Sli_3), we deduce that

$$L \le C_1 \left(d(x) + d(y) + \sum_{i=1}^{m-1} d_i \right), \tag{2.14}$$

where $C_1 = (\exp(C)-1) \vee (3(e \vee C)K)$. On the other hand, ($\mathrm{Sli}_1$) implies that

$$\forall\, \lambda \in \Gamma(x,y),\, 0 < i < m : \qquad \mathrm{len}_d(\lambda \cap S_i) \ge d_i/C,$$

while (Sli_4) and the fact that $d(x,y) > d(x)/4K$ together imply that

$$\mathrm{len}_d(\lambda \cap S_0) \ \ge d(x)/(4K \vee C),$$
$$\mathrm{len}_d(\lambda \cap S_m) \ge d(y)/(4K \vee C).$$

Thus

$$\forall\, \lambda \in \Gamma(x,y) : \qquad \mathrm{len}_d(\lambda) \ge \sum_{i=0}^{m} \mathrm{len}_d(\lambda \cap S_i) \ge C_2^{-1} \left(d(x) + d(y) + \sum_{i=1}^{m-1} d_i \right),$$

where $C_2 = 8K \vee 2C$. Combining this inequality with (2.14), we deduce a C'-Gehring-Hayman condition with $C' = C_1 C_2$. □

Theorem 2.10 proves one part of Theorem 2.3. The other part follows from the following theorem.

Theorem 2.15. *Suppose (X,d) is a minimally nice metric space that satisfies a C-slice condition. Then (X,d) satisfies a C_1-ball separation condition, where $C_1 = 2 \vee (C^2 \exp(C))$.*

Proof. Fix two distinct points $x, y \in X$, and let $\gamma = [x,y] : [0,L] \to X$ be a geodesic for x, y, and let $w = \gamma(t)$ be a point on γ. Without loss of generality, we assume that $d(y) \le d(x)$. If $k(x,y) < \log 2$ and $u \in [x,y]$, then by (2.6), we have $x, y \in B_d(u, 2d(u))$, as desired. We may suppose therefore that $k(x,y) \ge \log 2$.

We fix a point $w = \gamma(t)$, and let $\{S_i, d_i\}_{i=0}^m$ be the slices and slice diameters associated with $x, y, [x, y]$. Since there are only finitely many slices S_i, we can choose an index j and numbers $t_n \in [0, L]$ such that $\lim_{n\to\infty} t_n = t$ and $w_n = \gamma(t_n) \in S_j$.

Let $\gamma_j \equiv \gamma \cap S_j$. By ($\mathrm{Sli}_2$) and (2.9), it readily follows that

$$\mathrm{len}_d(\gamma_j)/d(z) \le C\exp(C), \qquad z \in \gamma_j. \tag{2.16}$$

Combining this with (Sli_1), we deduce that if $0 < j < m$, then $d_j \le C^2\exp(C)d(z)$. Taking $z = w_n$ and passing to a limit, we see that $d_j \le C_1 d(w)$. Thus $\overline{S_j} \subset B(w, C_1 d(w))$, and the C_1-ball separation condition for the data $\{x, y, [x, y], w\}$ follows from (Sli_1). The endpoint cases $j \in \{0, m\}$ are even easier since then γ_j must include either x or y, and so a (C_1/C^2)-ball separation condition follows for the data. □

Proof of Theorem 2.4. To prove the first statement suppose we are given separation data x, y, $[x, y]$, and $w \in [x, y]$, with $k(x, y) > \log 2$. It follows from (2.7) that $\mathrm{len}_d([x, y]) \ge d(w)/2$. Now if we were to choose S_w to be $B_d(w, 2CC'd(w))$, then (Sep_1) and (Sep_2) would readily follow, but (Sep_3) would typically fail.

We instead define the separation set S_w by

$$S_w = B_d(w, 4CC'd(w)) \setminus \big((X \setminus B_k(w, \log 2)) \cap [x, y] \big).$$

We claim that S_w satisfies the C_1-separation condition, where $C_1 = 4CC'$. Clearly (Sep_1) holds, and (Sep_3) is just as obvious once we note that $\log(C_1 + 1) > 2\log 2$. However the fact that S_w satisfies (Sep_2) is far from obvious, so let us fix an arbitrary path $\lambda \in \Gamma(x, y)$, and suppose for the sake of contradiction that $\mathrm{len}_d(A) < d(w)/C_1$, where $A = \lambda \cap S_w$. We may as well assume that $\lambda : [0, L] \to X$ is parametrized by arclength.

We define w_- to be the point in $[x, w]$ with $k(w_-, w) = (\log 2) \wedge k(x, w)$, and w_+ to be the point in $[w, y]$ with $k(w_+, w) = (\log 2) \wedge k(w, y)$, so that the part of $[x, y]$ that lies in S_w

is (w_-, w_+) plus perhaps one or both endpoints of this geodesic segment (an endpoint e is included precisely when $k(e, w) < \log 2$, and so $e \in \{x, y\}$).

Since $k(w_-, w_+) \geq \log 2$, $\operatorname{len}_d([w_-, w_+]) \geq d(w)/2$. With this notation, A is the set of points in $\lambda \cap B_d(w, 4CC'd(w))$ that lie in neither $[x, w_-]$ nor $[w_+, y]$.

Also let $t_- \in [0, L)$ be defined by the equation

$$t_- = \sup\{t \in [0, L] \mid \lambda(t) \in [x, w_-]\},$$

let $t_+ \in [t_-, L]$ be defined by

$$t_+ = \inf\{t \in [t_-, L] \mid \lambda(t) \in [w_+, y]\},$$

so that $v_- \equiv \lambda(t_-) \in [x, w_-]$, $v_+ \equiv \lambda(t_+) \in [w_+, y]$. It is clear that

$$\{\lambda(t) : t \in (t_-, t_+)\} \cap B_d(w, 4CC'd(w)) \subseteq A.$$

Set $\tilde{\lambda} := \{\lambda(t) : t \in (t_-, t_+)$. Suppose first that $\tilde{\lambda} \subseteq B_d(w, 4CC'd(w))$. In this case $\tilde{\lambda} \subseteq A$. Gehring-Hayman implies that

$$\begin{aligned} \operatorname{len}_d \tilde{\lambda} = t_+ - t_- \geq \frac{1}{C'} \operatorname{len}_d([\lambda(t_-), \lambda(t_+)]) \geq \\ \geq \frac{1}{C'} \operatorname{len}_d([w_-, w_+]) \geq \frac{d(w)}{2C'} > d(w)/C_1. \end{aligned} \tag{2.17}$$

This leads to a contradiction with $\operatorname{len}_d(A) < d(w)/C_1$.

In the second case, when $\tilde{\lambda}$ has a point outside of $B_d(w, 4CC'd(w))$ we apply (BS) for the data $\{v_-, v_+, w \in [v_-, v_+]\}$. This implies that $\tilde{\lambda}$ must intersect the smaller ball $B_d(w, Cd(w))$ and so $\tilde{\lambda}$ has a subarc $\beta \subseteq B_d(w, 4CC'd(w))$ such that $\operatorname{len}_d \beta \geq (4CC' - C)d(w)$. Since $\beta \subseteq A$ we obtain again a contradiction to $\operatorname{len}_d(A) < d(w)/C_1$. □

One might wonder if there is an implication one way or the other between the Gehring-Hayman and ball-separation conditions, at least in the setting of bounded Euclidean domains. We do not know whether ball-separation (or separation) implies Gehring-Hayman,

but we now give an example of a quasiconvex planar domain G for which the (inner) Euclidean metric satisfies a Gehring-Hayman condition, but not a ball-separation condition. To construct G, we "weld" the square $(-1,1)^2$ to the sequence of squares $Q_j = (a_j, a_j + l_j) \times (1, 1 + l_j,)$, where $a_j = 2^{-j}$, and $l_j = 4^{-j}$, $j \in \mathbb{N}$, and the welding process consists of adding all points on the interval whose coordinates are of the form $(x, 0)$ for some $a_j < x < a_j + l_j$, except for all points of the form $(a_j + il_j/j)$, $i \in \mathbb{N}$, $i < j$. Taking z_j to be the center of Q_j, the large number of passages forces any C-ball-separation condition to fail in the vicinity of the jth weld for any geodesic from the origin to z_j whenever $j \geq j_0 = j_0(C)$. A case analysis shows that (G, d) satisfies a Gehring-Hayman condition.

Let us finish this section by commenting on a minor difference between the ball-separation condition we are using and the separation condition defined at the end of Section 7 of [BoHeKo]. The latter is the formally stronger condition

$$\forall\, x, y \in X \ \ \forall\, [x, y] \ \ \forall\, w \in [x, y],\ u \in [x, w), v \in (w, y] \ \ \forall\, \lambda \in \Gamma(u, v) : \\ \lambda \cap B_d(w, Cd(w)) \neq \emptyset. \tag{SBS}$$

In fact these conditions are equivalent. To see this, suppose that (X, d) satisfies (BS) and let x, y, w, u, v, λ be as in (SBS). Since the arc $[u, v] \subseteq [x, y]$ connecting $u, v \in X$ is itself a geodesic, we can use (BS) with data $\{u, v, w \in [u, v]\}$ to deduce that $\lambda \cap B_d(w, Cd(w)) \neq \emptyset$, as required.

3. Gromov hyperbolicity implies Rough Starlikeness

In the next section we prove that, in a general class of metric spaces, kG-hyperbolicity implies a slice condition. This class of spaces include bounded domains in complete metric measure spaces that are Q-regular and Q-Loewner. This includes a large variety of non-Euclidean spaces such as Carnot groups, sub-Riemannian and Riemannian manifolds which

all satisfy the Loewner condition. We refer to [HeKo] for a detailed exposition on analysis in Loewner spaces. We recall the definition and basic properties in the Appendix.

In proving that Gromov hyperbolicity implies a slice condition the property that (X, k) is roughly starlike also plays a crucial role. This property has not been treated in as much detail as the Loewner property. Therefore we pause and devote this section to this condition. We show that rough starlikeness follows essentially from kG-hyperbolicity in a rather general setting.

It is shown in Lemma 7.8 of [BoHeKo] that if a proper subdomain X of $\overline{\mathbb{R}^n}$ is kG-hyperbolic with respect to the inner spherical metric, then (X, k) is roughly starlike with respect to any basepoint w that maximizes the spherical distance to ∂X. In this section, our main theorem generalizes that result, in the process providing a large class of spaces in which this implication is valid, although we delay until the end of the next section a discussion of what spaces are covered by the theorem.

If (X, d) is a minimally nice metric space, then (X, k) is said to be *K-roughly starlike*, $K \geq 0$, if there exists a basepoint w such that every point in the space is within a distance K of some geodesic ray emanating from w.

Throughout the rest of this section, we define $A(z, r, C, C')$ to be the annulus $\overline{B}_{\overline{d}}(z, Cr) \setminus B_{\overline{d}}(z, r/C')$, whenever $z \in \overline{X}$, $r > 0$, $C, C' \geq 1$, and $(\overline{X}, \overline{d})$ is the metric completion of a metric space (X, d). As a special case, we write $A(z, r, C) = A(z, r, C, C)$.

Given $C \geq 1$, we say that (X, d) is in the class $QL(C)$ if it is minimally nice and it satisfies the following three conditions:

(a) If $x, x' \in X$, $z \in \partial X$, $x, x' \in A(z, r, 1, 8C)$, and $A(z, r, 8C) \subset X$, then there exists a path $\lambda \in \Gamma(x, x'; A(z, r, 8C^2))$ such that $\operatorname{len}_d(\lambda) \leq 2Cr$.

(b) If $x \in X$, $z \in \partial X$, $x \in A(z, r, 2)$, and $A(z, r, 4C) \cap \partial X \neq \emptyset$, then there exists a point $y \in A(z, r, 4C) \cap \partial X$ and a path $\eta \in \Gamma(x, y; A(z, r, 8C^2))$ such that $\operatorname{len}_{\overline{d}}(\eta) \leq 6C^2 r$.

(c) For each $x \in X$, there exists a path $\nu : [0, s] \to X$ with $\mathrm{len}_d(\nu) \leq Cd(x)$, $\nu(0) = x$, and $d(\nu(t)) > d(\nu(s)) = d(x)/2$ for all $0 \leq t < s$.

We chose the notation $QL(C)$ because the three parts of this condition are combinations of special cases of two well-known conditions that are usually termed quasiconvexity and LLC-2. Let us also write $QL(C)$(a), $QL(C)$(b), $QL(C)$(c), to refer to the three individual conditions above.

It is straightforward to verify that (inner) Euclidean and (inner) spherical domains are in the class $QL(1)$. It is similarly easy to prove from Remark 7.2 that a bounded, upper Q-regular, Q-Loewner minimally nice metric measure space is in $QL(C)$, where C depends quantitatively on the assumptions (see the proof of Proposition 4.19 for more details).

We are now ready to state the main theorem of this section.

