SINGULAR MEASURES AND THE KEY OF G

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0. Introduction

A non-zero Borel measure ν is said to be *doubling* if there is a constant $C \geq 1$ such that

$$C^{-1} \leq \frac{\nu(I)}{\nu(J)} \leq C, \tag{1}$$

whenever I, J are adjacent intervals of the same length. We call the smallest $C = C_\nu$ for which this condition holds, the *doubling constant* of ν. A measure is a multiple of Lebesgue measure if and only if its doubling constant is 1.

It was shown in [BHM] that if $U \subset [0,1]^n$ is open and $|\partial U| = 0$, then $\nu_n(U) \to |U|$ whenever ν_n is a sequence of probability measures on $[0,1]^n$ whose doubling constants tend to 1. In particular, if U is an open subset of $[0,1]$ of full measure, then $\nu_n(U) \to 1$. We will show, amongst other things, that there exists a G_δ set G in $[0,1]$ of full measure, and a sequence ν_n of measures whose doubling constants tend to 1, yet $\nu_n(G) = 0$ for all n. We can even choose the measures to be "renormalizations" of a single measure ν which "fit the gaps in G" as a key fits a lock.

We wish to thank the referee for drawing our attention to the paper of Kakutani.

1. Definitions and basic results

There is an easy way, essentially due to Kahane [K], to generate doubling measures. Let $\mathcal{Q}$ consist of all intervals on $[0,1)$ of the form $[m4^{-k}, (m+1)4^{-k})$, where m, k are non-negative integers, and set $\mathcal{Q}(j)$ to be the subset of $\mathcal{Q}$ consisting of those intervals of length 4^{-j}.

For any $I \in Q$ the four children are labeled I_0, I_1, I_2, I_3, moving from left to right. Now consider

$$H_I(x) = \begin{cases} 1, & x \in I_1 \\ -1, & x \in I_2 \\ 0, & \text{otherwise.} \end{cases}$$

The product $\prod_{I \in \mathcal{Q}} (1 + a_I H_I)$ converges weak-$*$ to a doubling, probability measure μ, provided that $\sup_{I \in \mathcal{Q}} |a_I| < 1$. We call any such measure μ a *Kahane measure* and write $\|\mu\|_K =$

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$\sup_{I\in\mathcal{Q}}|a_I|$. Furthermore, the doubling constant C_μ tends to 1 as $\|\mu\|_K$ tends to 0; in fact, if $\|\mu\|_K \le 1-\epsilon$, then there is a constant c_ϵ, dependent only on ϵ, such that $C_\mu \le 1 + c_\epsilon\|\mu\|_K$ whenever for some $\epsilon > 0$.

For our purposes it will be sufficient to consider Kahane measures for which all of the coefficients a_I at any given scale are equal and $\|\mu\|_K \le 1-\epsilon$ for some $\epsilon > 0$, which we assume to be fixed from now on. We denote this class of measures by $\mathcal{M}_\epsilon$, or simply $\mathcal{M}$. Then every measure in $\mathcal{M}$ is of the form $\prod_{j=1}^\infty(1+a_jR_j)$ where $R_j = \sum_{I\in Q(j)} H_I$; it is convenient to introduce the notation $c_j(\mu) \equiv a_j$. We will focus on those measures $\mu \in \mathcal{M}$ for which $c_j(\mu) \to 0$ as $j \to \infty$, and we label these $\mathcal{M}_0$. For $\mu \in \mathcal{M}$ and $n = 0, 1, 2, \ldots$ the measure $\mu_n \in \mathcal{M}$ henceforth denotes the element of $\mathcal{M}$ with $c_j(\mu_n) = c_{j+n}(\mu)$, $j \in \mathbb{N}$. The measures μ_n are "renormalized" versions of μ; in fact, if $S \subset [0, 1)$ is a measurable set and f_S is the periodic function with period 1 whose restriction to $[0, 1)$ is the characteristic function of S, then $\mu_n(S) = \int_0^1 f_S(4^n t)\, d\mu(t)$. Given $\mu \in \mathcal{M}_0$, it follows from the estimate in the last paragraph that the sequence of doubling constants (C_{μ_n}) has limit 1. Thus every $\mu \in \mathcal{M}_0$ is *optimally doubling at small scales* in the sense that $\nu = \mu$ satisfies (1) with $C = C_{\mu_n}$ whenever I, J are adjacent intervals with $|I| = |J| \le 4^{-n}$.

