On the Approximation of Moments for Nonlinear Systems

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Abstract—**Model reduction by moment-matching relies upon the availability of the so-called** *moment*. **If the system is nonlinear, the computation of moments depends on an underlying specific invariance equation, which can be difficult or impossible to solve. This article presents four technical contributions related to the theory of moment matching: first, we identify a connection between moment-based theory and weighted residual methods. Second, we exploit this relation to provide an approximation technique for the computation of nonlinear moments. Third, we extend the definition of nonlinear moment to the case in which the generator is described in explicit form. Finally, we provide an approximation technique to compute the moments in this scenario. The results are illustrated by means of two examples.**

Index Terms—**Moment matching, moments, nonlinear systems, steady state, weighted residual methods.**

I. INTRODUCTION

The theory behind model order reduction by moment-matching relies upon the notion of *moment*, which was originally conceived within an interpolation framework for linear systems described by differential equations, see, e.g., [1]. Subsequently, the definition of moment has been extended to a wider class of systems, see [2], [3], including nonlinear systems. A comprehensive review of the state of the art on model reduction by moment matching, including connections between the notion of moment introduced in [2] and [3] and previously established definitions (such as those related to Krylov methods), can be found in [2] and [4]. Note that, apart from the notion of moment arising in systems theory, a different notion can be found in the field of probability (see [5], [6]). While the latter is not considered within the scope of this article, we note that recent efforts have been presented in [7] to bridge the gap between the notions of moment in probability theory and in systems theory.

The moment as defined in [2] and [3] is strongly related to the steady-state output response of the interconnection between the system under analysis and a signal generator. When the system is nonlinear, the moment is essentially defined in terms of the solution of an invariance equation, which can be difficult or impossible to solve. In other words, the computation of a reduced model by moment-matching depends upon the availability of a suitable technique to approximate the corresponding moment.

In this article, the moment-based theory as introduced in [2] and [3] is connected to the classical formulation of weighted residual methods (WRMs), i.e., spectral (*Galerkin*) and pseudospectral (*collocation*) methods, see, e.g., [8], [9]. The family of WRMs aims to compute approximate solutions of differential equations by expanding the system variables in a set of basis functions to then minimize a particular (approximation) error function termed *residual*. These methods have been successfully applied to a variety of problems in different applications, including, for instance, numerical approximation of solutions for the Navier–Stokes equations [8], [10].

This article provides four technical contributions related to the framework of moment matching. First, we formalize a connection between moment-based theory and the family of WRMs. Second, inspired by this result, we propose a method to approximate the moment of a nonlinear system driven by signal generators in *implicit* form (loosely speaking, generators described by differential equations). We note that approximation methods for the moment of a nonlinear system driven by this class of inputs have been studied in [11]. However, since [11] relies on computations on the steady-state response, [11] assumes local exponential stability of an equilibrium point of the underlying system, which is not required for the methods proposed in this article. The third contribution of the article is to extend the moment-based framework to *nonlinear* systems driven by signal generators in *explicit* form (loosely speaking, generators not necessarily described by differential equations). In particular, we focus on periodic and potentially discontinuous inputs (motivated by the existence of a large number of applications in which this class of signals occurs). Note that, up until this point, this explicit framework was only defined for linear systems (see [12]). Finally, combining all the previous results, we propose a method to approximate the moment of a nonlinear system driven by generators in explicit form.

We briefly mention that while our contribution is technical and strictly related to the definition and computation of moments, these technical contributions allow computing and defining new classes of reduced order models. These models, which are essential for a variety of engineering applications (see, e.g., [1]), directly motivate this article. Moreover, we note that the contributions of this article can also be potentially used to compute approximate solutions to optimal control problems, following the moment-based control framework presented in [13] and [14].

The remainder of this article is organized as follows. Section II briefly recalls both the theory behind moments for systems driven by implicit signal generators and the theory of WRMs. Then, a connection between

those methodologies is formalized. Section III proposes a method to approximate the moment of a nonlinear system driven by an implicit signal generator. Section IV formalizes the definition of the moment of a nonlinear system driven by an explicit signal generator and proposes a method to approximate such a moment, with particular focus on periodic discontinuous signals. Finally, Section V concludes this article.