Theorem 3.1. *Let $(X, d) \in QL(C)$ be C'-kG-hyperbolic and $M \equiv \sup_{x \in X} d(x) < \infty$. Then there exists $K = K(C, C')$ such that (X, k) is K-roughly starlike with respect to any basepoint w for which $d(w) > M/2$.*

The following lemma, which actually only uses minimal niceness and part (c) of the definition of $QL(C)$ is the main tool in our proof of Theorem 3.1.

Lemma 3.2. *Suppose that $(X, d) \in QL(C)$. Then for each $u_0 \in X$, there exists a rough quasigeodesic ray $\gamma : [0, L) \to X$ emanating from u_0, and parametrized by d-arclength such that $d(\gamma(L-t)) \geq t/4C$ for all $0 < t \leq L$, $\gamma(t)$ d-converges to some point $z \in \partial X$ as $t \to L^-$, and*

$$\mathrm{len}_k(\gamma[u, v]) \leq 2C[1 + \log_2(3C) + |\log_2(d(u)/d(v))|], \quad \textit{for all points } u, v \textit{ lying on } \gamma. \tag{3.3}$$

Proof. Suppose $u_0 \in X$ is given. We define γ by stringing together a sequence of paths $\gamma_i \in \Gamma(u_{i-1}, u_i; X)$, $i \in \mathbb{N}$, where $d(u_i) = 2^{-i} d(u_0)$, $i \in \mathbb{N}$. We define γ_1 to be the path

ν given by $QL(C)$(c) for $x = u_0$, and let $u_1 = \nu(s)$. Inductively, for each $i \in \mathbb{N}$, let ν and s be the data given by $QL(C)$(c) for $x = u_{i-1}$. We then define $\gamma_i : [0, l_i] \to X$ to be the d-arclength reparametrization of ν, and u_i to be $\nu(s)$; note that $l_i \le 2^{-i+1}Cd(u_0)$. Writing $L_i = \sum_{j=1}^{i} l_j$, $i \in \mathbb{N}$, and $L = \sum_{j=1}^{\infty} l_j$, we define the path $\gamma : [0, L) \to \overline{X}$ by the equations $\gamma(t) = \gamma_i(t - L_{i-1})$, for all $L_{i-1} \le t \le L_i$. Then γ is parametrized by d-arclength, $L \le 4Cd(u_0)$, and clearly $\gamma(t)$ d-approaches some $z \in \partial X$ as $t \to L^-$. The estimate $d(\gamma(L - t)) \ge t/4C$ is also clear.

Suppose $u = \gamma(s)$ and $v = \gamma(t)$, where $s < t$, $L_{i-1} \le s \le L_i$, and $L_{j-1} \le t \le L_j$, for some $i \le j \in \mathbb{N}$. Since all points in γ_i are of distance at least $2^{-i}d(u_0)$ from ∂X, and $l_i \le 2^{-i+1}Cd(u_0)$, it follows that $\mathrm{len}_k(\gamma_i) \le 2C$, and so $\mathrm{len}_k(\gamma[u, v]) \le 2C(j - i + 1)$. On the other hand,

$$2^{j-1}\frac{1}{3C} \le \frac{d(u)}{d(v)} \le 2^{j-i}3C,$$

and so $j - i \le |\log(d(u)/d(v))| + \log_2(3C)$. The estimate (3.3) is now clear.

We now record the following elementary but useful estimate, whose simple proof we leave to the reader.

$$k(u, v) \ge \log\left(1 + \frac{l(u, v)}{d(u) \wedge d(v)}\right). \tag{3.4}$$

This implies the inequality $k(u, v) \ge |\log(d(u)/d(v))|$, which together with (3.3) proves that γ is a rough quasigeodesic. □

Let us pause to recap some basics of Gromov hyperbolicity in the context of a quasihyperbolic space (X, k) which we assume to be Gromov hyperbolic, proper, and geodesic; recall that the latter two assumptions follow from minimal niceness of (X, d). For proofs of the results in the next two paragraphs and much more on hyperbolicity, we refer the reader to [GhHa], [CoDePa]; see also [BoSc] and [BoHeKo].

We define the *Gromov boundary of* X, denoted $\partial_G X$, to be the set of all geodesic rays emanating from some fixed point $w \in X$, modulo the equivalence relation given by taking any two rays which lie within a bounded Hausdorff distance of each other as equivalent; this definition is independent of w. Morally, $\partial_G X$ consists of the "points at infinity" of (X, k). It may or may not be homeomorphic to the boundary ∂X defined previously. For example if X is a Euclidean slit disk (e.g., the unit disk in the complex plane less the positive real axis), then $\partial_G X$ is not homeomorphic to ∂X, but rather to the unit circle; each point on the positive real axis corresponds to two points in $\partial_G X$.

Proof of Theorem 3.1. We assume without loss of generality that $C \geq 2$. Suppose $x \in X$. Let $\gamma : [0, L) \to X$ be a rough quasigeodesic ray emanating from x, as in Lemma 3.2, such that $\gamma(t)$ d-converges to $z \in \partial X$. We define the path $\gamma_1 : (0, L] \to X$ by the equation $\gamma_1(t) = \gamma(L - t)$, for all $0 < t \leq L$. Letting $r = d(z, x)$, we certainly have $x \in A(z, r, 2)$.

Suppose $A(z, r, 2C) \cap \partial X$ is non-empty. Let y and η be as in part (b) of the $QL(C)$ definition. As we move along move along η from x to y, let x' be the first point with $d(x') = r/(32C^3)$, and let γ_2 be the segment of η from x to x'. Finally, let $\gamma_3 : [0, L) \to X$ be a rough quasigeodesic ray from x' that converges to some point $z' \in \partial X$ and satisfies the conclusions of Lemma 3.2.

We claim that γ is a rough quasigeodesic line. Since γ_1 and γ_3 are rough quasigeodesic rays and

$$\mathrm{len}_k(\gamma_2) \leq \frac{\mathrm{len}_d(\lambda_2)}{r/(32C^3)} \leq 192C^5,$$

it suffices to show the quasigeodesic property for a pair of points u on γ_1 and v on γ_3. Since $\mathrm{len}_k(\gamma[u, v]) \leq \mathrm{len}_k(\gamma_1[u, x]) + 192C^5 + \mathrm{len}_k(\gamma_3[x', v])$, it suffices to bound the first and last term in this sum by a constant plus $k(u, v)$.

We first consider $K_1 \equiv \mathrm{len}_k(\gamma_1[u,x])$. Note that

$$d(z,v) \geq d(z,x') - d(x',v) \geq \frac{r}{4C^2} - \frac{r}{8C^2} = r/(8C^2).$$

If $d(u,z) < r/(16C^2)$, then $d(u,v) > r/(16C^2)$ and so by (3.4) we have

$$k(u,v) \geq \log\left(1 + \frac{r/(16C^2)}{d(u) \wedge d(v)}\right). \tag{3.5}$$

But by (3.3),

$$K_1 \leq 2C\left(1 + \log_2(3C) + \left|\log\frac{d(x)}{d(u)}\right|\right) \leq 2C\left(1 + \log_2(3C) + \log\frac{r}{d(u)}\right).$$

Putting together the last two estimates, we get the required bound for K_1. On the other hand if $d(u,z) \geq r/(16C^2)$, then $d(u)/d(x) \leq 4C$ by Lemma 3.2. Moreover $d(x) \leq d(x,z) = r$ and again by Lemma 3.2

$$d(u) \geq \frac{\mathrm{len}_d(\gamma[u,z])}{4C} \geq \frac{d(u,z)}{4C} \geq \frac{r}{64C^3}.$$

This implies that $d(x)/d(u) \leq 64C^3$. So by (3.3), we see that

$$K_1 \leq 2C(7 + \log_2(3C) + 3\log_2 C).$$

We now consider $K_3 \equiv \mathrm{len}_k(\gamma_3[x',v])$. We first use (3.3) to get the inequality

$$K_3 \leq 2C\left(1 + \left|\log\frac{d(x')}{d(v)}\right|\right) \leq 2C\left(1 + \left|\log\frac{r}{32C^3 d(v)}\right|\right). \tag{3.6}$$

If $d(u,z) < r/(16C^2)$, the desired bound follows by combining (3.6) and (3.5). If instead $d(u,z) \geq r/(16C^2)$, and so $r/(16C^2) \geq d(u) \geq r/(64C^3)$, then the desired bound follows by combining (3.6) with the elementary estimate $k(u,v) \geq |\log(d(u)/d(v))|$ and the upper bound $d(v) \leq (2C+1)d(x') < r/(8C^2)$.

But rough quasigeodesics lines are never far from honest geodesic lines in a Gromov hyperbolic space according to [CoDePa, Theorem 3.3.1]. So x lies within a bounded quasihyperbolic distance of a geodesic line (a,b) joining two points $a,b \in \partial_G X$. Let ρ_a, ρ_b be geodesic rays from w to a and b, respectively. Now the "thin triangles" property (Hyp) is valid for arbitrary points $x,y,z \in X \cup \partial_G X$ and any associated geodesics, as long as we replace the hyperbolicity constant C by $24C$; see [CoDePa, 2.2.2]. Consequently x lies within a bounded distance of either ρ_a or ρ_b.

Finally, we must consider the case where there is no point $y \in \partial X$ which also lies in the annulus $A(z,r,2C)$. The part of γ_1 that is near z is within a bounded Hausdorff distance of some final segment of a geodesic ray θ emanating from w. Note that θ must necessarily approach z, and that $d(w,z) \geq d(w) \geq d(x)/2 \geq r/8C$. Let x' be the first point on θ such that $d(x',z) = r/8C$, and let λ be the path given by $QL(C)$(a). Since $\mathrm{len}_d(\lambda) < 2Cr$, λ stays in the annulus $\overline{B}(z,(C+1)r) \setminus B(z,r/(8C))$. It is now easy to deduce that $\mathrm{len}_k(\lambda)$ is bounded, so we are done. □

We end this section by giving an example of a quasiconvex bounded planar domain G where the quasihyperbolic metric k associated to the (inner) Euclidean metric is not roughly starlike (but (G,k) is not Gromov-hyperbolic either). We define $G = D \setminus \bigcup_{j=2}^{\infty} E_j$, where D is the unit disk and E_j consists of j equally spaced points on the boundary of the disk $D_j = B(z_j,r_j)$, where $z_j = (1-2^{-j},0)$, and $r_j = 4^{-j}$. It is not hard to see that there is some integer j_0 such that every geodesic ray emanating from the origin remains outside D_j for all $j \geq j_0$. Thus the points z_j are quasihyperbolically far from all geodesic rays. It is also interesting to note that when j is large, no geodesic segment from the origin to z_j can be continued outside of cD_j, where $c < 1/2$ tends to zero as we let j tend to infinity.

We also note that we cannot strengthen the conclusions of the main results in this section from roughly starlike to starlike (the latter meaning that geodesic rays pass through every

point, as they do in any Euclidean ball). To see this, we simply modify the previous example so that every E_j consists of exactly 1000 equally spaced points. The resulting domain is kG-hyperbolic, but the points z_j are still not on geodesic rays.

4. Gromov hyperbolicity implies Slice

Our main theorem in this section is as follows. The definitions of (local) regularity, (local) Loewner, and some related lemmas have been relegated to the appendix, Section 7.

Theorem 4.1. *Suppose (X, d, μ) is minimally nice, upper Q-regular, locally Q-regular, and locally Q-Loewner. Suppose also that (X, d) is a length space, and that (X, k) is Gromov hyperbolic and roughly starlike. Then (X, d) satisfies a slice condition, quantitatively.*

This theorem is applicable in particular if (X, d, μ) is a proper subdomain of $\overline{\mathbb{R}^n}$, with the inner spherical metric and Lebesgue measure attached. In this case, upper regularity, local regularity and the local Loewner property are clearly true and, since inner spherical domains lie in the class $QL(1)$, Theorem 3.1 implies that (X, k) is roughly starlike whenever it is Gromov hyperbolic. Thus the above theorem implies that a kG-hyperbolic inner spherical domain satisfies a slice condition. By Theorem 3.1 and the remarks that precede it, (X, k) is roughly starlike whenever (X, d, μ) is a minimally nice, bounded, upper Q-regular, Q-Loewner space, and (X, k) is Gromov hyperbolic. We generalize this result to domains in such spaces in Proposition 4.19. So the rough starlike assumption in Theorem 4.1 is readily available in most applications.

Let us briefly overview the contents of this section. We first introduce and discuss a variant slice condition slice$^+$ which is stronger than slice. Then we discuss some concepts that we need to prove the main theorem, including the uniform deformations (X, d_ϵ) of a metric space (X, d) satisfying (Hyp) that were introduced by Bonk, Heinonen, and Koskela

[BoHeKo]. The benefit of (X, d_ϵ) is that it is a uniform space provided that (X, k) is Gromov hyperbolic. Using this, we prove that (X, d_ϵ) satisfies a slice$^+$ condition, and then use a conformal modulus argument to deduce that (X, d) also satisfies a slice$^+$ condition. Finally, we explicitly describe a large class of spaces on which kG-hyperbolicity implies slice$^+$.