The following result is a special case of a result of Kakutani [Kk, Corollary 1].

Theorem A. *Let $\mu, \nu \in \mathcal{M}$, with $a_j = c_j(\mu)$, $b_j = c_j(\nu)$, for all $j \in \mathbb{N}$. If $(a_j - b_j)_{j=1}^\infty$ lies in l^2, the class of square summable sequences, then $\mu << \nu << \mu$, otherwise $\mu \perp \nu$.*

In fact, when ν is Lebesgue measure and $(a_n) \in l^2$ above, more is true: μ lies in the Muckenhoupt class A_∞, and in particular μ has density lying in $L^p([0, 1])$ for some $p > 1$; see [Bu] and [FKP].[1]

Kakutani proves this result by careful analysis, but let us pause to prove the singularity part of this result using the Lyapunov version of the Central Limit Theorem [Bi, Theorem 27.3] which we now state.

Theorem B. *Suppose that $\{X_n\}_{n=1}^\infty$ is a sequence of independent random variables, and that the moments $E(X_n) = e_n$, $E(X_n - e_n)^2 = \sigma_n^2 \ne 0$, and $E|X_n - e_n|^3 = \tau_n^3$ are finite for each n. Let*

$$s_n = \left(\sum_{i=1}^n \sigma_i^2\right)^{1/2}, \qquad t_n = \left(\sum_{i=1}^n \tau_i^3\right)^{1/3}.$$

If $\lim_{n\to\infty} t_n/s_n = 0$, then $Y_n \equiv \sum_{i=1}^n (X_i - e_i)/s_n$ converges in distribution to the standard normal distribution.

In this paragraph we employ the notation of Theorem A. The functions R_n are independent as random variables on $[0, 1]$ with respect to ν, and so the functions $f_n = \log[(1+a_nR_n)/(1+b_nR_n)]$

[1] These references only say that μ lies in dyadic A_∞ but, since μ is a doubling measure, this implies that $\mu \in A_\infty$.

are also independent. A little calculation with the power series expansion for $\log(1+t)$ gives

$$E_\nu(f_n) \equiv e_n = -\frac{(a_n-b_n)^2}{4(1-b_n^2)} + O(|a_n-b_n|^3),$$
$$E_\nu(f_n-e_n)^2 \equiv \sigma_n^2 = \frac{(a_n-b_n)^2}{2(1-b_n^2)} + O(|a_n-b_n|^3),$$
$$E_\nu|f_n-e_n|^3 \equiv \tau_n^3 = \frac{|a_n-b_n|^3}{2}\,\frac{1+b_n^2}{(1-b_n^2)^2} + O(|a_n-b_n|^4).$$

Thus if s_n, t_n are as in Theorem B, $\lim_{n\to\infty}|a_n-b_n| = 0$, and $(a_n-b_n)_{n=1}^\infty \notin l^2$, then $t_n^3/\sum_{i=1}^n |a_i-b_i|^3$ and $s_n^2/\sum_{i=1}^n (a_i-b_i)^2$ are bounded above and below by positive, finite constants that are independent of n. It is then routine to deduce that $\lim_{n\to\infty} t_n/s_n = 0$; one simply splits the sum at a point beyond which $|a_n-b_n|$ is very small and uses the estimate $\|\cdot\|_{l^3} \le \|\cdot\|_{l^2}^{2/3}\|\cdot\|_{l^\infty}^{1/3}$. Thus Theorem B is applicable in the case $X_n = f_n$. Since $\sum_{i=1}^n e_i$ is much larger than s_n for large n, it follows that Y_n tends to $-\infty$ in ν-measure and thus $\prod_{n=1}^\infty (1+a_nR_n)/(1+b_nR_n)$ converges in ν-measure to the zero function. Set $\{P_N\}$ to be the partial products of this infinite product. We have just seen that this sequence of functions converges to zero in ν-measure. However, $P_N(x) = \mu(I_N(x))/\nu(I_N(x))$, where $I_N(x)$ is the unique element of $\mathcal{Q}(N)$ containing x and so, by the Radon-Nikodym theorem, $\{P_N\}$ converges ν-a.e. to the Radon-Nikodym derivative of μ with respect to ν. Consequently, the Radon-Nikodym derivative is zero ν-a.e., and so $\mu \perp \nu$ whenever $(a_j-b_j) \notin l^2$

