DOUBLING FOR GENERAL SETS

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Abstract. We investigate doubling conditions defined in terms of measurable bounded sets and find a simple characterization of quasisymmetrically thick Cantor sets on the line.

§0. Introduction

In this paper we look at two seemingly unrelated questions, which broadly put are the following:

1. If we use an arbitrary set E in place of a cube when defining the doubling condition for a measure, what effect does this have on the class of measures which satisfy the doubling condition?

2. Given a subset E of $\mathbb{R}$ such that $|E| > 0$, when is it possible to find a quasisymmetric function f which "kills" E (i.e. such that $|f(E)| = 0$)?

Surprisingly enough, we will show that there is a close connection between these two questions. In the remainder of this introduction we describe these questions more precisely and end the section by stating a theorem which gives a clear description of this connection. We begin with the first question.

Doubling conditions for measures in Euclidean space are usually defined using nice open sets such as cubes or balls; we are interested in studying what happens when the doubling condition is defined using much more irregular bounded sets E. There are two basic problems that we shall tackle. Firstly, we shall investigate whether doubling with respect to E is implied by, or implies, doubling with respect to cubes. Secondly, we shall investigate whether "nearly optimal" doubling with respect to E is implied by, or implies, "nearly optimal" doubling with respect to cubes.

Thoughout this paper, Q_0 is the centered unit cube $(-1/2, 1/2)^n \subset \mathbb{R}^n$, and $E \subset \mathbb{R}^n$ is a non-empty bounded Borel measurable set. Since we are really interested in the family of all translated dilates of E, rather than just E itself, we normalize E so that it contains the origin, lies in $\overline{Q_0}$, and has diameter between 2^{-1} and $\sqrt{n}$. Whenever we have a set X with a distinguished "center" point $x \in X$, and $\lambda > 0$, we denote by λX the *concentric dilate of* X by the factor λ (with respect

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to the center x). For any of our sets E that we are considering, we take 0 to be its "center", so that λE has its usual vector space meaning. However, when considering $E' = fE$, where f is some specified bilipschitz mapping (usually an affine map), $f0$ is taken to be the center of E'; in particular, we always consider cubes and balls as having their usual centers. In any case, it should always be clear from the context which point is the center of a given set. We use C in proofs of theorems to refer to a generic constant that depends only on the allowed parameters; we shall also write $A \lesssim B$ (or $B \gtrsim A$) if $A \le CB$ for some such C, and we write $A \approx B$ if $A \lesssim B \lesssim A$.

We define doubling measures for E as follows: first, we say that two sets E_1 and E_2 are *neighboring copies of* E (or simply that E_2 is a neighbor of E_1) if there exist $\lambda > 0$ and $x_i \in \mathbb{R}^n$ such that $E_i = \lambda E + x_i$ and $|x_1 - x_2| \le \lambda$. More generally, if $N > 0$, we say that E_1 and E_2 are N*-neighboring copies of* E (or that E_2 is an N-neighbor of E_1), if there exist $\lambda > 0$ and $x_i \in \mathbb{R}^n$ such that $E_i = \lambda E + x_i$ and $|x_1 - x_2| \le N\lambda$. For any $\epsilon \ge 0$, $\mathcal{D}(E, \epsilon)$ is the set of all Borel measures μ such that $\mu(E) > 0$ and

$$(1+\epsilon)^{-1} \le \frac{\mu(E_1)}{\mu(E_2)} \le 1 + \epsilon, \tag{0.1}$$

whenever E_1, E_2 are neighboring copies of E. When $E = Q_0$, the unit cube, we will abbreviate this set as $\mathcal{D}(\epsilon)$. Note that by iteration, we get that if N is a positive integer and $\mu \in \mathcal{D}(E, \epsilon)$, then the ratio $\mu(E_1)/\mu(E_2)$ is at most $(1+\epsilon)^N$ for any N-neighboring copies of E. We call the smallest ϵ for which (0.1) is valid, the E*-doubling constant* of μ (or simply the doubling constant of μ if $E = Q_0$). As we shall see in the next section, $\mathcal{D}(E, \epsilon)$ is empty unless $|E| > 0$ (in which case Lebesgue measure always lies in $\mathcal{D}(E, \epsilon)$), so we restrict our attention to sets E of positive Lebesgue measure.

Doubling with respect to a general set E is always stronger than doubling with respect to cubes, as revealed by the following theorem (which we prove in the next section).

Theorem 0.2. *For all $\epsilon > 0$ there exists $\delta = \delta(\epsilon, E) > 0$ such that $\mathcal{D}(E, \epsilon) \subseteq \mathcal{D}(\delta)$. Furthermore, we can choose δ to tend to zero as ϵ tends to zero.*

Corollary 0.3. *$\mathcal{D}(E, 0)$ consists solely of multiples of Lebesgue measure.*

Proof. By Theorem 0.2, $\mathcal{D}(E, 0) \subseteq \mathcal{D}(0)$. It is clear that each measure in $\mathcal{D}(0)$ is a multiple of Lebesgue measure. □

If we reverse the roles of E and Q_0 in Theorem 0.2, both statements in the above theorem may become false. Let us therefore introduce the following properties of sets E:

$\mathbf{P_1}$: $\forall \epsilon > 0 \, \exists \, \delta = \delta(\epsilon, E) > 0 : \ \mathcal{D}(\epsilon) \subseteq \mathcal{D}(E, \delta)$.
$\mathbf{P_2}$: $\forall \delta > 0 \, \exists \, \epsilon = \epsilon(\delta, E) > 0 : \ \mathcal{D}(\epsilon) \subseteq \mathcal{D}(E, \delta)$.

We write $E \in P_i$ if E satisfies property P_i, $i = 1, 2$. A set E satisfies P_1 if all measures which are doubling with respect to cubes are doubling with respect to E, while E satisfies P_2 if all measures with a sufficiently small cube-doubling constant also have a small E-doubling constant. We shall see that there are sets that satisfy both, neither, or just one of these properties (in fact all four logical possibilities are realized).