A. Notation and Preliminaries

Standard notation is used throughout this article, with some exceptions detailed in this preliminary section. $\mathbb{R}^+$ ($\mathbb{R}^-$) denotes the set of nonnegative (nonpositive) real numbers. $\mathbb{C}^0$ denotes the set of purely imaginary complex numbers and $\mathbb{C}_{<0}$ denotes the set of complex numbers with a negative real part. The symbol $\mathbb{N}_q$ indicates the set of all positive natural numbers up to q, i.e., $\mathbb{N}_q = \{1, 2, \ldots, q\}$. The symbol 0 stands for any zero element, according to the context. The symbol $\mathbb{I}_n$ denotes the identity matrix of size n. If x is a real-valued row/column vector, then $x_i \in \mathbb{R}$ denotes the ith element of x. We write a matrix $X \in \mathbb{R}^{n \times m}$ elementwise as $X = [x_{ij}]^{n,m}$. The spectrum of a matrix $A \in \mathbb{R}^{n \times n}$, i.e., the set of its eigenvalues, is denoted as $\sigma(A)$. The symbol $\bigoplus_{i=1}^n$ denotes the direct sum of n matrices, i.e., $\bigoplus_{i=1}^n A_i = \text{blkdiag}(A_1, A_2, \ldots, A_n)$. If $F \in \mathbb{R}^{n \times n}$ is a symmetric matrix, the expression $F \succ 0$ means that F is positive definite. The symbol $\mathcal{L}\{f(t)\}$ denotes the Laplace transform of the function f (provided that f is Laplace transformable) and, abusing the notation, $\sigma(\mathcal{L}\{f(t)\})$ denotes the set of poles of $\mathcal{L}\{f(t)\}$. The set of square integrable functions on the interval $\Xi \subset \mathbb{R}$ is denoted as $L^2(\Xi)$. The *Kronecker product* between two matrices M_1 and M_2 is denoted as $M_1 \otimes M_2$. The symbol $\varepsilon_n \in \mathbb{R}^{n \times 1}$ denotes a vector with entries in odd positions equal to 1 and even positions equal to 0. Finally, we recall a definition from [2].

Definition 1: Let x, with $x(t) \in \mathbb{R}^n$ be the state of the dynamical system[1] Σ, and u, $u(t) \in \mathbb{R}$, be the input of Σ. Let t_0 and $x_0 = x(t_0)$ be the initial time and the initial state, respectively. If there exists a function $\Phi : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times L^2(\mathbb{R}) \to \mathbb{R}^n$ such that

$$x(t) = \Phi(t, t_0, x_0, u_{[t_0, t)}), \quad \forall t \geq t_0 \tag{1}$$

we call (1), the *representation in explicit form*, or the *explicit model*, of Σ. Assume that $\Phi(t, t_0, x_0, u)$ has a continuous derivative with respect to t for every t_0, x_0 and u, and there exists a function $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ continuous for each t over $\mathbb{R}^n \times \mathbb{R}$ such that

$$\dot{x} = f(x, u). \tag{2}$$

We call the differential equation (2) the *representation in implicit form*, or the *implicit model*, of Σ.

II. MOMENT-BASED THEORY AND WEIGHTED RESIDUAL METHODS

This section briefly recalls the notion of *moment* for systems driven by generators in implicit form. We, then, specialize this notion to linear systems to formalize a connection between moment-based theory and the family of WRMs.

A. Notion of Moment

Consider a nonlinear, single input, single output, continuous time, system described by[2]

$$\dot{x} = f(x, u) = f_l(x, u) + f_{nl}(x, u)$$
$$y = h(x) = h_l(x) + h_{nl}(x) \tag{3}$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, and f and h analytic mappings defined in the neighborhood of the origin of $\mathbb{R}^n$. Assume that system (3) is minimal, i.e., *observable* and *accessible* (see [2, Ch. 2]), and suppose that $f(0,0) = 0$ and $h(0) = 0$. The mappings $f_l(x, u) = Ax + Bu$, with $A = \partial f / \partial x|_{(x,u)=(0,0)}$, $A \in \mathbb{R}^{n \times n}$, $B = \partial f / \partial u|_{(x,u)=(0,0)}$, $B \in \mathbb{R}^n$, and $C = \partial y / \partial x|_{(x,u)=(0,0)}$, $C \in \mathbb{R}^{1 \times n}$, describe the linearization of system (3) around the origin and $f_{nl}(x, u) = f(x, u) - f_l(x, u)$ and $h_{nl}(x) = h(x) - h_l(x)$ describe the nonlinear parts of f and h, respectively. Consider a signal generator, i.e., an external system "generating" the input u in (3) (see [16, Ch. 8] for further detail), described by