The main reason for proving the stronger slice$^+$ condition is that it immediately implies (if we take d to be the inner Euclidean metric) all of the slice-type conditions defined for a variety of purposes in [BuKo2], [BuSt1], and [BuSt2]. (The same is not true of our slice condition.) Thus it follows from the results in this section that a bounded kG-hyperbolic Euclidean domain satisfies all these slice-type conditions, and that slice$^+$ and slice are equivalent on inner spherical domains, or more generally on all length spaces that are minimally nice and satisfy conditions (a) and (b) in the discussion after Theorem 0.1.

The slice$^+$ condition involves the conformal modulus mod_Q, defined in Section 7, a concept that is central to our method of proof. Suppose (X, d, μ) is a minimally nice, upper Q-regular metric space, and that $0 < c < 1$, $C \geq 6$. Given $x, y \in X$ and $\gamma \in \Gamma(x, y)$, suppose there exist pairwise disjoint open subsets $\{S_i\}_{i=0}^m$ of X, $m \geq 0$, with $d_i \equiv \mathrm{dia}_d(S_i) < \infty$, and points $\{x_i\}_{i=0}^m$ on γ, such that $x_i \in S_i$, with $d_i/C \leq d(x_i) \leq Cd_i$, and $x_0 = x$, $x_m = y$. We then write $(x, y; \gamma) \in \mathrm{slice}^+(C, c)$ if additionally (Sli_2) and (Sli_3) both hold and, defining $E_i \equiv \overline{B}_d(x_i, cd(x_i))$, the following two conditions also hold:

(Sli_1^+) For $0 < i < m$, and every $\lambda \in \Gamma(x, y)$, there exists a closed subpath λ_i of λ such that $\lambda_i \in \overline{S_i}$ and $\mathrm{dia}_d(\lambda_i) \geq d_i/C$.

(Sli_4^+) For $0 \leq i \leq m$, the ball $6E_i$ is contained in S_i.

We say that (X, d, μ) satisfies the (C, c)-slice$^+$ condition if $(x, y; \gamma) \in \mathrm{slice}^+(C, c)$ for every $x, y \in X$ and every connecting geodesic segment γ. Clearly the (C, c)-slice$^+$ condition implies a $(C', 0)$-slice$^+$ condition, where $C' = C \vee c^{-1}$.

We shall actually prove the following stronger version of Theorem 4.1.

Theorem 4.2. *Suppose (X, d, μ) is minimally nice, upper Q-regular, locally Q-regular, and locally Q-Loewner. Suppose also that (X, d) is a length space, and that (X, k) is Gromov hyperbolic and roughly starlike. Then (X, d) satisfies a slice$^+$ condition, quantitatively.*

We define one more slice-type condition. Specifically, we define the mod-slice(C, c) condition to be identical to the slice$^+(C, c)$ condition, except that we replace (Sli_1^+) by the following modulus variant:

(Sli_5) For $0 < i < m$, and every $\lambda \in \Gamma(x, y)$, there exists a closed subpath λ_i of λ such that $\lambda_i \in \overline{S_i}$ and $\mathrm{mod}_Q(\lambda_i, E_i) \geq C^{-1}$.

The meaning of the condition "$(x, y; \gamma) \in$ mod-slice(C, c)" is as one would expect.

We suspect that mod-slice may not be implied by Gromov hyperbolicity under the hypotheses of Theorem 4.1, but it is nevertheless rather useful in our proof of Theorem 4.2. It is clear that in a (global) Loewner space, (Sli_1^+) implies (Sli_5) quantitatively, and so slice$^+$ implies mod-slice. The converse implication is true under weaker assumptions, as the following lemma indicates.

Lemma 4.3. *Suppose (X, d, μ) is a minimally nice, upper Q-regular, metric measure space, and that $(x, y; \gamma) \in$ mod-slice(C, c) for some $x, y \in X$, $\gamma \in \Gamma(x, y)$. Then there exists C' dependent only on C, c, Q, and $C^+(\mu, Q)$ such that $(x, y; \gamma) \in$ slice$^+(C', c)$.*

Proof. We need to prove (Sli_1^+). If λ_i intersects $5E_i$, it suffices to appeal to (Sli_4^+). If, on the other hand, λ_i is disjoint from $5E_i$, we let $E = E_i$, $F = \lambda_i$, and apply Lemma 7.3 to (Sli_5). □

The following lemma formulates the interdependence of the constants C and c in the mod-slice(C, c) condition as the constant c is allowed to shrink.

Lemma 4.4. *Suppose (X, d, μ) is a minimally nice, locally $(Q, \kappa, \delta_0, \psi)$-Loewner, metric measure space. Let $0 < c' \le c \le \delta_0/5$. If $(x, y; \gamma) \in \text{mod-slice}(C, c)$ for some $x, y \in X$, $\gamma \in \Gamma(x, y)$, then $(x, y; \gamma) \in \text{mod-slice}(C', c')$, where*

$$C' = 3^Q[C \vee \psi(6)^{-1} \vee \psi(3c/c')^{-1}].$$

Proof. Except for (Sli_5), all the defining conditions for (C', c')-mod-slice follow from those for (C, c)-mod-slice. For (Sli_5), we choose the same data $\{S_i, d_i, x_i\}_{i=0}^m$ for both choices of constants. Fixing $\lambda \in \Gamma(x, y)$, we also choose the same subpaths λ_i unless otherwise stated. Fix $0 < i < m$, and write $E_i = \overline{B}_d(x_i, cd(x_i))$, $E_i' = \overline{B}_d(x_i, c'd(x_i))$. Suppose first that λ_i intersects $3E_i$. In view of (Sli_4^+), we may select a segment λ_i' of λ_i that connects $3E_i$ with $\partial(4E_i)$ and lies inside $5E_i$. Since $\text{dia}_d(\lambda_i') \ge cd(x_i)$, the local Loewner property with $E = E_i'$ and $F = (\lambda_i')$ implies that

$$\psi(3c/c') \le \text{mod}_Q(E, F; B(\epsilon\kappa d(x))) \le \text{mod}_Q(E, F),$$

as required (note that ψ is nonincreasing by definition).

Suppose instead that λ is a path for which $\lambda_i \cap 3E_i = \emptyset$. We first prove this case under the added assumption $c' \le c/6$. Then the data

$$(E, E', F, x, r) = (\overline{E_i \setminus (5/6)E_i}, E_i', \lambda_i, x_i, cd(x_i))$$

satisfy the assumptions of Lemma 7.4. This is clear except for the inequality $\text{dia}_d(E) \le 3\,\text{dist}_d(E, E')$, which follows from the estimates $\text{dia}_d(E) \le 2r$ and $\text{dist}_d(E, E') \ge 2r/3$.

Now, $\text{dia}_d(E) \ge r/6$. To see this, we simply choose an initial path η from x_i to a point x' with $d(x') < d(x_i)/2$, and extract a subpath $\nu = \eta[u, v]$ that lies in E, with $d(u, x_i) = 5r/6$ and $d(v, x_i) = r$. Then $\text{dia}_d(E) \ge \text{dia}_d(\nu) \ge r/6$.

It follows similarly that $\operatorname{dia}(E') \geq c'd(x)$, and it is clear that $\operatorname{dist}_d(E, E') < r$. Thus

$$\Delta_d(E, E') \leq \frac{cd(x_i)}{c'd(x_i)} = \frac{c}{c'}.$$

Using Lemma 7.4 and the estimate $\operatorname{mod}_Q(\lambda_i, E_i) \geq C^{-1}$, we deduce that $\operatorname{mod}_Q(E', F) \geq 3^{-Q}[\psi(3) \wedge C^{-1} \wedge \psi(c/c')]$. This gives the desired conclusion for such paths λ with a constant

$$C' = 3^Q[C \vee \psi(3)^{-1} \vee \psi(c/c')^{-1}],$$

which is stronger than the desired conclusion.

It remains to prove it for this second type of path when $c' > c/6$. Since the modulus in (Sli_5) increases as c' increases, any C' that works for $c' = c/6$ also works for $c' > c/6$. But for $c' = c/6$, and the type of path that we need to consider, we can take

$$C' = 3^Q[C \vee \psi(3)^{-1} \vee \psi(6)^{-1}] = 3^Q[C \vee \psi(6)^{-1}].$$

The desired estimate thus follows for larger c'. □

Suppose $C \geq 1$. Given points $x, y \in X$, and a path $\gamma \in \Gamma(x, y)$, $\gamma : [0, L] \to X$, we say that γ is a *C-uniform path* for x, y if $\operatorname{len}_d(\gamma) \leq Cd(x, y)$ and

$$\forall\, t \in [0, L]: \qquad d_X(\gamma(t)) \geq C^{-1}[\operatorname{len}_d(\gamma|_{[0,t]}) \wedge \operatorname{len}_d(\gamma|_{[t,L]})].$$

If we assume, as we may, that γ is parametrized by arclength, the second condition above takes the simpler form

$$\forall\, t \in (0, L): \qquad d_X(\gamma(t)) \geq C^{-1}[t \wedge (L - t)]. \tag{4.5}$$

We say that X is a *C-uniform space* if there is a C-uniform path for every pair of points $x, y \in X$. It is also useful to extend the concept to paths $\gamma : [0, L] \to \overline{X}$ parametrized by

arclength. We say that such a path is a C-uniform path if $L \leq C d_{\overline{X}}(\gamma(0), \gamma(L))$ and (4.5) holds. Equivalently, modulo a controlled change of constants, γ is a uniform path if for some constant C', the path segments $\gamma|_{[\epsilon, L-\epsilon]}$ are C'-uniform for all $0 < \epsilon < L/C'$.

We now show that uniform spaces satisfy a slice$^+$ condition; a similar but simpler proof for a somewhat different slice condition in the Euclidean setting can be found in [BuKo2, Theorem 3.2].

Theorem 4.6. *Suppose (X, d) is a C-uniform space. Then it also satisfies a (C', c')-slice$^+$ condition, for some C', c' dependent only on C. Additionally, if $x, y \in X$ and $\gamma \in \Gamma(x, y)$ is a geodesic segment, then $(x, y; \gamma) \in \mathrm{slice}^+(C'', c'')$, for some C'', c'' dependent only on C.*

Proof. Let us fix $x, y \in X$. By symmetry, we may assume that $d(y) \leq d(x)$. Let $\gamma = [x, y]$ be a geodesic path. By [BoHeKo, Theorem 2.10], γ is a C_1-uniform path for the pair x, y, with $C_1 = C_1(C) \geq 1$. We assume, as we may, that $\gamma : [0, l] \to X$ is parametrized by arclength and that $C_1 \geq C$. Since we shall only use the fact that γ is a C_1-uniform path, the first conclusion in the theorem thus follows from the second which we now prove.

Let $\alpha = 5/4$ and let n_0 be the least integer such that $\alpha^{n_0} > 3C_1$; in particular, $n_0 > \log_\alpha 3 > 4$. For $z \in \{x, y\}$, define d-balls $B_i^z = B(z, r_i^z)$ for each non-negative integer i, where each r_i^z is a number chosen so that

$$\frac{9}{10} \cdot \frac{\alpha^i d(z)}{25C} \leq r_i^z \leq \frac{\alpha^i d(z)}{25C}$$

and

$$\mathrm{len}_d(\gamma \cap \partial B(z, r_i^z)) = 0.$$

Such numbers r_i^z exist; indeed, the finiteness of $\mathrm{len}_d(\gamma)$ implies that the length of the part of γ lying on the sphere $\partial B_d(z, r) = \{w \in X \mid d(w, z) = r\}$, is zero for almost every $r > 0$. Note that

$$\frac{9\alpha^j}{10} \leq \frac{r_{i+j}^z}{r_i^z} \leq \frac{10\alpha^j}{9}. \tag{4.7}$$

We also define the associated annuli

$$A_i^z = \begin{cases} B_0^z & i = 0, \\ B_i^z \setminus \overline{B_{i-1}^z}, & i \in \mathbb{N}. \end{cases}$$

We may assume that $d(x,y) \geq d(x)/2C$, since otherwise the single slice $S_0 = B(x, r_N^x)$ suffices, where N is any integer such that $\alpha^N > 20$. For each $i \geq 1$, there is a non-negative integer $g(i)$ such that B_i^x intersects B_j^y precisely when $j \geq g(i)$. A little calculation shows that $r_{g(i)}^y / r_i^x \geq \alpha$ for $i \leq 2$, while $r_{g(i)}^y / r_i^x < 1$ for all sufficiently large i. Let $i_0 \geq 3$ to be the least integer i for which $r_{g(i)}^y / r_i^x < 5$. Note that $r_{g(i_0-1)-1} < d(x,y)$, and so $r_{g(i_0-1)}^y < 25d(x,y)/18$ and $r_{i_0-1}^x < 5d(x,y)/18$. Using (4.7) again, we see that $r_{i_0}^x < d(x,y)/2$. Since we always have $r_i^x + r_{g(i)}^y > d(x,y)$, it follows that $r_{g(i_0)}^y > d(x,y)/2$, and so $r_{g(i_0)}^y / r_{i_0}^x \in (1,5)$. Clearly also $g(i_0) \geq 3$. Let $m = i_0 + g(i_0)$ and define the slices

$$S_i = \begin{cases} A_i^x, & 0 \leq i \leq i_0, \\ A_{m-i}^y, & i_0 + 2 \leq i \leq m \\ B_{g(i_0)+n_0}^y \setminus (S_{i_0} \cup S_{i_0+2}), & i = i_0 + 1. \end{cases}$$

We choose $x_0 = x$, $x_m = y$, and pick $x_i \in \gamma$ so that $d(x_i, x) = (r_i^x + r_{i-1}^x)/2$ for $0 < i \leq i_0$, and $d(x_{m-i}, y) = (r_i^y + r_{i-1}^y)/2$ for $0 < i \leq g(i_0) - 1$.