We now turn our attention to the second question mentioned above. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is quasisymmetric, i.e. f is an increasing homeomorphism and there is a real number $\lambda \geq 1$ such that

$$\frac{1}{\lambda} \leq \frac{f(x+t) - f(x)}{f(x) - f(x-t)} \leq \lambda,$$

for all $x \in \mathbb{R}$ and $t > 0$. If we wish to be more specific, we will refer to such a function as being λ-quasisymmetric and we let $\mathcal{QS}(\lambda)$ denote the class of all λ-quasisymmetric functions defined on $[0, 1]$.

There is a nice 1-1 correspondence between quasisymmetric functions and doubling measures on $\mathbb{R}$. Namely, if $\mu \in \mathcal{D}(\epsilon)$, then the function f defined by $f(x) - f(0) = \int_0^x d\mu$ is $(1 + \epsilon)$-quasisymmetric. Conversely, if f is λ-quasisymmetric and the measure μ is defined by $\mu([a, b]) = f(b) - f(a)$ for intervals $[a, b]$, then $\mu \in \mathcal{D}(\lambda - 1)$. Throughout this paper we will make use of this correspondence without referring to it explicitly. The basic question that we consider is the following: Given a set $E \subset [0, 1]$ with $|E| > 0$ when is it possible to find a λ-quasisymmetric function f such that $f(E) = 0$? Furthermore, for which sets E is it possible to find such f with λ arbitrarily close to 1? We restrict our attention to Cantor sets. Since only sets with empty interior have a chance to be killed by a quasisymmetric mapping, this restriction is not as severe as it might seem at first glance.

By a *Cantor set*, we shall mean a compact set $K \subset [0, 1]$ which is the intersection of the nested compact sets K_i, where $K_0 = [0, 1]$ and K_i is defined by deleting an open interval of length $c_i|I|$ from the middle of every component I of K_{i-1}, and $0 < c_i \leq 1/2$ for all $i > 0$. Thus there is a bijection between our class of Cantor sets and the set of sequences with values in $(0, 1/2]$, and the usual Cantor set corresponds to the case $c_i = 1/3$; we write $(c_i) = \mathrm{SEQ}(K)$. Note that $0 \in K$ for each such Cantor set K but $K \not\subset \overline{Q_0} = [-1/2, 1/2]$, so by our formalism we should really replace K by $K' = \{x : 2x \in K\}$ when discussing doubling with respect to a Cantor set. However, the normalization of Cantor sets employed here is so standard, that we shall stick with it.

We subdivide Cantor sets into two main classes, TC and FC (thin and fat Cantor sets): $K \in TC$ if $|K| = 0$ and otherwise $K \in FC$. We further partition FC into the subclasses MFC, FFC, VFC (minimally fat, fairly fat, and very fat), defined as follows:

- $K \in MFC$ if $|K| > 0$ but, for every $\lambda > 1$, there exists $f \in \mathcal{QS}(\lambda)$ such that $|f(K)| = 0$.
- $K \in FFC$ if there exists $1 < \lambda_1 < \lambda_2$ such that K is non-null for all $f \in \mathcal{QS}(\lambda_1)$, but $|f(K)| = 0$ for some $f \in \mathcal{QS}(\lambda_2)$.
- $K \in VFC$ if it is non-null for all quasisymmetric functions f.

Thus, K is minimally fat if K can be killed by λ-quasisymmetric f with λ arbitrarily close to 1, K is fairly fat if K can only be killed by a λ-quasisymmetric function with sufficiently large λ and K is very fat if K cannot be killed by any quasisymmetric function.

We are now ready to state the main result advertised at the beginning of this section.

Theorem 0.4. *Suppose that K is a fat Cantor set (i.e. $|K| > 0$) and that $(c_i) = \mathrm{SEQ}(K)$.*

(a) $K \in MFC \iff (c_i) \notin \bigcup_{0<r<1} l^r \iff K \in P_1^c \cap P_2^c$.
(b) $K \in FFC \iff \exists\, 0 < r < s < 1 : (c_i) \in l^s \setminus l^r \iff K \in P_1^c \cap P_2$.
(c) $K \in VFC \iff (c_i) \in \bigcap_{0<r} l^r \iff K \in P_1 \cap P_2$.

Thus, Theorem 0.4 says that our subclasses of fat Cantor sets can be characterized very neatly, either by simple summability criteria on (c_i) or in terms of membership status with respect to P_1 and P_2. The characterization in terms of (c_i) makes it easy to produce examples of each type of Cantor set. For instance,

$$\begin{aligned} c_i = 1/(2i\log^2(i+2)) &\Rightarrow K \in MFC, \\ c_i = 1/2i^2 &\Rightarrow K \in FFC, \\ c_i = 1/2i! &\Rightarrow K \in VFC. \end{aligned}$$

Note that the 2's above are simply designed to ensure that $c_i \le 1/2$ in each instance.

Theorem 0.4 bears some similarity to the work of Wu [W] who characterised the "very thin" Cantor sets, i.e. those Cantor sets which are null sets for every doubling measure. More closely related is the work of Staples and Ward [SW], who considered the problem of characterizing very fat Cantor sets in terms of summability criteria on (c_i). They considered a slightly larger class of Cantor-like sets which also have the property that they cannot be killed by quasisymmetric mappings and referred to such sets as being quasisymmetrically thick. In particular, they showed that the middle condition in (c) implies that $K \in VFC$ and also that the reverse implication holds for a restricted class of quasisymmetrically thick sets which does not include any of the sets in VFC. In fact, they asked whether the Cantor set K with $c_i = 1/(i+1)^2$ is quasisymmetrically thick. Theorem 0.4 gives a negative answer to their question.

Section 1 contains some basic results, including a proof of Theorem 0.2. In Section 2, we reformulate properties P_1 and P_2. Section 3 contains a proof of Theorem 0.4. In Section 4 we look at some examples. Finally, we consider various extensions and related results in Section 5.

§1. Basic results

In order to prove Theorem 0.2, we fix $\mu \in \mathcal{D}(E,\epsilon)$ and define the function $F_\delta(x) = \mu(\delta E + x)$ for any $\delta > 0$. This function has the following nice properties (note that (ii) implies that $\mathcal{D}(E,\epsilon)$ is empty if $|E| = 0$).