$$\dot{\omega} = S\omega, \quad u = L\omega \tag{4}$$

with $\omega(t) \in \mathbb{R}^\nu$, $S \in \mathbb{R}^{\nu \times \nu}$, and $L \in \mathbb{R}^{1 \times \nu}$. Consider now the interconnected system

$$\dot{\omega} = S\omega, \quad \dot{x} = f(x, L\omega), \quad y = h(x) \tag{5}$$

Following [2] and [3], we consider a set of assumptions to formalize the definition of moment.

Assumption 1: There exists a mapping π, locally[3] defined in a neighborhood W of $\omega = 0$, with $\pi(0) = 0$, which satisfies the following differential equation:

$$\frac{\partial \pi(\omega)}{\partial \omega} S\omega = f(\pi(\omega), L\omega) \tag{6}$$

for all $\omega \in W$.

Assumption 2: The triple $(L, S, \omega(0))$ is minimal.[4]

Definition 2 ([2, Definition 2.13] [3]): Consider the system (3) and the signal generator (4). Suppose Assumptions 1 and 2 hold. The mapping $h \circ \pi$ is the moment of system (3) at (S, L).

Finally, we recall a result which, introducing additional assumptions, connects the definition of moment with the steady-state response of the output of the interconnected system (5).

Assumption 3: The signal generator (4) is such that all eigenvalues of S are simple and with zero real part.

Theorem 1 ([2], [3]): Consider the system (3) and the signal generator (4). Suppose Assumptions 2 and 3 hold and that the zero equilibrium of the system (3) is locally exponentially stable, i.e., $\sigma(A) \subset \mathbb{C}_{<0}$. Then, Assumption 1 holds and the moment of system (3) at (S, L) computed along a trajectory $\omega(t)$ coincides with the steady-state response[5] of the output $y_{ss}(t)$ of the interconnected system (5), i.e., $y_{ss}(t) = h(\pi(\omega(t)))$.

Remark 1: While moments can be naturally defined for *nonlinear* signal generators in implicit form (see [2], [3]), this is beyond the scope

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[1]We adopt the standard definition (and associated notation) of a dynamical system arising in the field of systems theory (see [15, Ch. 2]).

[2]From now on, and aiming to simplify the notation, the dependence on t is dropped when clear for the context.

[3]All statements are local, although global versions can be straightforwardly derived.

[4]Minimality of the triple $(L, S, \omega(0))$ implies observability of the pair (S, L) and excitability of the pair $(S, \omega(0))$ (see [17, Definition 2] for further detail). Note that, for linear systems, excitability is equivalent to reachability, i.e., with $\omega(0)$ playing the role of the input matrix, see [17, Sec. III].

[5]See [18] for a formal definition of steady-state response.

of this article: the class of input signals, which motivates the technical contributions presented herein are captured by the linear generator (4) (see also Remark 3), and the implicit form generator (33), discussed in Section IV.

B. Special Case of Linear Systems

Suppose $f_{nl}(x, u) = 0$ and $h_{nl}(x) = 0$ in (3). The assumptions required in the nonlinear case to formalize the definition of moment are less restrictive when the mapping f is purely linear, as detailed in [2] and [3] and briefly recalled in the following.

Assumption 4: The matrices A and S are such that $\sigma(S) \cap \sigma(A) = \emptyset$.

Lemma 1 ([2], [3]): Suppose Assumption 4 holds. Then, there exists a unique matrix $\Pi \in \mathbb{R}^{n \times \nu}$, which solves the following Sylvester equation:

$$\Pi S = A\Pi + BL. \tag{7}$$

Definition 3 ([2], [3]): Consider the system (3), with $f_{nl}(x, u) = 0$ and $h_{nl}(x) = 0$, and the signal generator (4). Suppose Assumptions 2 and 4 hold. Then, we call the matrix $C\Pi$ the moment of system (3) at (S, L).