Condition (Sli_4^+) is easy to deduce in all cases. As for (Sli_3), we first combine (4.7) with the definition of n_0 and the fact that $r_{g(i_0)} > d(x,y)/2$, to deduce that $B_{g(i_0)+n_0}^y$ has radius larger than $C_1 d(x,y)$. It therefore fully contains γ, and (Sli_3) is now easy to verify.

Let us consider (Sli_2) for $1 \leq i \leq i_0$. Define t_i^- and t_i^+ to be the values of t for which $\gamma(t)$ is the point of first entry into, and last exit from, $\overline{S_i}$, respectively. Uniformity ensures that $\gamma(t)$ cannot lie in $B_d(x, r_{i-1}^x)$ for any $t \geq C_1(d(x) + r_{i-1}^x)$, so $t_i^- \leq C_1(d(x) + r_{i-1}^x) \lesssim r_i^x$. Similarly, $t_i^+ \lesssim r_i^+$, and so $t_i^+ - t_i^- \lesssim r_i^x$. But trivially $t_i^- \geq r_{i-1}^x$ and, since $r_{i_0}^x < d(x,y)/2$, we also have $l - t_i^+ > d(x,y)/2 \gtrsim r_i^x$. Uniformity therefore ensures that $d(\gamma(t)) \gtrsim r_i^x$ for all $t \in [t_i^-, t_i^+]$. This lower bound together with the upper bound on $t_i^+ - t_i^-$ imply (Sli_2).

Condition (Sli_1^+) follows readily from the construction. The "middle slice" S_{i_0+1} deserves separate attention. Here it is important that the succeeding slices are annuli A_i^y, where $i \leq g(i_0) - 2$; the gap between this upper index and $g(i_0)$ ensures (Sli_1^+) in this case. In each case, it is in fact clear that every path $\lambda \in \Gamma(x, y)$ contains a closed subpath λ_i whose diameter is comparable with the diameter of the slice; for instance if $i \neq i_0 + 1, 0, m$ and so S_i is an annulus, λ_i is any subpath that connects the inner spherical boundary of S_i to its outer spherical boundary. □

We now pause to recall some recent results of Bonk, Heinonen, and Koskela [BoHeKo] that we shall need. If (X, d) is a minimally nice metric space, and $w \in X$ is a *base point*, we define the densities $\rho_\epsilon(x) = \exp(-\epsilon k(x, w))$ and $\sigma_\epsilon(x) = \rho_\epsilon(x)/d(x)$, for all $\epsilon > 0$. This gives rise to the metric space[1] $X_\epsilon = (X, d_\epsilon)$ with

$$d_\epsilon(x, y) = \inf_{\gamma \in \Gamma(x,y)} \int_\gamma \rho_\epsilon(z)\, |d_k z| = \inf_{\gamma \in \Gamma(x,y)} \int_\gamma \sigma_\epsilon(z)\, |dz| \tag{4.8}$$

where $|d_k z| = |dz|/d_X(z)$ denotes the quasihyperbolic distance element. According to [BoHeKo], the resulting space is incomplete and so we can associate a boundary distance $d_\epsilon(\cdot)$ and a quasihyperbolic metric k_ϵ with d_ϵ, as we did with d.

If μ is a Borel measure on X, and $Q \geq 1$ is fixed, we also attach the Borel measure $d\mu_\epsilon(z) = \sigma_\epsilon(z)^Q d\mu(z)$ to (X, d_ϵ); the parameter Q is omitted from this notation, since it will always be given by a (local or global) Q-Loewner condition. The point of this definition is that the Q-modulus of a path family is invariant under the identity map $i_\epsilon : X \to X_\epsilon$, as is immediate from the definition. We write X_ϵ as a shortcut for either (X, d_ϵ) and $(X, d_\epsilon, \mu_\epsilon)$.

We now state Theorem 6.39 of [BoHeKo] as a lemma. In this lemma, the metric measure space (X, d_ρ, μ_ρ) is derived from the metric measure space (X, d, μ) via some density ρ. We have already defined the metric d_ρ in the introduction, and $d\mu_\rho$ is simply $\rho^Q\, d\mu$. Also the notation $d_\rho(x)$ stands for the d_ρ-distance to the boundary of a point $x \in X$.

[1] For consistency with Section 0, we would need to write d_{σ_ϵ} instead of d_ϵ. For ease of notation, however, we insist that d_ϵ is henceforth defined as in (4.8).

Lemma 4.9. *Let (X, d, μ) be a locally compact, incomplete, quasiconvex locally Q-Loewner space. Assume that $\rho : X \to (0, \infty)$ is a continuous function such that the following two conditions hold for some $C \geq 1$:*

$$\frac{1}{C} \leq \frac{\rho(x)}{\rho(y)} \leq C, \qquad x, y \in B_d(z, d(z)/2),\ z \in X, \tag{4.10}$$

and

$$\frac{1}{C}\rho(x)d(x) \leq d_\rho(x) \leq C\rho(x)d(x), \qquad x \in X. \tag{4.11}$$

Then (X, d_ρ, μ_ρ) is a locally Q-Loewner space, quantitatively.

We actually need a stronger "more local" version of this lemma. Specifically, we want a version with the same conclusion but with only a local version of the quasiconvexity assumption, and with the ball in (4.10) replaced by a smaller ball $B_d(z, cd(z))$ for some fixed but arbitrary $c > 0$. It is a routine matter to modify the proof in [BoHeKo] to prove this stronger version of the lemma.

Our next lemma is a combination of Propositions 4.5 and 4.37 of [BoHeKo].

Lemma 4.12. *Suppose (X, d) is minimally nice, and that (X, k) is both C-Gromov hyperbolic and K-roughly starlike, for some $C \geq 1$, $K > 0$. Then X_ϵ has diameter at most $2/\epsilon$ and there are positive numbers C', ϵ dependent only on C, K such that X_ϵ is C'-uniform for all $0 < \epsilon \leq \epsilon_0$. Furthermore, there exists $c = c(C, K) \in (0, 1)$ such that the quasihyperbolic metrics k and k_ϵ satisfy the quasi-isometric condition*

$$c\epsilon k(x, y) \leq k_\epsilon(x, y) \leq e\epsilon k(x, y).$$

Using Theorem 4.6 and Lemma 4.12, we know that X_ϵ satisfies a slice$^+$ condition if ϵ is sufficiently small. We shall transport this condition over to X by means of a modulus argument. For such an argument to work, we need to show that X_ϵ is a Loewner space.

Lemma 4.13. *Suppose that (X, d, μ) is minimally nice, locally Q-regular, and locally Q-Loewner, and that (X, k) is Gromov hyperbolic and roughly starlike. Then X_ϵ is Q-Loewner for all $0 < \epsilon \leq \epsilon_0$, where ϵ_0 depends quantitatively on the hypotheses, and the Loewner data depend quantitatively on ϵ and the hypotheses.*

Proof. According to Theorem 6.4 of [BoHeKo], a uniform and local Q-Loewner space is Q-Loewner, quantitatively. But by Lemma 4.12, X_ϵ is uniform when ϵ is sufficiently small. Thus it suffices to show that X_ϵ is locally Q-Loewner when ϵ is small enough. The local Loewner condition will in turn follow by applying the stronger version of Lemma 4.9 (defined after the statement of that lemma) to the density $\rho = \sigma_\epsilon$. We must therefore show that this density satisfies the hypotheses of that lemma.

A regular Loewner space is quasiconvex, as proven in [HeKo, Theorem 3.13]. This proof is readily modified to show that a locally regular, locally Loewner space is locally C_1-quasiconvex, where $C_1 \geq 1$ is dependent only on the local regularity and Loewner data.

Let us fix an arbitrary point $x \in X$, and write $B = B_d(x, d(x)/4C_1)$. We next show that

$$\frac{k(y,z)}{2C_1} \leq \frac{d(y,z)}{d(x)}, \qquad y, z \in B \tag{4.14}$$

Suppose $y, z \in B$. Since $d(y,z) \leq d(x)/2C_1$, it follows by quasiconvexity that there is a path $\lambda \in \Gamma(y,z)$ such that $\mathrm{len}_d(\lambda) \leq C_1 d(y,z)$. Since λ is of length at most $d(x)/2$, the set λ must be contained in $B' \equiv B(x, d(x)/2)$. But $d(z) \geq d(x)/2$ on B', so $\mathrm{len}_k(\lambda) \leq 2\,\mathrm{len}_d(\lambda)/d(x)$, which implies (4.14).

Assuming that $\epsilon \leq \epsilon_0 \leq 1/e$ is small enough that Lemma 4.12 is valid, and taking $y = x$ in (4.14), we see that $B \subset B_k(x, 1/2) \subset B_{k_\epsilon}(x, 1/2)$. By (2.8), we see that $d(z)/d(x)$ lies in the interval $(e^{-1/2}, e^{1/2})$ when $z \in B$. Using the triangle inequality we see that the density function ρ_ϵ satisfies the estimate

$$\exp(-\epsilon k(u,v)) \leq \frac{\rho_\epsilon(u)}{\rho_\epsilon(v)} \leq \exp(\epsilon k(u,v)), \qquad u, v \in X.$$

Since $\epsilon \le 1/e$, it follows that $\rho_\epsilon(z)/\rho_\epsilon(x) \in [e^{-1/(2e)}, e^{1/(2e)}]$ for all $z \in B$. We deduce that

$$\frac{1}{e^2} \le \frac{\sigma_\epsilon(y)}{\sigma_\epsilon(z)} \le e^2, \qquad y, z \in B. \tag{4.15}$$

Since (X, k) is K-roughly starlike, Gromov hyperbolic, and proper, it follows from Lemma 4.16 of [BoHeKo] that there are constants $0 < c_2 < C_2$, dependent only on ϵ and K, such that

$$c_2 \rho_\epsilon(x) \le d_\epsilon(x) \le C_2 \rho_\epsilon(x), \qquad x \in X,$$

which can be rewritten as

$$c_2 \sigma_\epsilon(x) d(x) \le d_\epsilon(x) \le C_2 \sigma_\epsilon(x) d(x), \qquad x \in X. \tag{4.16}$$

We have now verified all of the hypotheses of the stronger version of Lemma 4.9, and so the proof can be completed as indicated above. □

By Theorem 4.6 and the uniformity of X_ϵ for small $\epsilon > 0$, we already know that X_ϵ satisfies a slice$^+$ condition for appropriately chosen $\epsilon > 0$. The proof of Theorem 4.2, which we now present, consists of showing that the slice$^+$ condition on X_ϵ induces a slice condition on X. In this proof, mod_Q and $\mathrm{mod}_{\epsilon,Q}$ denote conformal modulus with respect to d and d_ϵ, respectively.

Note. There are quite a few constants in this proof. It is convenient to denote each of them as C_* or c_*, possibly with some prime superscripts, where $*$ is some suggestive *symbol* (rather than a *variable*, as is the case elsewhere in this paper). For example, C_ϵ and C_1 are completely unrelated constants, irrespective of whether or not the related variable ϵ takes on the value 1.

Proof of Theorem 4.2. We wish to give slice$^+$ data for a fixed but arbitrary pair of points $x, y \in X$ and k-geodesic $[x, y]$. Applying Lemmas 4.12 and 4.13, we choose $\epsilon_0 > 0$ so

small that X_ϵ is both uniform and Q-Loewner for all $0 < \epsilon \leq \epsilon_0$. Lemma 4.12 also implies that $[x, y]$ is a $(k_\epsilon; C_{\mathrm{QG}})$-quasigeodesic; C_{QG} depends only on the Gromov hyperbolicity and rough starlikeness constants, C_{G} and C_{S}, respectively.

Fixing $\epsilon = \epsilon_0$, Theorem 4.6 allows us to choose data $\{S_{i;\epsilon}, x_{i;\epsilon}\}_{i=0}^m$ associated with the slice$^+(C_\epsilon, c_\epsilon)$ condition for $x, y \in X_\epsilon$ and the k_ϵ-quasigeodesic $[x, y]$. By the Loewner assumption, the mod-slice$(C'_\epsilon, c_\epsilon)$ condition also follows for some $C'_\epsilon \geq 6$, and $c_\epsilon = \delta_0/4$, where δ_0 is the third local Loewner parameter. We then define the C_d-slice$^+$ data for $x, y \in X$ and the k-geodesic $[x, y]$ to be $\{S_i, x_i\}_{i=0}^m$, where $x_i = x_{i;\epsilon}$, $S_i = S_{i;\epsilon} \cap B_d(x_i, C_d d(x_i))$; $C_d \geq 6$ and $0 < c_d < 1$ will be specified later. Consistent with the slice$^+$ definition, we also write $E_{i;\epsilon} = \overline{B}_{d_\epsilon}(x_i, c_\epsilon d_\epsilon(x_i))$ and $E_i = \overline{B}_d(x_i, c_d d(x_i))$.