Lemma 1.1. *Suppose that $s > 0$ and that Q is a cube of sidelength $l > 0$.*

(i) $(1+\epsilon)^{-1}F_s(x) \le F_s(y) \le (1+\epsilon)F_s(x)$ *whenever* $|x-y| \le s$.
(ii) *If* $0 < s < 1$, *then* $\int_{(1-s)Q} F_{ls} \le \mu(Q)|lsE| \le \int_{(1+s)Q} F_{ls}$.

Proof. Part (i) follows from the doubling property; a short computation using Fubini's Theorem suffices to prove (ii). □

Proof of Theorem 0.2. Fix $\delta > 0$. Choose $0 < s < 1$ so that $\dfrac{(1+s)^n}{(1-s)^n} = 1+\delta/2$. Suppose that Q_1 and Q_2 are neighboring copies of Q_0. From Lemma 1.1 (ii) we see that $\mu(Q_1)|sE| \le \int_{(1+s)Q_1} F_s$

and that $\int_{(1-s)Q_2} F_s \le \mu(Q_2)|sE|$. Thus,

$$\mu(Q_1) \le \left(\frac{\int_{(1+s)Q_1} F_s}{\int_{(1-s)Q_2} F_s} \right) \mu(Q_2) \le \frac{|(1+s)Q_1|}{|(1-s)Q_2|} \left(\frac{⨍_{(1+s)Q_1} F_s}{⨍_{(1-s)Q_2} F_s} \right) \mu(Q_2)$$
$$\le (1+\delta/2) \left(\frac{⨍_{(1+s)Q_1} F_s}{⨍_{(1-s)Q_2} F_s} \right) \mu(Q_2).$$

It follows easily from Lemma 1.1 (i) that the ratio of integrals in this last term is bounded by $(1+\epsilon)^M$, where M depends only on n and s. By symmetry, we also get the inequality $\mu(Q_2) \le (1+\delta/2)(1+\epsilon)^M \mu(Q_1)$. Consequently, $\mu \in \mathcal{D}(\delta)$, provided that $\epsilon > 0$ is small enough. This proves the second claim of the theorem and a slight modification of the proof gives the first claim as well. □

We have now proven the easy part, that doubling for general sets is stronger than doubling for cubes. The remainder of the paper is devoted to the converse problem: when do P_1 and P_2 hold? The next proposition is well-known, so we omit the simple proof (which essentially reduces to the fact that every open set contains a little cube).

Proposition 1.2. *If $E \subset Q_0$ contains an open subset then $E \in P_1$.*

Thus, P_1 is of interest only when E has empty interior. However, the obvious proof provides rather crude estimates that are insufficient to prove that any open set (other than a cube) satisfies P_2. Korey [Ko] showed that balls satisfy P_2. We shall further investigate which open sets satisfy P_2 in Sections 2 and 4.

Finally in this section, we state a version of the well-known Whitney decomposition, as given in [S]. By the Whitney cubes of a domain Ω, denoted by $\mathcal{W}(\Omega)$, we shall always mean the collection of such cubes with $A = 10$.

Lemma 1.3. *Given $A \ge 1$, there is $C = C(A,n)$ such that if Ω is a proper subdomain of $\mathbb{R}^n$, then $\Omega = \bigcup_j Q_j$, where the Q_j are disjoint cubes satisfying*

(i) $5A \le \operatorname{dist}(Q_j, \partial\Omega)/\operatorname{diam} Q_j \le 15A$.
(ii) $\sum_j \chi_{AQ_j} \le C\chi_\Omega$ *(where χ_S denotes the characteristic function of a set S).*

§2. P_1 and P_2: Generalities

In this section, we shall find conditions that are necessary and sufficient for E to have properties P_1 or P_2; these conditions, although not explicitly geometric, reformulate our problems in a way that greatly facilitates our later investigation and in particular, the proof of Theorem 0.4. We begin with a localization result.

Lemma 2.1. *There is a constant C, depending only on n, such that if $\mu \in \mathcal{D}(\epsilon)$ with $\mu(Q_0) = |Q_0|$, then there exists $\nu \in \mathcal{D}(C\epsilon)$ which equals μ on Q_0, and equals Lebesgue measure on $(2Q_0)^c$.*

Proof. We denote by $\mathcal{W}_I$ the Whitney decomposition of the interior of Q_0 and by $\mathcal{W}_O$ the Whitney decomposition of the exterior of Q_0. We define a "partner" function from $\mathcal{W}_O$ to $\mathcal{W}_I$ by the rule $p(Q) = Q'$ if Q' is the nearest cube in $\mathcal{W}_I$ to Q whose sidelength equals, or is as close as possible to, that of Q; there might be several such cubes Q' in which case any one of them suffices as the definition of $p(Q)$.

Now let ν coincide with μ on Q_0 and be given elsewhere by $d\nu(x) = w(x)\,dx$, where the weight w is defined on Q_0^c by

$$w(x) = \begin{cases} w_0(x), & x \in 2Q_0 \setminus Q_0, \\ 1, & x \in (2Q_0)^c. \end{cases}$$

where $w_0(x) = \mu(p(Q))/|p(Q)|$ for all $x \in Q \in \mathcal{W}_O$. We leave to the reader the routine verification that ν has the required properties—the only non-trivial part is to check that the doubling constant of ν is controlled in the desired manner, and this ultimately follows from the fact that if Q_1 and Q_2 are adjoining cubes in $\mathcal{W}_O$, then their partners are comparable in size and each partner is contained in a fixed dilate of the other. □

Let us define

$$\Delta(E,\epsilon) = \sup_{\mu_1,\mu_2 \in \mathcal{D}(\epsilon)} \frac{\mu_2(E)}{\mu_2(Q_0)} \left(\frac{\mu_1(E)}{\mu_1(Q_0)} \right)^{-1}$$

With this notation in hand, we are ready to characterize the classes P_1, P_2.

Theorem 2.2.

(i) *$E \in P_1$ if and only if $\Delta(E,t) < \infty$ for all $t > 0$.*
(ii) *$E \in P_2$ if and only if $\Delta(E,t) \to 1$ as $t \to 0^+$.*

Proof. We first prove (ii). Assuming $\Delta(E,t) \to 1$ as $t \to 0^+$, we wish to show that $E \in P_2$. We fix $\delta > 0$, choose $r, t > 0$ so small that $\Delta(E,t) < (1+r) < \sqrt{1+\delta}$, and let $\epsilon \equiv \min\{t, \sqrt{1+\delta}-1\}$. Given any $\mu \in \mathcal{D}(\epsilon)$, we shall show that $\mu \in \mathcal{D}(E,\delta)$.