C. Weighted Residual Methods and Moments

Let Ξ be a closed interval in $\mathbb{R}$. The basic idea behind the family of WRMs relies on the selection of a complete set $\{\psi_k\}$ of orthogonal functions $\psi_k : \Xi \to \mathbb{R} : t \mapsto \psi_k(t)$ defined on a function space $\mathcal{H}$ with domain Ξ, and with the inner product on the space $\mathcal{H}$ defined as[6]

$$\langle p(t), l(t) \rangle = \int_{\Xi} p(\tau) l(\tau) w(\tau) d\tau \tag{8}$$

where $p \in \mathcal{H}$, $l \in \mathcal{H}$, and $w : \Xi \to \mathbb{R}$ is a weighting function. The standard assumption in WRMs (see [8, Ch. 1]) is that the state vector and the control input in (3) admit the expansions

$$x_i(t) = \sum_{j=1}^{M} \hat{x}_{ij} \psi_j(t), \quad u(t) = \sum_{j=1}^{M} \hat{u}_j \psi_j(t) \tag{9}$$

where $\hat{x}_{ij} \in \mathbb{R}$, $\hat{u}_j \in \mathbb{R}$, with $j \in \mathbb{N}_M$, denote, respectively, the coefficients of the expansion of x_i and u. Note that M may be infinity. Defining the set $\{\psi_k(t)\}_{k=1}^{N}$ with $N < M$, it is possible to write the corresponding N-dimensional approximations of x and u, denoted x^N and u^N, respectively, as

$$x_i(t) \approx x_i^N(t) = X_i \Psi(t), \quad u(t) \approx u^N(t) = U\Psi(t) \tag{10}$$

where $X_i \in \mathbb{R}^{1 \times N}$, $U \in \mathbb{R}^{1 \times N}$, $x^N(t) \in \mathbb{R}^n$, and the vector $\Psi(t) \in \mathbb{R}^N$ is defined as

$$\Psi(t) = \begin{bmatrix} \psi_1(t), & \dots & , \psi_N(t) \end{bmatrix}^\mathsf{T}. \tag{11}$$

Defining the matrix $X = [X_1^\mathsf{T}, \dots, X_N^\mathsf{T}]^\mathsf{T} \in \mathbb{R}^{n \times N}$, and substituting (10) into the dynamic equation (3), the *residual function* [8], [9]

$$R(X, U, \Psi(t)) := \dot{x}^N(t) - f(X\Psi(t), U\Psi(t)) \tag{12}$$

can be defined, in which the approximated time derivative of the state $\dot{x}^N$ is given by

$$\dot{x}^N(t) = X\dot{\Psi}(t). \tag{13}$$

Then, given values of U, the approximated state trajectory is computed in terms of the nN unknown coefficients of X as the solution of the system of $n^2 N^2$ algebraic equations[7]

$$\langle R_i(X, U, \Psi(t)), \zeta_j(t) \rangle = 0 \tag{14}$$

for $i \in \mathbb{N}_n$ and $j \in \mathbb{N}_N$, where $R = [R_1^\mathsf{T}, \dots, R_n^\mathsf{T}]^\mathsf{T}$ and the test functions ζ_j, assumed to be sufficiently regular, form an orthogonal set $\{\zeta_j\}_{j=1}^{N}$. If the test functions ζ_j are elements of the same set as the basis functions approximating the state, that is $\zeta_j = \psi_j$, then, the method is known as *spectral* or *Galerkin* method. If the test functions are translated Dirac-delta functions $\delta_{t_j} = \delta(t - t_j)$, then, the method is known as *pseudospectral* or *collocation* method, and the points t_j are called *collocation* points. From now on, we focus our study on the Galerkin method, since the collocation approach can be made equivalent to Galerkin method by an appropriate selection of the set $\{t_j\} \subset \Xi$ (see, e.g., [8, Ch. 4]).

We now present a connection between WRMs and moment-based theory in terms of the solution of the Sylvester equation (7). To this end, we assume that in (3) $f_{nl}(x, u) = 0$ and $h_{nl}(x) = 0$, i.e., the system dynamics are linear.