We are mainly interested in Theorem A when ν is Lebesgue measure. In this case if the sequence $(c_j(\mu))$ has limit zero but does not lie in l^2, then μ is a singular measure which is optimal doubling at small scales. The mere existence of such a measure may seem a little surprising and was only recently established (using different techniques) by Cantón [C] and Smith [S].

There is an obvious bijection, A, between $\mathcal{Q}$ and the set of finite sequences whose terms lie in $\{0,1,2,3\}$. We will refer to $A(I)$ as the *address* of I. The jth term in the address is $A_j(I)$. For $I \in Q$, we let $E(I)$ consist of the union of the intervals $J \in \mathcal{Q}$ for which $A_{2j}(J) = A_j(I)$ for all j. So the odd terms in $A(J)$ are arbitrary and the even terms are specified. If $I \in \mathcal{Q}(j)$, $E(I)$ consists of 4^j elements of $\mathcal{Q}(2j)$. For $n = 0,1,2,\ldots$ and $I \in Q$, $T_n(I)$ consists of those intervals $J \in \mathcal{Q}$ for which $A_{n+j}(J) = A_j(I)$ for all j. So the first n terms of J are arbitrary and the remainder are specified. When $I \in \mathcal{Q}(j)$, $T_n(I)$ consists of 4^n elements of $\mathcal{Q}(j+n)$. Note that if I and J are disjoint, then $E(I)$ and $E(J)$ are disjoint, as are $T_n(I)$ and $T_n(J)$. For any set B that is a union of disjoint elements I of $\mathcal{Q}$, we define $E(B)$ to be the union of the $E(I)$, and we define $T_n(B)$ similarly. It is easy to check that $|E(B)| = |B|$ and that $|T_n(B)| = |B|$.

Let Σ_j be the collection of subsets of $[0,1)$ that are unions of elements of $Q(j)$. Any set $B \in \Sigma_m$ is said to be *j-indifferent* if whenever $B \supset I \in Q(m)$ and J is one of the three elements of $Q(m)$ for which $A(J)$ and $A(I)$ differ only in the jth place, then $J \subset B$. Equivalently if $S(B)$ is the set of sequences of length m given by $A(I)$ for each $I \in \mathcal{Q}(m)$, $I \subset B$, then B is j-indifferent precisely if $S(B)$ is measurable with respect to the σ-algebra generated by the sets

$$S_{k,l} = \{(a_i)_{i=1}^m : a_k = l\}, \qquad 1 \le k \le m,\ k \ne j,\ l \in \{0,1,2,3\}.$$

The point of this definition is that if B is j-indifferent, then $\mu(B)$ does not depend on the $c_j(\mu)$. In particular, if $B \in \Sigma_m$, then $E(B)$ is j-indifferent for all odd numbers j and all even $j > 2m$, and $T_n(B)$ is j-indifferent for all $j \le n$ and all $j > n+m$.

2. Construction of μ and G

Our main result is as follows.

Theorem 1. *There exists a measure $\mu \in \mathcal{M}_0$ on the interval $[0,1)$ and a G_δ set G contained in $[0,1)$ which have the following properties:*

(a) $\mu([0,1)) = 1$, $|G| = 1$ *and* $\mu(G) = 0$.
(b) $\mu_n(G) = 1$ *for all odd* $n \in \mathbb{N}$ *and* $\mu_n(G) = 0$ *for all even* $n \in \mathbb{N}$.