Let E_1 and E_2 be neighboring copies of E. There are affine maps τ_i that take E to E_i, $i = 1,2$, and $E_i \subseteq Q_i$ where the cubes $Q_i = \tau_i(Q_0)$ are neighboring copies of Q_0. Since $\mu \in \mathcal{D}(\epsilon)$, we know that

$$(1+\epsilon)^{-1} \le \frac{\mu(Q_1)}{\mu(Q_2)} \le 1+\epsilon. \tag{2.3}$$

We define $\mu_i \in \mathcal{D}(\epsilon)$ to be the pullbacks of μ with respect to the maps τ_i, so that

$$\frac{\mu_i(E)}{\mu_i(Q_0)} = \frac{\mu(E_i)}{\mu(Q_i)}$$

Now, applying the estimate $\Delta(E,t) < 1+r$ to μ_i, and using (2.3), we deduce that

$$(1+r)^{-1}(1+\epsilon)^{-1} \le \frac{\mu(E_2)}{\mu(E_1)} \le (1+r)(1+\epsilon)$$

It follows that $\mu \in \mathcal{D}(E, \delta)$ as required.

For the converse direction, assume that $E \in P_2$. Then given $\delta > 0$, we choose t so that $\mu \in \mathcal{D}(t) \Rightarrow \mu \in \mathcal{D}(E, \delta)$. By rescaling, we can assume that $\mu(Q_0) = |Q_0|$ and, appealing to Lemma 2.1, we can also assume that μ equals Lebesgue measure on $(2Q_0)^c$.

Clearly, there exists a 3-neighbor E_1 of E, such that $E_1 \subseteq (2Q_0)^c$. Since μ is an element of $\mathcal{D}(E, \delta)$, we can then apply the doubling property iteratively to obtain

$$\frac{\mu(E)}{|E|} \in [(1+\delta)^{-3}, (1+\delta)^3],$$

which gives that $\Delta(E, t) \to 1$ as $t \to 0^+$, as desired.

We omit the proof of (i), as it is similar to that of (ii). □

As a first application of Theorem 2.2, we show that open sets automatically satisfy P_2 unless their boundary has positive measure (as already mentioned, all open sets satisfy P_1).

Theorem 2.4. *If $U \subset Q_0$ is open, and $|\partial U| = 0$ then $U \in P_2$.*

Proof. It suffices to show that $\Delta(U, t) \to 1$ as $t \to 0^+$. Let us fix $0 < \delta < 1$, and let D_k denote the class of all dyadic (closed) cubes of sidelength 2^{-k}. For every $k > 0$, let U_k be the union of all $Q \in D_k$, $Q \subset U$, and let U'_k be the union of all $Q \in D_k$ which intersect $\overline{U}$ but are not contained in U. Clearly (U_k) is a nested increasing sequence of sets whose union is U, and (U'_k) is a nested decreasing sequence of sets whose intersection is ∂U. Since $|\partial U| = 0$, we must have $|U'_k| \to 0$ as $k \to \infty$. Let us therefore fix k so large that $|U'_k| < \delta|U|/8$, and choose $\epsilon > 0$ so small that $(1+\epsilon)^k < 1 + \delta/2$ and $(1-\epsilon)^k > 1 - \delta/2$.

If μ has a very small doubling constant for cubes, then the μ-measure of the double-dilate of a cube must be almost exactly 2^n times the μ-measure of the cube itself (since we can tile the larger cube using 2^n smaller cubes). Thus there exists $0 < t < 1$ such that $2^n\mu(Q_1)/\mu(Q_2) \in [1-\epsilon, 1+\epsilon]$ whenever $\mu \in \mathcal{D}(t)$, $Q_1 \subset Q_2$ are cubes, and the sidelength of Q_2 is double that of Q_1.

Let $\mu \in \mathcal{D}(t)$ be normalized so that $\mu(Q_0) = 1$. Iterating our last estimate, we see that

$$\mu(Q)/|Q| \in [(1-\epsilon)^k, (1+\epsilon)^k], \qquad \text{for every } Q \in D_k.$$

Since U_k and U'_k are unions of dyadic cubes of size 2^{-k}, we see that

$$1 - \delta/2 < (1-\epsilon)^k < \mu(U_k)/|U_k| < (1+\epsilon)^k < 1 + \delta/2$$

and that

$$\mu(U'_k) < (1+\delta/2)|U'_k| < \delta(1+\delta/2)|U|/8 < \delta|U|/4.$$

Since $U_k \subset U \subset U_k \cup U'_k$, it readily follows that $\mu(U)/|U| \in (1-\delta, 1+\delta)$, as required. □

§3. Proof of Theorem 0.4

Throughout this section, $\mathcal{D}_k$ is the 4-adic subintervals of $[0, 1]$ of length 4^{-k}. We start by recalling a well-known method of constructing doubling measures. This type of construction originated with Kahane, see [Ka], although this particular lemma is not mentioned there. The proof of the lemma is straightforward and follows by much the same method used in Section 3 of [Ka].

Lemma 3.1. *Fix $\epsilon > 0$ and let $\{f_k\}_{k=0}^{\infty}$ be a sequence of positive functions on $[0,1]$ with the following properties:*

(1) *Each f_j is constant on each element of $\mathcal{D}_j$.*
(2) *If I and J are two adjacent elements of $\mathcal{D}_j$ having the same parent (i.e., contained in the same element of $\mathcal{D}_{j-1}$), then*

$$\frac{1}{1+\epsilon} \leq \frac{f_j|_I}{f_j|_J} \leq 1+\epsilon.$$

(3) *If I and J are two adjacent elements of $\mathcal{D}_j$ with different parents, then*

$$\frac{f_j|_I}{f_j|_J} = \frac{f_{j-1}|_I}{f_{j-1}|_J}.$$

(4) *f_0 is identically 1.*
(5) *If $I \in \mathcal{D}_{j-1}$, then $\int_I f_j = \int_I f_{j-1}$.*

Then the measures μ_j on $[0,1]$ defined by $d\mu_j = f_j\,dx$ are probability measures with doubling constant $C\epsilon$, where C is a universal constant. Furthermore, these measures converge, in the weak-$*$ sense, to a probability measure μ which is doubling with constant $C\epsilon$.