Remark 2: If $f_{nl}(x, u) = 0$ and $h_{nl}(x) = 0$, then, the approximated output of system (3) can be computed as $y^N(t) = Y\Psi(t)$ where $Y = CX$, with X solution of (14).

Proposition 1: Consider system (3) with $f_{nl}(x, u) = 0$ and $h_{nl}(x) = 0$, and the signal generator (4). Suppose that Assumption 2 holds and that u admits an expansion as in (10). Let $\Psi(t) = \omega(t)$. Then, the coefficients of the solution Y computed using the Galerkin method coincide with the elements of $C\Pi$.

Proof: Consider system (3) with $f_{nl}(x, u) = 0$ and $h_{nl}(x) = 0$, and the approximating state and input vectors $X\Psi(t)$ and $U\Psi(t)$, respectively. The residual equation defined in (12) can be written as

$$R(X, U, \Psi(t)) := X\dot{\Psi}(t) - AX\Psi(t) - BU\Psi(t) \tag{15}$$

and the approximating trajectory x^N can be computed in terms of X solving the equation

$$\langle X\dot{\Psi}(t) - AX\Psi(t) - BU\Psi(t), \Psi(t)^\mathsf{T} \rangle = 0. \tag{16}$$

Assume now that the vector of basis functions $\Psi(t)$ belongs to the class of functions generated by (4), i.e., we assume $\Psi(t) = \omega(t)$. Then, $\dot{\Psi}(t) = \dot{\omega}(t) = S\omega(t)$ and, considering the superposition property of the inner product, (16) can be written as

$$(XS - AX - BU) \langle \omega(t), \omega(t)^\mathsf{T} \rangle = 0. \tag{17}$$

Defining $\Omega = \langle \omega(t), \omega(t)^\mathsf{T} \rangle$, the proof follows once noted that $0 \notin \sigma(\Omega)$ since $\Omega = \Omega^\mathsf{T} \succ 0$ under the excitability condition on the pair $(S, \omega(0))$ (see Assumption 2). In particular, $Y = CX = C\Pi$, where Π is the unique solution of (7). ∎

Inspired by Proposition 1, we present, in the following sections, a general framework to approximate the moment of nonlinear dynamical systems driven by a wide class of input signals. Furthermore, we show that, under additional assumptions, the methods described in this article can be used to approximate the steady-state behavior of system (3) driven by this general class of inputs.

Remark 3: The class of input signals defined by (4) contains some of the most widely-used basis functions, such as polynomials and trigonometric functions. For instance, the first ν polynomial functions can be generated with S in (4) such that $S = \mathcal{N}_\nu$, where $\mathcal{N}_\nu \in \mathbb{R}^{\nu \times \nu}$ is a matrix with ones in the upper diagonal and zeros elsewhere. To address

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[6]We adopt the inner-product definition given by (8)

[7]The reader is referred to, for instance, [8, Ch. 2] for a discussion on computational aspects of WMRs.

more general cases, let $\underline{\lambda} \in \lambda(S) \subset \mathbb{C}$, and let p be the dimension of the largest Jordan block associated with $\underline{\lambda}$. Then, for each $\underline{\lambda}$, the signal generator (4) can generate linear combinations of the set of functions $\{t^q e^{\underline{\lambda} t}\}_{q=0}^{p-1}$.

Remark 4: The methods proposed in Sections III and IV compute the moment for a particular trajectory $\omega(t)$. We note that, if required, the methods can be modified to incorporate different $\omega(t)$, following an analogous procedure to that used in the so-called "$\mathfrak{U}/\mathfrak{N}$" variation proposed in [11].

III. APPROXIMATION OF MOMENTS: IMPLICIT SIGNAL GENERATOR CASE

In this section, we propose a method to approximate the moment of the nonlinear system (3) driven by the signal generator (4). For this purpose, we introduce an additional signal generator. We note that the assumptions considered in this section on the function π [solution of (6)] resemble those in [11].