Taking $\nu_n = \mu_{2n}$, we immediately get

Corollary 2. *There exists a G_δ set G in $[0,1]$ of full measure and a sequence ν_n of probability measures on $[0,1]$ whose doubling constants tend to 1 and for which $\nu_n(G) = 0$ for all n.*

The oscillatory behaviour of $\mu_n(G)$ described in Theorem 1(b) is all the more remarkable since the measures μ_n are renormalized versions of a single measure μ whose doubling constants are tending to one. The idea is to construct G from sets that are indifferent at odd levels n (and thus treat such μ_n like Lebesgue measure), but which are concentrated in areas where μ_n is small whenever n is even.

Proof of Theorem 1. Let b be any number strictly between 0 and 1. Define ν_k to be the element of $\mathcal{M}$ whose coefficients are all 2^{-k}. This measure is singular with respect to Lebesgue measure. It follows that for sufficiently large n_k, there exists $A_k \in \Sigma_{n_k}$ for which $|A_k| \geq 1 - b^k$ and $\nu_k(A_k) \leq b^k$. We can assume that the n_k are increasing to ∞.

Divide the natural numbers into consecutive blocks $B_1, B_2, \ldots$ of length $2n_1, 2n_2, \ldots$. Set $a_j = 2^{-k}$ whenever j is an even number in block B_k, and 0 otherwise. Define $\mu \in \mathcal{M}_0$ by the equations $c_j(\mu) = a_j$.

Now let $m_k = 2n_1 + \cdots + 2n_{k-1}$ for $k > 1$ and $m_1 = 0$. Thus m_k is the total length of the blocks $B_1, \ldots B_{k-1}$. Define H_k to be $T_{m_k}(E(A_k))$. Then $H_k \in \Sigma_{m_k+2n_k}$ and is j-indifferent for all j except even numbers larger than m_k and no larger than $m_k + 2n_k$, i.e., all even numbers in B_k. Remove the endpoints of the intervals that make up H_k to get an open set U_k. The sets U_k and H_k differ only by a countable number of points. Thus any doubling measure gives them the same measure (doubling measures on the line are non-atomic). Set $G_m = \bigcup_{k=m}^{\infty} U_k$ and $G = \bigcap_{m=1}^{\infty} G_m$. This set G is a G_δ set.

We have $|H_k| = |A_k| \geq 1 - b^k$ for all k, hence $|G_m| = 1$ for all m, and $|G| = 1$. If n is odd and j is even, then $c_j(\mu_n) = 0$. But H_k is j-indifferent for all odd j, so it follows that $\mu_n(H_k) = |H_k|$. As a result, $\mu_n(G) = 1$ whenever n is odd.

The set H_k is j-indifferent for all j except even j in B_k and $c_j(\mu) = 2^{-k}$ for these exceptional integers. Thus $\mu(H_k) = \nu_k(A_k) \leq b^k$. Consequently, $\mu(G_m) \leq b^m(1-b)^{-1}$ for all m, and so $\mu(G) = 0$.

Suppose $n - m$ is even. Then $c_j(\mu_n) = c_j(\mu_m)$ for "most" values of j in the sense that for each k the number of places where the coefficients of size 2^{-k} do not match up is bounded independently of k, indeed by $n - m$. It follows readily from Theorem A that $\mu_n << \mu_m << \mu_n$. In particular, $\mu_n(G) = 0$ for all even n. □

Finally, we note two facts about the relationship between μ_n and μ_m. First, if $n - m$ is odd, then one of n, m is odd and the other is even. Thus one of the measures gives full measure to G, while the other gives G zero measure. In particular, $\mu_n \perp \mu_m$. Secondly, when $n-m$ is even, the absolute continuity mentioned in the last paragraph of the proof can be strengthened: there exists a constant C, dependent only on $n-m$, such that $C^{-1}\mu_m(E) \le \mu_n(E) \le C\mu_m(E)$. It suffices to prove this last estimate for $E \in \mathcal{Q}$, in which case the estimate follows from the fact, that $c_j(\mu_n) = c_j(\mu_m)$ for "most" values of j. We leave the details to the reader.

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