Proof of Theorem 0.4.

Recall that we are assuming that $|K| > 0$, which is well-known to be equivalent to $(c_i) \in l^1$. We first show that the left-hand conditions in (a), (b), and (c) are equivalent to the center conditions. As MFC, FFC, and VFC partition the fat Cantor sets, it is enough to prove the following implications:

1. $(c_i) \in l^s$ for some $0 < s < 1 \Rightarrow K \in FFC \cup VFC$
2. $(c_i) \in l^s$ for all $0 < s < 1 \Rightarrow K \in VFC$.
3. $(c_i) \notin l^s$ for all $0 < s < 1 \Rightarrow K \in MFC$.
4. $(c_i) \notin l^s$ for some $0 < s < 1 \Rightarrow K \in MFC \cup FFC$.

To begin with we assume that $(c_i) \in l^s$ for some $0 < s < 1$. We will show that there exists an $\epsilon = \epsilon(s) > 0$ such that $\mu(K) > 0$ for all $\mu \in \mathcal{D}(\epsilon)$. Moreover, we show that we can take $\epsilon \to \infty$ as s tends to 0. To this end assume that $\mu \in \mathcal{D}(\epsilon)$, where $\epsilon = \epsilon(s)$ is a positive number to be specified later. Suppose also that I is an interval and that J is the concentric open subinterval of I whose length is $\lambda|I|$. Since translate-doubling for cubes gives us control over dilate-doubling for cubes, we see that $\mu(J)/\mu(I) \leq (1+\epsilon')|J|/|I|$ if $\lambda = 1/2$, where $1 > \epsilon' > 0$ and ϵ' tends to 0 as $\epsilon \to 0$. Iterating this inequality for $\lambda = 1/2$, we get that $\mu(J) \leq (1+\epsilon')^k \mu(I)/2^k$ if $\lambda = 2^{-k}$. Straightforward estimation now gives us that, for arbitrary $0 < \lambda \leq 1/2$, $\mu(J)/\mu(I) \leq (1+\epsilon')(|J|/|I|)^{\alpha}$, where $0 < \alpha < 1$ and α tends to 1 as $\epsilon \to 0$. In the case where I is a component of K_{i-1}, and J is the concentric open subinterval of I of length $c_i|I|$, we therefore deduce that $\mu(K_i)/\mu(K_{i-1}) \geq 1 - Cc_i^{\alpha} \equiv d_i$. For large enough i, $d_i < \frac{1}{2}$ and so $\mu(K) > 0$ if $\sum_{k=1}^{\infty} c_i^{\alpha} < \infty$. Hence, we need only choose $\epsilon > 0$ small enough so that $\alpha > s$ and we are done. Moreover, if $(c_i) \in l^s$ for all $0 < s < 1$, then we get $\mu(K) > 0$ regardless of the size of ϵ. Thus we have proven parts 1 and 2 above.

Now suppose that $(c_i) \notin l^s$ for all $0 < s < 1$. Fix $0 < \epsilon < 1/10$. We will show that there is a universal constant C and measure $\mu \in \mathcal{D}(C\epsilon)$ for which $\mu(K) = 0$, i.e. $K \in MFC$.

Denote by $\mathcal{G}$ the collection of components of $[0,1] \setminus K$. These are the "gaps" in the Cantor set K. For each $A \in \mathcal{G}$ there exists n such that A is contained in K_n but not in K_{n+1}. In other words, the gap A first appears at level $n+1$. We set $\tilde{A}$ to be the component of K_n that contains A. We will refer to this interval as the parent of A. Define $\mathcal{G}_k$ to be those elements A of $\mathcal{G}$ for which $|\tilde{A}| \geq 10(4^{-k})$. The constraint $c_i \leq 1/2$ guarantees that the elements of $\mathcal{G}_k$ are at least a distance $5(4^{-k})/2$ apart. Consequently, the union of any two adjacent elements of $\mathcal{D}_k$ can intersect at most one of the gaps A in $\mathcal{G}_k$. Now consider the subset $\mathcal{A}_k$ of $\mathcal{G}_k$ consisting of those A for which $|A| \leq (1/5)4^{-k}$. We will refer to these as the active gaps. The active gaps have relatively small length compared to the elements of $\mathcal{D}_k$ and have parents that are relatively long compared to the elements of $\mathcal{D}_k$. Next, we divide up the elements of $\mathcal{D}_k$. Set

$$\mathcal{Z}_k = \{I \in \mathcal{D}_k : I \text{ intersects no element of } \mathcal{A}_k\}$$
$$\mathcal{W}_k = \{I \in \mathcal{D}_k : I \text{ intersects some element of } \mathcal{A}_k\}$$

As mentioned above, the union of any two adjacent elements of $\mathcal{D}_k$ can intersect at most one of the gaps $\mathcal{G}_k$. Consequently,

i) Every interval in $\mathcal{W}_k$ actually only intersects one element of $\mathcal{A}_k$
ii) If two elements of $\mathcal{W}_k$ are adjacent, then their common endpoint lies in an element of $\mathcal{A}_k$.

We are now ready to build the measure. Set f_0 to be 1 on $[0,1]$. Let us assume that $\{f_j\}_{j=0}^{k}$ satisfies the hypotheses of Lemma 3.1 with ϵ replaced by 3ϵ. In our definition of f_{k+1} we need to be careful to ensure that condition (3) in the lemma is satisfied. Suppose that $I \in \mathcal{W}_k$. We let f_{k+1} have value $(1+\epsilon)f_k$ on any child that intersects an element of $\mathcal{A}_k$. The condition $|A| \leq (1/5)4^{-k}$ guarantees that there are at most two such children. We then assign the values f_k or $(1-\epsilon)f_k$ to the remaining children so as to ensure that $\int_I f_{k+1} = \int_I f_k$. Now suppose that $I \in \mathcal{Z}_k$. For $M \in \mathcal{D}_{k+1}$, take $\sigma(M)$ to be $1, 0,$ or -1 depending on whether f_{k+1} differs from f_k by a factor of $1+\epsilon, 1,$ or $1-\epsilon$ on M. Let L be the leftmost child of I, J be the lefthand neighbour of I, and M be the lefthand neighbour of L. If $J \in \mathcal{Z}_k$, we define f_{k+1} on L to be f_k. Otherwise we set $f_{k+1} = (1+\sigma(M)\epsilon)f_k$ on L. We define f_{k+1} analogously on the rightmost child of I. As above, we assign the values f_k or $(1-\epsilon)f_k$ to the remaining children so as to ensure that $\int_I f_{k+1} = \int_I f_k$.