By Lemma 4.12, there exists a constant $c_1 \geq 1$ such that $\mathrm{dia}_k([x, y] \cap S_{i;\epsilon}) \leq c_1$. Appealing to (2.7), we see that there exists a constant C_1 such that $[x, y]$ is contained in $\bigcup_{i=0}^m B_d(x_i, C_1 d(x_i))$, where $C_1 = \exp(c_1) - 1 \geq c_1$. Conditions ($\mathrm{Sli}_2$) and ($\mathrm{Sli}_3$) are now clearly true for any choice of constant $C_d \geq C_1$.

Using (2.7), Lemma 4.12, and (2.9) in that order, we see that there are constants $c_{2,\epsilon}, c_2, c_d \in (0, 1)$, dependent only on ϵ, C_ϵ, C_{G}, C_{S}, such that

$$6E_{i;\epsilon} \supset B_{k_\epsilon}(x_i, c_{2,\epsilon}) \supset B_k(x_i, c_2) \supset \overline{B}_d(x_i, 6c_d d(x_i)) \supset 6E_i. \tag{4.17}$$

This immediately implies (Sli_4) for any choice of constant $C_d \geq c_2^{-1}$ (as well as (Sli_4^+) for any $c_d \leq c_2$, but this does not concern us).

It remains only to prove (Sli_1^+) for some choice of C_d. In a similar fashion to (4.17), we can show that there exists a constant $c_3 \in (0, 1)$, dependent on the same parameters as c_d, such that $c_3 E_{i;\epsilon} \subset E_i$. Assume therefore that a path $\lambda \in \Gamma(x, y)$ is given, and let $\lambda_{i;\epsilon}$ be the subpath provided by (Sli_5). By Lemma 4.4, there exists some $c' \in (0, 1)$ such

that $\mathrm{mod}_{\epsilon,Q}(\lambda_{i;\epsilon}, E_i) \geq \mathrm{mod}_{\epsilon,Q}(\lambda_{i;\epsilon}, c_3 E_{i;\epsilon}) \geq c'$. Using Lemma 7.3, we see that there exists some C_2 dependent only on allowed parameters such that $\lambda_{i;\epsilon}$ must intersect $B(x, C_2 d(x_i))$. We now finally declare $C_d = C_1 \vee c_2^{-1} \vee 2C_2$.

If $\lambda_{i;\epsilon}$ lies fully in $B_d(x_i, C_d d(x_i))$, then take $\lambda_i = \lambda_{i;\epsilon}$. By Lemma 4.4, we have $\mathrm{mod}_{\epsilon,Q}(\lambda_i, E_i) \geq \mathrm{mod}_{\epsilon,Q}(\lambda_i, C_d^{-1} E_{i;\epsilon}) \geq c'$, for some $c' > 0$ dependent only the data in the hypotheses. The identity map from (X, d) to (X, d_ϵ) is locally a dilation composed with a controlled bilipschitz distortion and so it clearly satisfies (7.6). By Lemma 7.5, it follows that $\mathrm{mod}_Q(\lambda_i, E_i) \geq c''$ for some $c'' > 0$ dependent only the data in the hypotheses. Condition (Sli_1^+) now follows as in Lemma 4.3.

Alternatively if $\lambda_{i;\epsilon}$ does not lie fully in $B_d(x_i, C_d d(x_i))$, then we choose a subpath λ_i of $\lambda_{i;\epsilon}$ which connects points v, w, where $d(v, x_i) = C_2 d(x_i)$ and $d(w, x_i) = 2C_2 d(x_i)$, and so (Sli_1^+) is clearly true. □

We now give a more explicit description of a class of spaces for which kG-hyperbolicity implies a slice condition.

Theorem 4.18. *Suppose (Y, d_Y, μ_Y) is a complete, Q-regular, Q-Loewner metric measure space, and that $X \subsetneq Y$ is a bounded open connected subset of Y. Let d denote the inner metric (on X) associated with $d_X \equiv d_Y|_{X\times X}$ and let μ be the restriction of μ_Y to X. If (X, d) is kG-hyperbolic, then it satisfies a slice$^+$ condition, quantitatively.*

As we already indicated at the beginning of the previous section, there are many examples of metric measure spaces (Y, d_Y, μ_Y) that satisfy the assumptions of this theorem. Examples include n-regular Riemannian manifolds of non-negative Ricci curvature, Carnot-Carathéodory spaces such as the Heisenberg group; see [HeKo, Section 6]. Also noteworthy are the exotic examples with non-integer dimension Q given by Bourdon and Pajot [BoPa] and by Laakso [Lk].

It is easily verified that, under the hypotheses of Theorem 4.18, upper regularity, local regularity, and the local Loewner property are inherited by (X, d_X, μ) and hence by (X, d, μ). Additionally, Y (and hence X) is locally compact because it is complete and Q-regular. Thus, in view of Theorem 4.1, the proof of Theorem 4.18 reduces to proving the following result.

Proposition 4.19. *Let (X, d, μ) be as in Theorem 4.18 and let $M = \sup_{z \in X} d(z)$. Then there exists a number K, dependent quantitatively on the hypotheses, such that (X, k) is K-roughly starlike with respect to any basepoint w for which $d(w) > M/2$.*

Proof. By Theorem 3.1, it suffices to verify a $QL(C)$ condition. Condition $QL(C)$(a) follows readily from the second conclusion in Lemma 7.1 applied to the ambient space Y; note that since γ does not intercept ∂X, it lies in X. Condition $QL(C)$(b) is almost as easy: we make some tentative choice of $y \in A(z, r, 4C) \cap \partial X$ and again apply the second conclusion in Lemma 7.1. Since we do not want γ to wander outside $\overline{X}$, we cut it off when it first reaches a point on ∂X, and redefine y to be that point. For $QL(C)$(c), we apply the first conclusion in Lemma 7.1 with $y \in \partial X$, $d(x, y) < 2d(x)$, and cut off a near minimal-length connecting path when it first reaches a point x' with $d(x') = d(x)/2$. For all parts, we can take $C = 2C_0$. Thus Proposition 4.19 follows from Theorem 3.1.

5. Basic Properties of GHS Spaces

We begin by introducing some extra notation that we shall use in these two sections. We say that (X, d) is a C-GHS space, $C \geq 1$, if it is an incomplete locally compact length space (and so minimally nice) and it satisfies both a C-Gehring-Hayman and a C-separation condition. We denote by $\lambda_{x,y}$ any path from x to y such that $\mathrm{len}_d(\lambda_{x,y})$ is "close to" $d(x, y)$.

Lemma 5.1. *Suppose that (X, d) satisfies a C-Gehring-Hayman condition. Then whenever $x, y \in X$, and w lies on a geodesic $[x, y]$, we have*

$$d(x, y) \geq 2d(x, w)/(C + 1). \tag{5.2}$$

Proof. If the conclusion were false, then

$$\mathrm{len}_d([x,w]) \geq d(x,w) > \frac{C+1}{2}\, d(x,y)$$

and

$$\mathrm{len}_d([w,y]) \geq d(w,y) \geq d(x,w) - d(x,y) > \frac{C-1}{2} d(x,y)\ .$$

Combining the two inequalities we have

$$\mathrm{len}_d([x,y]) = \mathrm{len}_d([x,w]) + \mathrm{len}_d([w,y]) > \left(\frac{C+1}{2} + \frac{C-1}{2}\right) d(x,y) = Cd(x,y),$$

contradicting (GH). □

Lemma 5.3. *Suppose that X is a C-GHS space and that $x, y, z \in X$. Suppose further that $[a,b] \subset [x,z]$, $[a',b'] \subset [z,y]$. Then*

$$\mathrm{len}_d([a,b]) \leq 2Cd_1 + C^2(C+1)d_2/2,$$

$$\mathrm{len}_d([a',b']) \leq C(C+5)d_1/2 + C^2(C+1)d_2/2,$$

where $d(a,a') \vee d(b,b') = d_1$ and $d(a') = d_2$.

Note. As mentioned in Section 1, the inclusions $[a,b] \subset [x,z]$ and $[a',b'] \subset [z,y]$ are meant with the ordering convention coming from the parametrization of the geodesics by the real interval, so that $b \in [a,z]$ and $b' \in [a',y]$.

Proof of Lemma 5.3. We choose a path $\lambda_{b',b} \in \Gamma(b',b)$ such that $\mathrm{len}_d(\lambda_{b',b}) \leq d_1 + \epsilon$, where $\epsilon > 0$ is fixed but arbitrary. By (Sep_2), there exists a point $a'' \in S_{a'}^{[y,z]} \cap \lambda$, where $\lambda = [y,b'] + \lambda_{b',b} + [b,z] \in \Gamma(y,z)$. By ($\mathrm{Sep}_1$), we have $d(a',a'') < Cd_2$. If $a'' \in [y,b']$ then by (GH)

$$\mathrm{len}_d([a',b']) \leq \mathrm{len}_d([a',a'']) < C^2 d_2,$$

which proves the second inequality in our claim. Using (GH) and (5.2), it follows that

$$\begin{aligned}\operatorname{len}_d([a,b]) \le Cd(a,b) &\le C(d(a,a') + d(a',b') + d(b',b)) \\ &\le C[d_1 + C(C+1)d_2/2 + d_1],\end{aligned}$$

as required by the first inequality.

If instead $a'' \in \lambda_{b',b}$, then

$$d(a,b) \le d(a,a') + d(a',a'') + d(a'',b) < d_1 + Cd_2 + d_1 + \epsilon.$$

and

$$d(a',b') \le d(a',a'') + d(a'',b') < Cd_2 + d_1 + \epsilon.$$

Consequently, $\operatorname{len}_d([a,b]) \le 2Cd_1 + C^2d_2 + C\epsilon$ and $\operatorname{len}_d([a',b']) \le Cd_1 + C^2d_2 + C\epsilon$. Letting ϵ tend to 0, the desired inequality follows.

Finally, if $a'' \in [b,z]$, then $d(a,a'') \le d(a,a') + d(a',a'') \le d_1 + Cd_2$, from which it follows that

$$\operatorname{len}_d([a,b]) \le \operatorname{len}_d([a,a'']) \le C(d_1 + Cd_2).$$

Furthermore, using (5.2) again we see that $d(a,b) \le (C+1)(d_1 + Cd_2)/2$. By (GH) this implies that

$$d(a',b') \le d(a',a) + d(a,b) + d(b,b') \le (C+5)d_1/2 + C(C+1)d_2/2$$

and so $2\operatorname{len}_d([a',b']) \le C(C+5)d_1 + C^2(C+1)d_2$. □

6. GHS Implies Gromov Hyperbolicity

In this section, we aim to prove (Hyp) for GHS spaces. Our main theorem is as follows.

Theorem 6.1. *Every C-GHS space (X, d) is K-kG-hyperbolic, where $K = 120C^{11}$.*

The constant K is given for the convenience of the reader, but it is not optimal even for our proof (where we need K to be larger than several polynomials in C of degree at most 11, so for simplicity we replaced all lower powers of C by C^{11} before taking a maximum).

We assume that $x, y, z \in X$, $w \in [x, y]$, and write $\gamma = [x, z] + [z, y]$, $A = [x, z] \cup [z, y]$. Our purpose is to prove that w is at a bounded quasihyperbolic distance from A.

Before starting the proof let us fix some notational conventions. We push the ordering of the real line forward to A via γ and, for $x \in A$, $S \subset A$, we write $u \leq S$ (or $u \geq S$) if $u \leq v$ (or $u \geq v$, respectively) for all $v \in S$. For $\emptyset \neq S \subset A$, $\sup S$ and $\inf S$ are defined in the obvious way; additionally, we define $\sup \emptyset = x$ and $\inf \emptyset = y$. For any $u \in A$, S_u will mean $S_u^{[x,z]}$ or $S_u^{[z,y]}$, depending on whether $u \leq z$ or $u > z$. We give the data $(x, y, z, [x, y], [x, z], [y, z])$ the collective name D.

Before we tackle the full strength version of Theorem 6.1, we shall prove a couple of weaker versions as lemmas. Our first (rather lengthy) lemma concerns the case when one of the sides of the geodesic triangle is very short.

Lemma 6.2. *Let (X, d) be a C-GHS space, $x, y, z \in X$, $w \in [x, y]$ and assume that $\operatorname{len}_d([z, y]) \leq d(w)/2C(C+1)^2$. Then there exists $u_0 \in [x, z]$ such that $k(w, u_0) \leq 4C^4(C+1)^3$.*

Proof. We write $d_0 = d(w)/2C(C+1)^2$, and assume for the sake of contradiction that $\operatorname{len}_d([z, y]) \leq d_0$ and $k(w, [x, z]) > 4C^4(C+1)^3$. In particular, $k(x, y) \geq \log 2$, so we can apply (Sep_2) for the pair x, y. We start with the following:

Claim.