To begin with, consider the "extended" signal generator described by the set of first order differential equations

$$\dot{\omega}_e = S_e \omega_e, \quad u = [L \ 0]\omega_e = L_e \omega_e \tag{18}$$

where $\omega_e(t) = [\omega^\intercal(t), \omega_e^{*\intercal}(t)]^\intercal \in \mathbb{R}^N$, $\omega_e^*(t) = [\omega_{e_{\nu+1}}(t), \ldots, \omega_{e_N}(t)]$, with $N \geq \nu$ integer, and the matrix $S_e \in \mathbb{R}^{N \times N}$ given by

$$S_e = \begin{bmatrix} S & 0 \\ 0 & S_e^* \end{bmatrix} \tag{19}$$

with any matrix S_e^* such that the pair $(S_e, \omega_e(0))$ is excitable (in line with Assumption 2).

Remark 5: The signals $u(t)$ generated by (4) and (18) coincide. Consequently, the moment of system (3) at (S, L) coincides with its moment at (S_e, L_e).

After introducing the implicit signal generator (18), we identify two sets of functions: $F_\omega = \{\omega_i\}_{i=1}^\nu$, with ω as in (4), and $F_{\omega_e} = \{\omega_{e_i}\}_{i=1}^N$, with ω_e as in (18). Note that the relation $F_\omega \subset F_{\omega_e}$ holds. We now introduce the following standing assumption.

Assumption 5: The elements of the set $F_{\omega_e} \subset \mathcal{H}$, where $\mathcal{H}$ is a complete inner-product space with (closed) domain $\Xi \subset \mathbb{R}$, are orthogonal in the interval Ξ. Moreover, each component of the function π, i.e., π_k, $k \in \mathbb{N}_n$, which solves (6), belongs to $\mathcal{H}$.

Remark 6: Under Assumption 5, one can always extend the set of functions F_{ω_e} to form an orthogonal basis of $\mathcal{H}$ by simply considering the orthogonal complement of the subspace spanned by the elements of F_{ω_e} (see, for instance, [19] and [20]).

Assumption 5, together with Remark 6, directly imply that π_k can be expanded as [19]

$$\pi_k(\omega) = \sum_{i=1}^N \alpha_{k_i}^\pi \omega_{e_i} + \epsilon_k(\omega_e) = \widetilde{\Pi}_k \omega_e + \epsilon_k(\omega_e) \tag{20}$$

where $\pi(\omega) = [\pi_1(\omega), \ldots, \pi_n(\omega)]^\intercal$, with $\alpha_{k_i}^\pi \in \mathbb{R}$, for $i \in \mathbb{N}_N$, $\widetilde{\Pi}_k = [\alpha_{k_1}^\pi, \ldots, \alpha_{k_N}^\pi]$, for some mapping $\epsilon_k : \mathbb{R}^N \to \mathbb{R}$. The existence of ϵ_k such that (20) holds follows directly from Remark 6.

Remark 7: Note that, under Assumption 5, we can always write π as the sum of two contributions, namely

$$\pi(\omega) = \widetilde{\Pi}\omega_e + E(\omega_e) \tag{21}$$

where $\widetilde{\Pi} = [\widetilde{\Pi}_1^\intercal, \ldots, \widetilde{\Pi}_n^\intercal]^\intercal$ and $E = [\epsilon_1(\omega_e), \ldots, \epsilon_n(\omega_e)]$.

In the following, we proceed to formulate a method, which allows the computation of an approximation of π in terms of the set F_{ω_e}, i.e., in terms of the N-dimensional expansion $\widetilde{\Pi}\omega_e$. We do this by minimizing

a residual equation analogously to what done for the family of WRMs (see Section II-C).

Proposition 2: Consider the nonlinear system (3) and the linear generator (4). Suppose that Assumptions 1, 2, and 5 hold. Then, $h(\widetilde{\Pi}\omega_e)$, where $\widetilde{\Pi}$ is the solution of the system of algebraic equations

$$(\widetilde{\Pi}S_e - A\widetilde{\Pi} - BL_e)\langle\omega_e, \omega_e^\intercal\rangle - \langle f_{nl}(\widetilde{\Pi}\omega_e, L_e\omega_e), \omega_e^\intercal\rangle = 0 \tag{22}$$

is the least-square minimizer of the residual $R(\widetilde{\Pi}, L_e, \omega_e)$, i.e., $h(\widetilde{\Pi}\omega_e)$ is a least-square approximation of the moment of system (3) at (S, L).