It is clear that $\{f_j\}_{j=0}^{k+1}$ satisfies all of the hypotheses of Lemma 3.1 with ϵ replaced by 3ϵ (since $(1+\epsilon)/(1-\epsilon) < 1+3\epsilon$), except perhaps for (3) being valid when $j = k+1$. To this end, let L and M be adjacent elements of $\mathcal{D}_{k+1}$ with different parents, I, J respectively. There are three cases to consider. If $I, J \in \mathcal{Z}_k$, then $f_{k+1} = f_k$ on L and on M, by definition. If $I \in \mathcal{Z}_k$ and $J \in \mathcal{W}_k$, then $f_{k+1} = (1+\sigma(M)\epsilon)f_k$ on M, by definition of σ, and so $f_{k+1} = (1+\sigma(M)\epsilon)f_k$ on L. If $I \in \mathcal{W}_k$ and $J \in \mathcal{W}_k$, then L and M both intersect an element of $\mathcal{A}_k$, by (ii) above. Thus $f_{k+1} = (1+\epsilon)f_k$ on L and on M. These are all the possible cases, and (3) holds for each. Hence, by induction, we obtain $\{f_j\}_{j=0}^{\infty}$ satisfying the hypotheses of Lemma 3.1 with ϵ replaced by 3ϵ. The measures $d\mu_j = f_j\,dx$ converge in the weak-$*$ sense to a probability measure μ which is doubling with constant $1+C\epsilon$. We will show that $\mu(K) = 0$. In fact, we will prove that there exists a positive constant α such that

$$\mu(K_{n+1}) \leq (1 - \alpha(c_n)^{1-\epsilon/2})\mu(K_n) \tag{3.2}$$

for every $n \geq 1$. This clearly yields $\mu(K) = 0$.

Let $A \in \mathcal{G}$ whose parent is a component of K_n. Then $|A| = c_n|\tilde{A}|$. If we can show that

$$\mu(A) \geq \alpha \left(\frac{|A|}{|\tilde{A}|} \right)^{1-\epsilon/2} \mu(\tilde{A}),$$

then we immediately obtain (3.2). The preceding inequality follows easily from the doubling of μ whenever $c_n \geq 1/100$. Thus we only need to consider the case $c_n \leq 1/100$. The gap A is active for all k satisfying $5|A| \leq 4^{-k} \leq |\tilde{A}|/10$. Set s to be the minimum such k and t to be the maximum such k. Now choose an interval J in $\mathcal{D}_t$ which intersects A and take I to be the element of $\mathcal{D}_s$ that contains J. The lengths of J and A and I and $\tilde{A}$ are comparable. As μ is doubling, we see that it is enough to prove that

$$\mu(J) \geq \left(\frac{|J|}{|I|} \right)^{1-\epsilon/2} \mu(I).$$

Note that $\mu(I) = \mu_s(I)$ and that $\mu(J) = \mu_t(J)$. As the gap A is active for $s \leq k \leq t$ and as J intersects A we must have

$$\mu_{k+1}(J) = (1+\epsilon)\mu_k(J) \text{ for } s \leq k \leq t-1.$$

So,

$$\begin{aligned} \mu(J) = \mu_t(J) = (1+\epsilon)^{t-s}\mu_s(J) &= (1+\epsilon)^{t-s}\frac{|J|}{|I|}\,\mu_s(I) \\ = (1+\epsilon)^{t-s}\frac{|J|}{|I|}\,\mu(I) &= \left(\frac{1+\epsilon}{4} \right)^{t-s} \mu(I) \\ \geq \left(\left(\frac{1}{4} \right)^{t-s} \right)^{1-\epsilon/2} \mu(I) &= \left(\frac{|J|}{|I|} \right)^{1-\epsilon/2} \mu(I). \end{aligned}$$

This completes the proof that $K \in MFC$ when $(c_i) \notin l^s$ for all $0 < s < 1$.

To conclude the first half of the proof we must show that if $(c_i) \notin l^r$ for some $0 < r < 1$, then $K \in MFC \cup FFC$. The estimates are now somewhat different (in particular, ϵ is now close to, but less than, 1, and the doubling constant is some fnuction of ϵ), but the proof is nevertheless obtained easily by modifying the above argument, so we leave the details to the reader.

For the second half of the proof we need to show that the left-hand conditions in (a), (b), and (c) are equivalent to the right-hand conditions.

We note first of all that if $\mu(K) = 0$, then μ is automatically disqualified from being doubling with respect to K. It follows easily that $K \in MFC \Rightarrow K \in P_1^c \cap P_2^c$ and $K \in FFC \Rightarrow K \in P_1^c$. Once again appealing to the fact that MFC, FFC, and VFC partition the fat Cantor sets, we see that the second half of the proof will be complete if we show that $K \in VFC \Rightarrow K \in P_1$ and $K \in VFC \cup FFC \Rightarrow K \in P_2$. Actually, we give a careful proof of the second statement and leave the similar, but simpler proof of the first statement to the reader.

So, assume that $K \in VFC \cup FFC$. By Theorem 2.2 it suffices to show that $\Delta(K,t) \to 1$ as $t \to 0^+$. Let $\epsilon > 0$ and $\mu \in \mathcal{D}(t)$, normalized so that $\mu([0,1]) = 1$. We need to show that

$$(1+\epsilon)^{-1} < \frac{\mu(K)}{|K|} < 1+\epsilon \tag{3.3}$$

if t is small enough.