Let $u \in [x,z]$, $u' \in [x,w]$. *Then*

$$d(u,u') \leq C(C+1)d_0 \;\Rightarrow\; (S_w \cap [x,y]) \subset [u',y] \tag{6.3}$$

Let $u \in [x,z]$, $u' \in [w,y]$. *Then*

$$d(u,u') \leq C(C+1)d_0 \;\Rightarrow\; (S_w \cap [x,y]) \subset [x,u'] \tag{6.4}$$

Proof. Suppose by contradiction that $d(u,u') \leq C(C+1)d_0$ for some $u \in [x,z]$ and $u' \in [x,w]$, but that $u' \in (v,y]$ for some $v \in S_w \cap [x,y]$. By (Sep_3) we can write

$$k(u',w) \leq k(v,w) \leq \log(C+1)\ .$$

By (2.8), the above inequality implies that $d(u') \geq d(w)/(C+1)$. This, together with

$$d(u,u') \leq C(C+1)d_0 = \frac{d(w)}{2(C+1)}$$

and the length space assumption imply that $k(u,u') \leq \log 2$. This gives

$$k(u,w) \leq \log(2C+2) < 4C^4(C+1)^3$$

a contradiction to the added assumption. The second implication is proven similarly. □

Now we continue the proof of Lemma 6.2. Let

$$p = \sup\{u \in [x,z] \mid d(u,u') \leq C(C+1)d_0 \text{ for some } u' \in [x,w]\},$$

and let $p' \in [x,w]$ be such that $d(p,p') \leq C(C+1)d_0$. We choose $\lambda_{p',p} \in \Gamma(p',p)$ such that $\mathrm{len}_d(\lambda_{p',p}) < C(C+2)d_0$, and we recall the subpath notation $\gamma[u,v]$ defined in Section 1. By (6.3), we see that $S_w \cap [x,y] \subset [p',y]$, and so by applying (Sep_2) to the path $[x,p'] + \lambda_{p',p} + \gamma[p,y]$, we deduce that

$$\mathrm{len}_d(\gamma[p,y]) > C^{-1}d(w) - C(C+2)d_0 > d_0\ .$$

This implies that $p \in [x, z)$.

Next let

$$q = \inf\{u \in [p, z] \mid d(u, u') \leq (C(C+1) - 1)d_0 \text{ for some } u' \in [w, y] \cup [y, z]\}.$$

Then $q \in [p, z]$, and there exists $q' \in [w, y] \cup [y, z]$ such that we have the inequality $d(q, q') \leq (C(C+1) - 1)d_0$.

Consequently, there exists $q'' \in [w, y]$ such that $d(q, q'') < (C^2 + C)d_0$. We choose $\lambda_{q,q''} \in \Gamma(q, q'')$ such that $\mathrm{len}_d(\lambda_{q,q''}) \leq (C^2 + C)d_0$.

By (6.3) and (6.4), we see that $(S_w \cap [x, y]) \subset [p', q'']$, and so by applying ($\mathrm{Sep}_2$) to the path

$$[x, p'] + \lambda_{p',p} + [p, q] + \lambda_{q,q''} + [q'', y],$$

we deduce that $\mathrm{len}_d(I_w) > C^{-1}d(w) - 2C(C+1)d_0 = d(w)/C(C+1)$, where $I_w \equiv \{u \in S_w \mid p < u < q\}$.

Let us fix $u \in I_w$. By the extremality of p and q we have

$$d(u, [y, z] \cup [w, y]) > (C(C+1) - 1)d_0,$$

and

$$d(u, [x, w]) > C(C+1)d_0 \ .$$

Since S_u surely intersects $[x, y] \cup [y, z]$, by (Sep) we must have $d(u) > Cd_0$.

Since $k(w, [x, z]) > 4C^4(C+1)^3$ and $\mathrm{len}_d([u, w]) \leq Cd(u, w) \leq C^2 d(w)$, there must be a point $r \in [w, u]$ with

$$d(r) < d(w)/4C^2(C+1)^3 = d_0/2C(C+1) \ .$$

There exist points $r' \in [w,x] \cup [x,u]$ and $r'' \in [w,y] \cup [y,z] \cup [z,u]$ which are also in $S_r^{[w,u]}$, and so strictly within a distance $d_0/(C+1)$ of each other.

Suppose first that $r' \in [w,x]$. Since $d(r',r'') < d_0/(C+1)$ the maximality of p implies that we cannot have $r'' \in [u,z] \cup [z,y]$. We shall now show that $r'' \in [w,y]$ leads also to a contradiction. To do so observe first that $d(r') \vee d(r'') \leq (C+1)d(r) < d_0/2C$. Assuming $r'' \in [w,y]$ leads to

$$\begin{aligned} \mathrm{len}_d([r',r'']) &= \mathrm{len}_d([r',w]) + \mathrm{len}_d([w,r'']) \\ &\geq 2[d(w) - (d(r') \vee d(r''))] \geq 2[d(w) - \frac{d_0}{2C}] > d(w) > Cd(r',r''), \end{aligned}$$

which contradicts (GH).

Thus we must have $r' \in [x,u]$. If $r'' \in [u,z]$, then

$$\mathrm{len}_d([r',r'']) \geq d(u) - d(r') \geq Cd_0 - d_0/2C > Cd(r',r''),$$

which contradicts again (GH).

If $r'' \in [z,y]$, then

$$d(r',z) \leq d(r',r'') + d_0 < (C+2)d_0/(C+1)\ .$$

On the other hand by the minimality of q we have $\mathrm{len}_d([r',u]) \geq Cd_0 - d_0/(2C)$ and

$$\mathrm{len}_d([u,z]) \geq \mathrm{len}_d([q,z]) \geq d(q,z) \geq [C(C+1)-1]d_0.$$

Thus

$$\begin{aligned} \mathrm{len}_d([r',z]) \geq [C(C+2) - 1 - 1/(2C)]d_0 &\geq [C(C+2)/(C+1)]d_0 \\ &> C[d(r',r'') + d_0] \\ &\geq Cd(r',z), \end{aligned}$$

again contradicting (GH).

In conclusion we must have $r'' \in [y, w]$. Since $d(r', r'') < \frac{d_0}{C+1}$ and $r' \leq u < q$, the minimality of q implies that $r' < p$. We shall now apply the *first* estimate from Lemma 5.3 with the choice of data

$$(a, a', b, b', d_1, d_2) = (r'', r', p', p, d(r'', r') \vee d(p', p), d(r')) \ .$$

Since $d(r', r'') \vee d(p', p) \leq C(C+1)d_0$ and $d(r') \leq d_0/(2C)$, we obtain

$$\begin{aligned} \mathrm{len}_d([r'', p']) = \mathrm{len}_d([a, b]) &\leq 2Cd_1 + \frac{C^2(C+1)}{2} d_2 \\ &\leq 2C^2(C+1)d_0 + \frac{C(C+1)}{4} d_0 = [2C^3 + (9/4)C^2 + C/4]d_0 \ . \end{aligned}$$

This gives a contradiction, since

$$\begin{aligned} \mathrm{len}_d([r'', p']) \geq \mathrm{len}_d([r'', w]) \geq d(w) - d(r'') > d(w) - d_0/2C = \\ = [2C^3 + 4C^2 + 2C - 1/(2C)]d_0 \end{aligned}$$

The proof of the lemma is completed. □

For $0 < \eta \leq \epsilon$, we define an (ϵ, η)-*shortcut set* for our data D to be a collection of points $P = \{p_m, p'_m\}_{m=0}^{m_0}$, $m_0 \in \mathbb{N}$, that satisfy the following five properties:

(i) $p_0 \in [x, w]$, $p'_0 \in A$, $p_{m_0} \in A$, $p'_{m_0} \in [w, y]$;

(ii) $p_m, p'_m \in \gamma$, for all $0 < m < m_0$;

(iii) $p'_{m-1} \leq p_m \leq p'_m \leq p_{m+1}$ for all $0 < m < m_0$;

(iv) $\sum_{m=0}^{m_0} d(p_m, p'_m) < \epsilon$;

(v) $d(p_m, p'_m) \leq \eta$, for all $0 \leq m \leq m_0$.

Given such a set P, we write $A_P = \bigcup_{m=1}^{m_0} (p'_{m-1}, p_m)$, $A'_P = S_w^{[x,y]} \cap A_P$, and $A''_P = S_w^{[x,y]} \cap ([x, p_0] \cup [p'_{m_0}, y])$. We use shortcut sets to travel from x to y by connecting bits of geodesics with shortcut paths from p_m to p'_m. If ϵ is small, we can then use (Sep_2) to get a lower bound for $\mathrm{len}_d(A'_P)$; in particular A'_P is non-empty in that case. In fact, we have the following technical lemma.

Lemma 6.5. *Suppose that (X,d) is a C-GHS space, $x,y,z \in X$, $w \in [x,y]$ and that P is an $(L_0, L_0/2)$-shortcut set for the data D, where $L_0 = 2d(w)/3C$. Suppose also that $k(u,w) \geq \log(3C+3)$ for all $u \in A$. Then $A''_P = \emptyset$ and $\operatorname{len}_d(A'_P) \geq d(w)/3C > 0$.*

Proof. By hypothesis, there are paths $\lambda_m \in \Gamma(p_m, p'_m)$ such that $\sum_{m=0}^{m_0} \operatorname{len}_d(\lambda_m) \leq L_0$. We define the joined path

$$\lambda_P = [x,p_0] + \lambda_0 + \sum_{m=1}^{m_0} ([p'_{m-1}, p_m] + \lambda_m) + [p'_{m_0}, y].$$

Applying (Sep_2) to λ_P, we see that $\operatorname{len}_d(A'_P) + \operatorname{len}_d(A''_P) \geq C^{-1}d(w) - L_0 = L_0/2 > 0$.

To show that $A''_P = \emptyset$ assume by contradiction that there is a point $p \in [x,p_0]$ such that $p \in S_w^{[x,y]}$. Using (Sep_3), we see that

$$k(p_0,w) \leq k(p,w) \leq \log(C+1),$$

which implies in particular that $d(p_0) \geq d(w)/(C+1)$. Because $d(p_0,p'_0) \leq d(w)/3C$, it follows from the length space assumption that $k(p_0,p'_0) \leq \log 3$. By the triangle inequality, we get $k(p'_0,w) \leq \log 3(C+1)$, contradicting the hypothesis.

Similarly, we must have $S_w^{[x,y]} \cap [p'_{m_0}, y] = \emptyset$. Thus $A''_P = \emptyset$, and consequently $\operatorname{len}_d(A'_P) \geq L_0/2$, as desired. □

We now come to our main lemma concerning shortcut sets. We are really interested in applying this lemma only in the cases $m_0 = 1$ and $m_0 = 2$, but stating it for arbitrary m_0 allows us to write down a proof simultaneously for both cases. Note that, since we are aiming to prove Gromov hyperbolicity, the assumption $k(u,w) \geq \log(3C+3)$ does not make the lemma any less useful.

Lemma 6.6. *Suppose that (X,d) is a C-GHS space, and $x,y,z \in X$, $w \in [x,y]$. Let $\epsilon = d(w)/3C$ and and $\eta = \epsilon/C_1$, where $C_1 = (C^3+3C^2+6C+6)/2$. Suppose that P is an (ϵ,η)-shortcut set for the data D, and $k(u,w) \geq \log(3C+3)$ for all $u \in A$. Then there exists a point $u \in A'_P$ such that $d(u,w) \leq Cd(w)$ and $d(u) \geq \eta/C$.*

Proof. By the previous lemma the set A'_P is not empty. Since A'_P is a subset of S_w, for $u \in A'_P$ we have $d(u,w) \leq Cd(w)$. For the purpose of contradiction let us assume that $d(u) < \eta/C$ for all $u \in A'_P$.

We first wish to modify P to define a new shortcut set Q which avoids as much of the initial and final parts of γ as possible. Let

$$a'_- = \sup\left(\{p'_0\} \cup \{u \in A'_P \mid S_u \cap [x,w] \neq \emptyset\}\right) \in \overline{A'_P}\ .$$

We define a_- by the rule: $a_- = p_0$ if $a'_- = p'_0$, and otherwise let a_- be any point on $[x,w]$ for which $d(a'_-, a_-) < \eta$; this last choice is possible because $d(u,u') \leq Cd(u) \leq \eta$ whenever $u \in A'_P$, $u' \in S_u \cap [x,w]$. Next we set

$$a_+ = \inf\left(\{p_{m_0}\} \cup \{u \in A'_P \mid u \geq a'_-, S_u \cap [w,y] \neq \emptyset\}\right) \in \overline{A'_P}\ .$$

We also define a'_+ by the rule $a'_+ = p'_{m_0}$ if $a_+ = p_{m_0}$, and otherwise a'_+ is any point on $[w,y]$ for which $d(a'_+, a_+) < \eta$; this last choice is possible for similar reasons to those for a_-. Note that $d(a'_-) \leq \eta/C$ if $a'_- \neq p'_0$, and that $d(a_+) \leq \eta/C$ if $a_+ \neq p_{m_0}$. In this case we have

$$d(a_-, a'_-) \leq Cd(a'_-) \leq \eta\ ,\ d(a_+, a'_+) \leq Cd(a_+) \leq \eta\ .$$

We now define the new shortcut set $Q = \{q_m, q'_m\}_{m=0}^{m_1}$ by replacing p_0, p'_0, p_{m_0}, and p'_{m_0} by a_-, a'_-, a_+, and a'_+, respectively, and also discarding any pairs p_m, p'_m which are not elements of $[a'_-, a_+]$ (i.e. we keep only those pairs (p_m, p'_m) for which $a'_- \leq p_m \leq p'_m \leq a_+$).