Since $K \in VFC \cup FFC$, we know by the first half of the proof that there exists an $\alpha < 1$ such that $\sum_{i=1}^{\infty} c_i^\alpha < \infty$. A simple induction argument shows that by choosing t small enough we may guarantee that

$$\frac{\mu(I)}{\mu(J)} \le C_1 \left(\frac{|I|}{|J|}\right)^\alpha, \tag{3.4}$$

where $C_1 \ge 1$ and I, J are any intervals with $I \subset J$. Now choose N so large that

$$\prod_{i=N+1}^{\infty} (1 - C_1 c_i^\alpha) > \frac{1}{\sqrt{1+\epsilon}}, \tag{3.5}$$

and hence

$$\prod_{i=N+1}^{\infty} (1 - c_i) > \frac{1}{\sqrt{1+\epsilon}}. \tag{3.6}$$

Since K_N is the union of intervals whose lengths are bounded below, we may also choose t so small that

$$\frac{1}{\sqrt{1+\epsilon}} < \frac{\mu(K_N)}{|K_N|} < \sqrt{1+\epsilon}. \tag{3.7}$$

Now note that (3.4) implies that

$$\frac{\mu(K_{i+1})}{\mu(K_i)} \ge 1 - C_1 c_i^\alpha$$

for all i and hence by (3.5), $1 > \frac{\mu(K)}{\mu(K_N)} > \frac{1}{\sqrt{1+\epsilon}}$ and similarly, by (3.6), $1 > \frac{|K|}{|K_N|} > \frac{1}{\sqrt{1+\epsilon}}$. Finally, using (3.7) we get (3.3), as desired. $\square$

§4. Examples

We can now show that all four logical possibilities involving P_1 and P_2 can occur. Theorem 0.4 shows that there are compact sets satisfying $P_1 \cap P_2$, $P_1^c \cap P_2$, and $P_1^c \cap P_2^c$. Finally, the $(0,1)$-complement of a minimally fat Cantor set lies in $P_1 \cap P_2^c$ as follows from Theorem 2.2 and the following paragraph.

The above characterization of Cantor sets K also allows us to completely solve the associated question of whether or not $U \equiv (0,1) \setminus K \in P_2$ (since U is open, we automatically know that $U \in P_1$). Specifically, $U \notin P_2$ if and only if $K \in MFC$. To see this, note that if K is a fat Cantor set, then it is clear from Theorem 2.2 and the fact that K and U partition $[0,1]$ that $K \in P_2$ if and only if $U \in P_2$. On the other hand, if K is thin then Theorem 2.4 shows that $U \in P_2$.

The following table summarizes when various types of Cantor sets K and their $(0,1)$-complements U satisfy P_1 or P_2 ("Y" and "N" indicate respectively that the property is or is not satisfied; "–" indicates that the set has Lebesgue measure zero and so is not of interest).

	TC		MFC		FFC		VFC	
	K	U	K	U	K	U	K	U
P_1	–	Y	N	Y	N	Y	Y	Y
P_2	–	Y	N	N	Y	Y	Y	Y

Having seen the examples above, it might seem reasonable to hope that the question of whether or not a given set satisfies P_1 or P_2 could be decided using a criterion depending only on the boundary. For example, one might guess from the above table and Theorem 0.4 that an open set E lies in P_2^c if and only if ∂E is a set of positive Lebesgue measure that is a null set for a doubling measure of arbitrary small cube-doubling constant. This guess is however wrong, and the following class of examples show that knowing ∂E alone cannot determine whether or not $E \in P_2$.

Given a Cantor set K as above, we partition its $(0,1)$-complement U into three pieces, $N \cup L \cup H$, where N is an countable set (and so null for all doubling measures on the line), and L and H are open sets (the Laurel and Hardy sets). The components of L and H are intermingled in such a way that $\partial L = \partial H$ (when considered as subsets of the topological space $(0,1)$), but if K is minimally fat then $L \in P_2$ and $H \notin P_2$. To define L and H, let us write U as a union of components $U_{i,k} = (a_{i,k}, b_{i,k})$, $i \in \mathbb{N}$, $k = 1, \ldots, 2^{i-1}$, where $U_{i,k}$ is one of the new components of U added at the ith stage of the construction of U. We then define $H = \bigcup H_{i,k}$ and $L = \bigcup L_{i,k}$, where $L_{i,k} = (a_{i,k}, c_{i,k})$, $H_{i,k} = (c_{i,k}, b_{i,k})$, and $c_{i,k} = a_{i,k} + 3^{-i}(b_{i,k} - a_{i,k})$ (N is, of course, the set of all points $c_{i,k}$). The later-stage components of the Laurel set are much thinner than those of the Hardy set. We leave it as an exercise to the reader to verify that, because $\sum_k |L_{i,k}|$ decreases so quickly as i increases the set L always satisfies P_2 (hint: look at the proof of Theorem 2.4, and use as approximating intervals the component intervals of L which are added prior to the i_0th stage of the construction, for some arbitrary i_0). If H also satisfies P_2, it follows from Theorem 2.2 that U satisfies P_2. Thus H does not satisfy P_2 if K is minimally fat.

§5. Further results

First in this section, we discuss variants of the classes $\mathcal{D}(E, \epsilon)$ of doubling measures. We define BL to be the class of all mappings from $\mathbb{R}^n$ to itself that fix the origin and satisfy the bilipschitz condition $|x-y|/2 \le |f(x) - f(y)| \le 2|x-y|$, for all $x, y \in Q_0$. If S is a subset of BL, we denote by $\mathcal{D}^S(E,\epsilon)$ the class of all Borel measures in $\mathcal{D}(E,\epsilon)$ satisfying the following additional conditions:

$$(1+\epsilon)^{-1} \le \frac{\mu(gE')}{|gE'|}\left(\frac{\mu(E')}{|E'|}\right)^{-1} \le 1+\epsilon,$$

whenever $E' = \lambda E + x$, $g(z) = AfA^{-1}z$, $Az = \lambda z + x$, $\lambda > 0$, $x \in \mathbb{R}^n$, $f \in S$. When $E = Q_0$ we will abreviate this set set as $\mathcal{D}^S(\epsilon)$. We also define LBL to be the class of linear maps $x \mapsto Bx$ that lie in BL. Notice that the functions g above are recentered and rescaled versions of f that satisfy the same bilipschitz condition. To better understand this condition, the reader may wish to consider $S = \{x \mapsto 2x\}$.