Now Q is an $(\epsilon + 2\eta, \eta)$-shortcut set and, since $\epsilon + 2\eta < 2\epsilon$, Lemma 6.5 implies that $\mathrm{len}_d(A'_Q) \geq \epsilon > 0$. Note also that $A'_Q \subset A'_P$. By symmetry of x and y, we may assume that $A'_Q \cap [x, z]$ is non-empty. Applying symmetry may seem dangerous since a'_- satisfies a stronger extremality property than a_+, but the only fact following from the extremality of these points that we use below is that S_u cannot intersect $[x, y]$ if $a'_- < u < a_+$, and this can safely be deduced whether or not we swap the roles of x and y.

Letting

$$T_1 = \{u \in [a'_-, z] \mid d(u) \leq (C+1)\eta/C,\ \exists\, u' \in [z, y] :\ d(u, u') \leq \eta, d(u') \leq (C+1)\eta/C\},$$

we claim that $A'_Q \cap [x, z] \subset T_1$. First observe that for $u \in A'_Q \setminus \{a'_-\}$ we have $u > a'_-$ which implies that $S_u \cap [x, w] = \emptyset$. Similarly, for $u \in A'_Q \setminus \{a_+\}$ we have $u < a_+$ and so $S_u \cap [w, y] = \emptyset$. We conclude that $S_u \cap [x, y] \subset \{a'_-, a_+\}$ for all $u \in A'_Q$. Thus by applying (Sep_2) for the slice S_u, $u \in A'_Q \cap [x, z]$, to the path $[x, y] + [y, z]$, we get a point $u' \in S_u \cap [z, y]$, and our claim readily follows. Defining $a = \inf T_1$, it follows that $a \leq A'_Q$.

Note also that $d(a) \leq (C+1)\eta/C$, and there exists $a' \in [z, y]$ with $d(a, a') \leq \eta$ and $d(a') \leq (C+1)\eta/C$. We distinguish two cases. Either $a' \in A_Q$, or $q_m < a' < q'_m$ for some $1 \leq m \leq m_1$. In the first case, we define $a'' = a'$, and in the second case we let $a'' = q'_m$.

We form a new shortcut set R from Q by inserting the pair a, a'' and discarding any pairs in Q that are contained in $[a, a'']$. We claim that $d(a, a'') \leq \epsilon - 2\eta$.

The claim is easy to see if $a'' = a'$ since then

$$d(a, a'') = d(a, a') \leq \eta \leq \epsilon - 2\eta$$

by the choice of the magnitude of ϵ and η from the statement.

If $a'' = q'_m$ we consider two cases: $q_m \geq z$ and $q_m < z$. In the first case the points $z \leq q_m < a' < q'_m = a''$ are situated on $[z, y]$. By applying (GH), we obtain

$$d(a', a'') \leq \mathrm{len}_d([a', a'']) \leq \mathrm{len}_d([q_m, q'_m]) \leq C\eta,$$

which implies that

$$d(a, a'') \leq d(a, a') + d(a', a'') \leq \eta + C\eta \leq \epsilon - 2\eta.$$

Consider the second case $q_m < z$. Since $a \leq A'_Q$, and $A'_Q \cap [x, z]$ is a non-empty subset of $[x, q_m]$, we must have $a < q_m$. Thus we may apply the second conclusion of Lemma 5.3 with data

$$(a, a', b, b', d_1, d_2) = (a, a', q_m, q'_m, \eta, (C+1)\eta/C)$$

to deduce that

$$\begin{aligned} d(a, a'') = d(a, q'_m) &\leq d(a, a') + \mathrm{len}_d([a', q'_m]) \leq \eta[1 + C(C+5)/2 + C(C+1)^2/2] \\ &= \eta[C^3 + 3C^2 + 6C + 2]/2 \leq \epsilon - 2\eta \end{aligned}$$

Our claim follows. Since Q is an $(\epsilon + 2\eta, \eta)$ shortcut set, it follows that R is a $(2\epsilon, \epsilon)$-shortcut set. Thus, using Lemma 6.5, $\mathrm{len}_d(A'_R) \geq \epsilon > 0$.

Since $a \leq A'_Q$ and $A'_R \subset A'_Q$, we have $a \leq A'_R$. On the other hand,

$$A'_R \cap [x, z] \subset A_R \cap [x, z] \subset (a'_-, a).$$

Thus $A'_R \cap [x, z] = \emptyset$ and so $A'_R \subset A_R \cap [z, y] \subset (a'', a_+)$. We choose $b' \in A'_R$ such that $0 < \mathrm{len}_d(A'_R \cap (b', a_+)) < \eta$.

Let λ_{a_-, a'_-} be a path from a_- to a'_- such that $\mathrm{len}_d(\lambda_{a_-, a'_-}) < 2\eta$. By ($\mathrm{Sep}_2$), there must be some point b in $S_{b'}^{[y,z]}$ that also lies on the path $[y, a_-] + \lambda_{a_-, a'_-} + [a_-, z]$. But

we cannot have $b \in [y, a_-]$ by the extremality of a'_- and a_+, so suppose $b \in \lambda_{a_-, a'_-}$. Since $d(a_-, b') \leq d(a_-, b) + d(b, b') < 3\eta$, we can define a new shortcut set U from Q by inserting the initial pair a_-, b' and discarding all other pairs except those that lie in $[b', y] \cup \{a'_+\}$. Since Q is an $(\epsilon + 2\eta, \eta)$-shortcut set, it follows that U is an $(\epsilon + 5\eta, \eta)$-shortcut set, and so a $(2\epsilon, \epsilon)$-shortcut set. Thus, as usual, $\mathrm{len}_d(A'_U) > \epsilon > 0$. Since the collection of pairs in U other than a_-, b' coincide with the collection of pairs u, u' in R for which $u > b$, it is clear that $A'_U = A'_R \cap (b', a_+)$. Thus $\mathrm{len}_d(A'_U) < \eta < \epsilon$, giving us a contradiction.

Finally, we consider the case where $b \in (a'_-, z]$. By minimality of a, we must have $b \geq a$. Since $b' \in A'_R \subset A'_Q$, we have $d(b') \leq \eta/C$, $d(b, b') \leq \eta$, and $d(b) \leq (C+1)\eta/C$. Thus $b \geq a$. Now $d(a, a') \vee d(b, b') \leq \eta$, $d(a') < (C+1)\eta/C$, and

$$\mathrm{len}_d([a', b']) \geq \mathrm{len}_d([a'', b']) \geq \mathrm{len}_d(A'_R \cap (a'', b')) \geq \epsilon - \eta,$$

which contradicts Lemma 5.3, so we are done. □

We are now ready to tackle Theorem 6.1.

Proof of Theorem 6.1. We set $d_0 = 2C^3 d(w)/K$ and let $e' = \sup T_1$, where

$$T_1 = \{x\} \cup \{u \in A \mid d(u) \leq d_0,\ \exists\, u' \in [x, w] :\ d(u, u') \leq d_0 \text{ and } d(u') \leq d_0\}.$$

If $e' = x$, let $e = x$ also. If $e' \neq x$, then $d(e') \leq d_0$, and we may choose $e \in [x, w]$ and a path $\lambda_{e,e'} \in \Gamma(e, e')$ such that $d(e, e') \leq d_0$, $d(e) \leq d_0$, and $\mathrm{len}_d(\lambda_{e,e'}) \leq 2d_0$. If $e' > z$ then, using the fact that $4C^5(C+1)^2 < K$, it follows that

$$\mathrm{len}_d([e, e']) \leq C d(e, e') < d(w)/2C(C+1)^2,$$

and we can apply Lemma 6.2 to the triangle $ee'y$. We deduce that

$$k(w, A) \leq k(w, [e', y]) \leq 4C^4(C+1)^3 < K,$$

and the theorem follows. We may therefore assume that $e' \le z$.

Next we let $f = \inf T_2$, where

$$T_2 = \{z\} \cup \{u \in [x,z] \mid d(u) \le d_0,\ \exists\, u' \in [z,y] :\ d(u,u') \le d_0 \text{ and } d(u') \le d_0\}.$$

If $f = z$, let $f' = z$. Otherwise, $d(f) \le d_0$ and we find $f' \in [z,y]$ such that $d(f,f') \le d_0$ and $d(f') \le d_0$.

There are two cases to consider. Suppose first that $f < e'$ (and so in particular, $f \ne z$ and $e' \ne x$). In this case we argue again using Lemma 6.2 to a smaller geodesic triangle containing w in one of its sides. To do that let us assume first that $\mathrm{len}_d([e',f]) > C^2 d_0$. Then (GH) implies that

$$d([e',z],f) \ge \mathrm{len}_d([e',f])/C \ge C d_0.$$

Since $d(f) \le d_0$ we have $S_f^{[x,z]} \subset B_d(f, Cd_0)$ which implies that $[e',z] \cap S_f^{[x,z]} = \emptyset$. Applying ($\mathrm{Sep}_2$) to the path $[x,e] + \lambda_{e,e'} + [e',z]$, we deduce that there exists $u \in S_f^{[x,z]} \cap ([x,e] + \lambda_{e,e'})$. Let $f'' = u$ if $u \in [x,e]$, and otherwise let $f'' = e$. In the first case, note that

$$d(f'',f') < d(f'',f) + d(f,f') \le Cd_0 + d_0.$$

In the second case we have

$$d(f'',f') < \mathrm{len}_d(\lambda_{e,e'}) + Cd_0 + d_0 \le (C+3)d_0\ .$$

Alternatively assume that $\mathrm{len}_d([e',f]) < C^2 d_0$ (recall still $f < e'$). Let $f'' = e$ and note that

$$d(f'',f') \le d(f'',e') + d(e',f) + d(f,f') < d_0[1 + C^2 + 1].$$

By Gehring-Hayman, we conclude in either case that w lies in a triangle $f''f'y$ with the property that $\mathrm{len}_d([f', f'']) < (C^3 + 3C)d_0$. Since $(C^3 + 3C)d_0 < d(w)/2C(C+1)^2$, the result then follows by Lemma 6.2.

We may now turn to the more difficult situation when $f \geq e'$.

Next we define $g = \inf T_3$, where

$$T_3 = \{y\} \cup \{u \in A \mid d(u) \leq d_0,\ \exists\, u' \in [w, y] :\ d(u, u') \leq d_0 \text{ and } d(u') \leq d_0\}.$$

If $g = y$, let $g' = y$ also. If $g \neq y$, then $d(g) \leq d_0$, and we may choose $g' \in [w, y]$ and a path $\lambda_{g,g'} \in \Gamma(g, g')$ such that $d(g, g') \leq d_0$, $d(g') \leq d_0$, and $\mathrm{len}_d(\lambda_{g,g'}) \leq 2d_0$. We also define $h = \sup T_4$, where

$$T_4 = \{z\} \cup \{u \in [z, y] \mid d(u) \leq d_0,\ \exists\, u' \in [x, z] :\ d(u, u') \leq d_0 \text{ and } d(u') \leq d_0\}.$$

As before, we may assume without loss of generality that $h \leq g$, and so $f' \leq h \leq g$.

We now assume for the sake of contradiction that $k(w, A) > K$. Letting $P = \{a_m, a'_m \mid m = 0, 1, 2\}$, where $(a_0, a'_0, a_1, a'_1, a_2, a'_2) = (e, e', f, f', g, g')$, we see that P is an (ϵ, η)-shortcut set for $\epsilon = d(w)/3C$, $\eta = d(w)/3CC_1 \geq d_0$. Note that we are including the possibility of degenerate shortcut sets such as $(e, e', f, f', g, g') = (x, x, z, z, y, y)$. By Lemma 6.6, there exists $v \in A'_P = (e', f) \cup (f', g)$ such that $d(v, w) \leq Cd(w)$ and $d(v) \geq d(w)/3C^2C_1 = Kd_0/6C^5C_1 > 5C^3d_0/2$.