Our somewhat abstract setup includes some rather interesting special cases—for example, we could choose S to consist of all rotations around the origin, or all dilations by factors between 1 and 2. The choice of 2 as the bound on the bilipschitz constant for functions in BL is not significant; any bilipschitz bound is sufficient to get the same results (of course the constants involved would then also depend on this bound).

Proposition 5.1. *If U is an open set and $S \subset BL$, the following statements are equivalent:*

(i) $\mu \in \mathcal{D}(U,\epsilon_1)$ *for some* $\epsilon_1 > 0$*;*
(ii) $\mu \in \mathcal{D}(\epsilon_2)$ *for some* $\epsilon_2 > 0$*;*
(iii) $\mu \in \mathcal{D}^S(U,\epsilon_3)$ *for some* $\epsilon_3 > 0$*;*
(iv) $\mu \in \mathcal{D}^S(\epsilon_4)$ *for some* $\epsilon_4 > 0$*.*

Furthermore, the constants ϵ_i depend only on each other, and the set U.

Proof. We first note that the implications (iii)⇒(i) and (iv)⇒(ii) hold trivially. Furthermore, (i) and (ii) are equivalent by Theorem 0.2 and Proposition 1.2. It therefore suffices to show that (ii) implies (iii) and (iv).

Fixing U, we note that plain cube doubling gives quantitative control over dilates of cubes by factors between one and two, because we can tile $2Q$ with 2^n copies of a cube Q. Iterating this inequality, we get that $\mu(2^kQ) \le C^k\mu(Q)$. Two neighboring copies of U contain cubes of the same size which have the same μ-measure up to a fixed factor C (since they are N-neighboring copies of Q_0 for some $N = N(U)$), and are contained in neighboring cubes of the same size that are larger than the interior cubes by some bounded factor (the bound depends only on U). It therefore follows that the μ-measures of the neighboring copies of U are comparable. Bilipschitz images of U are controlled in exactly the same fashion. This shows that (ii) implies (iii) and taking $U = Q_0$, we get (ii) implies (iv) as well. □

Obviously, Theorem 0.2 tells us that any of these new doubling conditions with respect to $E \subset \mathbb{R}^n$ and $S \subset BL$ implies translate-doubling for cubes (with the cube constant tending to zero as the E-constant tends to zero). As before, we can therefore ask what pairs (E,S) are such that the opposite implications hold true. To be more precise, we generalize P_1, P_2 in the obvious way:

$\mathbf{P_1^S}$: $\forall\epsilon > 0\,\exists\,\delta = \delta(\epsilon,E) > 0 :\ \mathcal{D}(\epsilon) \subseteq \mathcal{D}^S(E,\delta)$.
$\mathbf{P_2^S}$: $\forall\delta > 0\,\exists\,\epsilon = \epsilon(\delta,E) > 0 :\ \mathcal{D}(\epsilon) \subseteq \mathcal{D}^S(E,\delta)$.

A set E satisfies P_1^S if all measures which are doubling with respect to cubes are (E,S)-doubling, while E satisfies P_2^S if all measures with a sufficiently small cube-doubling constant also have a small (E,S)-doubling constant. As before, we also treat P_i^S as the set of all sets E satisfying the condition P_i^S, allowing us to write such things as "$E \in P_i^S$."

We now show that $P_1 = P_1^{BL}$ and that $P_2 = P_2^{LBL}$. We do not know if it is true that $P_2 = P_2^{BL}$ (although we suspect that this is false).

Proposition 5.2. *$P_1 = P_1^S$ for every $S \subset BL$. $P_2 = P_2^S$ for every $S \subset LBL$.*

Proof. Let us fix $E \in P_1$, $\epsilon > 0$, and $\mu_\epsilon \in \mathcal{D}(\epsilon)$. Also let $f \in BL$ and $g(z) = AfA^{-1}z$, where $Az = \lambda z + x$, $\lambda > 0$, and $x \in \mathbb{R}^n$. Defining the pullback measure $\mu'(U) = \mu(gU)$, it is easy to see that $\mu' \in \mathcal{D}(\epsilon')$ for some ϵ' dependent on ϵ. Thus $\mu' \in \mathcal{D}(E,\delta)$ for some $\delta > 0$ (independent of f and A). Since A is an arbitrary affine map and $f \in BL$ is arbitrary, it follows from Theorem 2.2 that $E \in P_1^{BL}$, which implies the first statement of our result.

The above argument does not work for P_2^S, $S = BL$, but it does if $S \subset LBL$, since any $f \in LBL$ gives rise to a function g which sends congruent cubes to congruent parallelpipeds. Applying the method of Theorem 2.4 to a decomposition of cube into parallelpipeds, we see that the measure $\mu \in \mathcal{D}(\epsilon)$ gives rise to $\mu' \in \mathcal{D}(\epsilon')$ where $\epsilon' \to 0^+ (\epsilon \to 0^+)$. The rest of the proof is easy. □

Even though the properties P_i^S are independent of $S \subset LBL$, it is not difficult to construct an individual measure μ which is doubling with respect to a pair (E,S) when $S = S_1$, but not when $S = S_2$ (of course E cannot satisfy P_1). For a very simple example exhibiting rather extreme behaviour of this type, let $E = E_1 \times (0,1)$, where E_1 is a minimally fat Cantor set, and let $\mu = m \times \mu_2$, where m is Lebesgue measure on the line and μ_2 is a doubling measure on the line which has E_1 as a null set. Then μ is (E,S)-doubling for $S = \emptyset$ (with zero doubling constant), but is not (E,S)-doubling if S includes a right-angle rotation about the origin.

Finally, let us briefly comment on "asymptotic doubling." Some papers, notably [Ko], concern themselves with *asymptotic doubling*, which means that the doubling constant ϵ in (0.1) can be taken to be very small at very small scales (i.e. when the associated scaling factor λ is very small). The proof of Theorem 0.2 can readily be modified to prove that asymptotic doubling with respect to any bounded set E implies asymptotic doubling with respect to cubes. Furthermore if P_3 is the set of all E such that asymptotic doubling with respect to cubes implies asymptotic doubling with respect to E, it is easy to see that $P_2 \subset P_3$. Thus, for example, Theorem 2.4 implies that if $U \subset Q_0$ is open, $|\partial U| = 0$, then $U \in P_3$.

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