Consider first the case $v \in (e', f)$. Since $k(v, w) > K$ and $\mathrm{len}_d([v, w]) \leq C^2 d(w)$, it follows that there exists a point $p \in [v, w]$ such that $d(p) < C^2 d(w)/K = d_0/2C$. Now $S_p^{[v,w]}$ must intersect $[w, x] \cup [x, v]$ at some point p' and $[v, z] \cup [z, y] \cup [y, w]$ at some point p''; note that $d(p') \vee d(p'') \vee d(p', p'') < d_0$. If $p' \in [x, w]$, we cannot choose the point p'' with the desired properties. In fact, $p'' \in [v, z]$ contradicts the maximality of e', while if $p'' \in [z, y]$,

then applying Lemma 6.2 to the triangle $p'p''y$ contradicts the lower bound for $k(w, A)$. Finally if $p'' \in [w, y]$, then

$$\mathrm{len}_d([p', p'']) \geq d(w) - d(p') > Cd_0 \geq Cd(p', p''),$$

contradicting (GH).

Consequently, we must have $p' \in [x, v]$. Now $p'' \notin [y, w]$ since then applying Lemma 6.2 to the triangle $p'p''x$ contradicts the lower bound for $k(w, A)$, and $p'' \notin [v, z]$ since then $\mathrm{len}([p', p'']) \geq d(v) - d(p') > Cd(p', p'')$. Finally if $p'' \in [z, y]$, then we get a contradiction to the minimality of f.

If instead $v \in (f', g)$, we find, as before, points $p' \in [w, y] \cup [y, v]$ and $p'' \in [v, z] \cup [z, x] \cup [x, w]$ such that $d(p') \vee d(p'') \vee d(p', p'') < d_0$. All possibilities are then ruled out as before with the exception that there is no extremality of f' to rule out the possibility that $p' \in [y, v]$, $p'' \in [x, z]$. However, the minimality of f ensures that $p'' \geq f$. We can therefore apply Lemma 5.3 to the data

$$(a, a', b, b', d_1, d_2) = (f', f, p', p'', d_0, d_0)$$

(with the roles of x, y, z switched among themselves) to deduce that

$$\mathrm{len}_d([f', p']) \leq (2C + C^2(C + 1)/2)d_0 \leq 3C^3 d_0 \ ,$$

which contradicts the fact that

$$\mathrm{len}_d([f', p']) \geq 2d(v) - d(p') - d(f') > 3C^3 d_0 \ .$$

□

7. Appendix: Background on Loewner spaces

In this appendix, we define some basic concepts related to the modulus, and develop some properties of Loewner spaces.

First a *metric measure space* (X, d, μ) is simply a metric space with a Borel measure μ attached. If additionally (X, d) is rectifiably connected, we define $A(\Gamma)$, the set of Γ-*admissable weights* for a family Γ of paths in X, to be the set of all Borel functions $\rho : X \to [0, \infty]$ such that $\int_\gamma \rho\,|dz| \geq 1$ for each path $\gamma \in \Gamma$. For $Q \geq 1$, we then define the Q-*modulus* of Γ as

$$\operatorname{mod}_Q(\Gamma) = \inf_{\rho \in A(\Gamma)} \int_X \rho^Q \, d\mu.$$

When $S \subseteq X$ is a domain in X and E, F are two disjoint compacta in S we denote by $\Gamma(E, F; S)$ the family of paths contained in S that begin in E and terminate in F. In this case we define $A(E, F; S) = A(\Gamma(E, F; S))$ and $\operatorname{mod}_Q(E, F; S) = \operatorname{mod}_Q(\Gamma(E, F; S))$; we omit S in this notation when $S = X$. We define $\Gamma(E, F; S) = \infty$ if E and F overlap.

Suppose (X, d, μ) is a metric measure space. Given $Q \geq 1$, we say that X is Q-*regular* if there exists a constant $C > 0$ such that

$$\forall\, x \in X,\ 0 < r \leq \operatorname{dia}_d(X): \qquad C^{-1} r^Q \leq \mu(B_d(x, r)) \leq C r^Q.$$

If only the upper bound above holds, we say that X is *upper* Q-*regular*. We denote the smallest such constant C for which the (upper) Q-regularity condition holds by $C(\mu, Q)$ (or $C^+(\mu, Q)$, respectively). We define *local* Q-*regularity* in a similar fashion except that the upper and lower bounds for $\mu(B_d(x, r))$ apply only when $r < d(x)/2$.

Recall that

$$\Delta_d(E, F) = \frac{\operatorname{dist}_d(E, F)}{\operatorname{dia}_d(E) \wedge \operatorname{dia}_d(F)}.$$

Writing $PC(S)$ for the set of pairs of disjoint non-degenerate continua in $S \subset X$, a metric measure space (X, d, μ) is said to be *Q-Loewner* if for all $t > 0$,

$$\phi(t) \equiv \inf\{\mathrm{mod}_Q(E, F) \,:\, (E, F) \in PC(X),\, \Delta_d(E, F) \le t\} > 0.$$

We say that (X, d, μ) is *locally Q-Loewner*, or *locally $(Q, \kappa, \delta_0, \psi)$-Loewner*, if $\kappa \ge 1$, $\delta_0 \in (0, \kappa^{-1}]$, and for all $t > 0$,

$$\psi(t) \equiv \inf\{\, \mathrm{mod}_Q(E, F; B(\kappa\delta d(x))) \,:\, (E, F) \in PC(B(x, \delta d(x))),\, \Delta_d(E, F) \le t,\, x \in X,\, 0 < \delta \le \delta_0\} > 0$$

A typical example of a locally regular, locally Loewner space is an open set in a regular Loewner space; see [BoHeKo, Theorem 6.47].

The following lemma, in the case $r' = r/2$, is a restatement of Remark 3.19 in [HeKo]; our more general statement follows simply by chaining together paths generated by this special case.

Lemma 7.1. *Suppose (Y, d, μ) is a bounded, Q-regular, Q-Loewner metric measure space. Then there exists a number $C_0 > 1$, dependent quantitatively on the hypotheses, such that if $x, y \in Y$, $x \ne y$, then $l(x, y) < C_0 d(x, y)$. Moreover if $0 < r' < r$, $z \in Y$, and $x, y \in B_d(z, r) \setminus B_d(z, r')$, then there exists a path $\gamma \in \Gamma(x, y; B_d(z, C_0 r) \setminus B_d(z, r'/C_0))$ such that $\mathrm{len}_d(\gamma) \le C_0 d(x, y)$.*

Remark 7.2. The two concluded properties of (Y, d) in Lemma 7.1 are inherited by $(\overline{Y}, \overline{d})$. This can be seen using a chaining argument in the case of both properties. Assume $x, y \in \overline{Y}$ are distinct. A path of length comparable to $\overline{d}(x, y)$ is constructed by chaining together paths between the points

$$\ldots, x_{n+1}, x_n, \ldots, x_1, y_1, \ldots, y_n, y_{n+1}, \ldots$$

where (x_n), (y_n) are sequences of points in Y that converge at a geometric rate to x, y, respectively. The second property is handled in a similar fashion, except that one should choose a point $z_1 \in Y$, $\overline{d}(z_1, z) < r'/3C_0$, and appropriate points $x_n, y_n \in B_d(z_1, 3r/2) \setminus B_d(z, 2r'/3)$, $n \in \mathbb{N}$, in order to construct a path in $\gamma \in \Gamma(x, y; B_d(z, 2C_0 r) \setminus B_d(z, r'/3C_0))$.

The next lemma is of a well-known type, but for completeness we sketch the proof.

Lemma 7.3. *Suppose (X, d, μ) is a minimally nice, upper Q-regular, metric measure space, and let $C_0 = 2^{2Q-1} C^+(\mu, Q)$. Then*

$$\operatorname{mod}_Q(E, F) \leq C_0 (\log_2 \Delta_d(E, F))^{-Q+1},$$

whenever $E, F \subset X$ are disjoint non-degenerate compacta with $\Delta_d(E, F) \geq 2$.

Proof. Without loss of generality, we assume that $r \equiv \operatorname{dia}_d(E) \leq \operatorname{dia}_d(F)$. Let us fix $x \in E$ and write $R = \operatorname{dist}_d(E, F)$, so that $\Delta_d(E, F) = R/r \geq 2$. Let $N = \lfloor \log_2(R/r) \rfloor$, and define a density

$$\rho(z) = \begin{cases} 1/(2^{i-1} N r), & z \in \overline{B}_d(x, 2^i r) \setminus B_d(x, 2^{i-1} r),\ 1 \leq i \leq N, \\ 0, & \text{otherwise.} \end{cases}$$

It is a rather routine task to verify that ρ is an admissible weight and that for this choice of ρ we have the inequality

$$\int_X \rho^Q \, d\mu \leq C_0 \left(\log_2 \Delta_d(E, F)\right)^{-Q+1}.$$

This estimate concludes the proof. □

The next result is a rather technical modulus comparison lemma used only in the proof of Lemma 4.4, where E is an annulus, E' is a ball concentric with E that lies inside E, and F is far from E.

Lemma 7.4. *Let (X, d, μ) be a minimally nice, locally $(Q, \kappa, \delta_0, \psi)$-Loewner, metric measure space, let E, E', F be three continua in X with $0 < \operatorname{dia}_d(E) \le 3 \operatorname{dist}_d(E, E')$. Suppose further that E' lies in a component G of $X \setminus E$, and that there exists some point $x \in X$, and a number $r \in (0, \delta_0 d(x)/4]$ such that $G \cup E \subset \overline{B_d}(x, r)$, and $F \subset X \setminus B_d(x, 3r)$. Then*

$$\operatorname{mod}_Q(E', F) \ge 3^{-Q}[\psi(3) \wedge \operatorname{mod}_Q(E, F) \wedge \psi(\Delta_d(E, E'))].$$

Proof. Suppose $\int_X \rho^Q < 3^{-Q}[\operatorname{mod}_Q(E, F) \wedge \psi(\Delta_d(E, E'))]$ for some $\rho \in A(E', F)$. Using the local Loewner condition, we have $\psi(\Delta_d(E, E')) \le \operatorname{mod}_Q(E, E')$, and so $\int_X (3\rho)^Q < \operatorname{mod}_Q(E, F) \wedge \operatorname{mod}_Q(E, E')$. Thus $3\rho \notin A(E, F) \cup A(E, E')$, and consequently there exist paths $\lambda_1 \in \Gamma(F, E)$ and $\lambda_2 \in \Gamma(E, E')$ such that $\int_{\lambda_i} \rho \, |dz| < 1/3$, $i = 1, 2$.

We may assume that $\lambda_2 \subset \overline{G}$, and that the domains of both λ_1 and λ_2 are $[0, 1]$. Given a path η from $\lambda_1(t_1)$ to $\lambda_2(t_2)$, $t_1, t_2 \in [0, 1]$, we define a new path $\gamma = \lambda_1|_{[0,t_1]} + \eta + \lambda_2|_{[t_2,1]}$. Since $\rho \in A(E', F)$ we have $\int_\gamma \rho \ge 1$ which implies that $\int_\eta \rho \, |dz| \ge 1 - 2/3 = 1/3$. Because $\eta \in \Gamma(\lambda_1, \lambda_2)$ was an arbitrary path we obtain that $3\rho \in A(\lambda_1, \lambda_2)$.

Let λ_3 be a segment of λ_1 which begins on $\partial B_d(x, 3r)$ and ends in E. Then λ_2 and λ_3 are continua in $B_d(x, \delta_0 d(x))$, with $\operatorname{dia}_d(\lambda_2) \ge \operatorname{dist}_d(E, E') \ge \operatorname{dia}_d(E)/3$, $\operatorname{dia}_d(\lambda_3) \ge 2r \ge \operatorname{dia}_d E$, and $\operatorname{dist}_d(\lambda_2, \lambda_3) \le \operatorname{dia}_d(E)$. Thus

$$\Delta_d(\lambda_2, \lambda_3) \le \frac{\operatorname{dia}_d(E)}{\operatorname{dia}_d(E)/3} = 3,$$

and so

$$\operatorname{mod}_Q(\lambda_2, \lambda_3) \ge \operatorname{mod}_Q(\lambda_2, \lambda_3; B(x, \delta_0 \kappa d(x))) \ge \psi(3).$$

But $3\rho \in A(\lambda_1, \lambda_2) \subset A(\lambda_3, \lambda_2)$, and so $\int_X \rho^Q \ge 3^{-Q}\psi(3)$. The desired modulus estimate follows. □

The following "metric quasiconformality implies geometric quasiconformality" type result of Tyson [Ty, Theorem 6.4] is needed in Section 4. It uses the notation

$$L_f(x,r) = \sup\{d(f(x),f(y)) : y \in \overline{B}(x,r)\},$$

$$l_f(x,r) = \inf\{d(f(x),f(z)) : z \in X \setminus B(x,r)\}.$$

Lemma 7.5. *Let X and Y be locally compact metric measure spaces which are locally Q-regular for some $Q > 1$. Let $f : X \to Y$ be a homeomorphism and let $H < \infty$ and $t > 1$ be constants such that*

$$\forall\, x \in X : \qquad \liminf_{r\to 0} \frac{L_f(x,tr)}{l_f(x,r)} \le H. \tag{7.6}$$

Then there exists K dependent only in t, H, Q, and the Q-regularity constants of X and Y such that

$$\operatorname{mod}_Q \Gamma \le K \operatorname{mod}_Q f\Gamma,$$

for every curve family Γ in X.

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