Proof: By Assumption 5, we can define a residual equation in terms of the extended signal generator (18) as $R(\widetilde{\Pi}, L_e, \omega_e)$ [similarly to (12)], replacing $\widetilde{\Pi}\omega_e$ in (6), i.e.,

$$R(\widetilde{\Pi}, L, \omega_e) = (\widetilde{\Pi}S_e - A\widetilde{\Pi} - BL_e)\omega_e - f_{nl}(\widetilde{\Pi}\omega_e, L_e\omega_e). \tag{23}$$

Then, we compute the approximating solution in terms of $\widetilde{\Pi}$ by forcing the residual (23) to be orthogonal to the N-dimensional space spanned by the set F_{ω_e}. Equation (22) follows after considering the superposition property of the inner product. ∎

Note that Proposition 2 makes explicit use of the extended signal generator (18) to compute an N-dimensional approximation of π in terms of the set F_{ω_e} by projecting the residual equation (23) into the same set of functions, analogously to what done in the family of WRMs. In other words, the extended signal generator, which defines the set F_{ω_e}, generates the function space used to approximate the corresponding moment (see also Remark 9).

Corollary 1: Let Assumptions 1, 2, and 5 hold, and suppose $E = 0$ in (21). Then

$$R(\widetilde{\Pi}, L_e, \omega_e) = 0 \iff \langle R(\widetilde{\Pi}, L_e, \omega_e), \omega_e^\intercal\rangle = 0. \tag{24}$$

Proof: ($\Rightarrow$) This implication is straightforward as the inner product of any function with the zero function is zero by definition.

($\Leftarrow$) Under Assumption 5, each component of the residual function $R_k(\widetilde{\Pi}, L_e, \omega_e) \in \mathcal{H}$, for $k \in \mathbb{N}_n$, where R_k denotes the kth row of R. Then, for all $k \in \mathbb{N}_n$

$$\left.\begin{array}{c} \langle R_k(\widetilde{\Pi}, L_e, \omega_e), \omega_{e_1}\rangle = 0 \\ \vdots \\ \langle R_k(\widetilde{\Pi}, L_e, \omega_e), \omega_{e_N}\rangle = 0 \end{array}\right\} \Rightarrow R_k(\widetilde{\Pi}, L_e, \omega_e) = 0 \tag{25}$$

as a direct consequence of the fact that the elements of the set F_{ω_e} are orthogonal in $\mathcal{H}$ [19]. ∎

Corollary 1 implies that, if $E = 0$ in (20), i.e., π can be exactly written as $\widetilde{\Pi}\omega_e$, then, the approach of Proposition 2 effectively recovers the exact solution $\widetilde{\Pi}$.

Remark 8: If, additionally, Assumption 3 holds and the zero equilibrium of the system $\dot{x} = f(x, 0)$ is locally exponentially stable, then $h(\widetilde{\Pi}\omega_e)$, with $\widetilde{\Pi}$ computed as in Proposition 2, approximates the steady-state response of the output of the interconnected system (5) (see Theorem 1).

We now rewrite the system of algebraic equations (22) of Proposition 2 in a more convenient form, in which the contribution of the linear and nonlinear parts of system (3) can be easily identified.

Corollary 2: Let Assumptions 1, 2, and 5 be satisfied. Then, the system of nonlinear algebraic equations (22) can be equivalently written as

$$(\widetilde{\Pi}S_e - A\widetilde{\Pi} - BL_e) - F_{nl}(\widetilde{\Pi}, L_e)\Omega^{-1} = 0 \tag{26}$$

where $\Omega = \langle \omega_e, \omega_e^\mathsf{T} \rangle$, the matrix $F_{nl}(\widetilde{\Pi}, L_e) \in \mathbb{R}^{n \times N}$ is given by

$$F_{nl}(\widetilde{\Pi}, L_e) = \begin{bmatrix} \langle f_{nl_1}(\widetilde{\Pi}\omega_e, L_e\omega_e), \omega_e^\mathsf{T} \rangle \\ \vdots \\ \langle f_{nl_n}(\widetilde{\Pi}\omega_e, L_e\omega_e), \omega_e^\mathsf{T} \rangle \end{bmatrix} \quad (27)$$

and f_{nl_i} is the ith row of the mapping f_{nl}.

Proof: Straightforward once noted that $0 \notin \sigma(\Omega)$, since $\Omega = \Omega^\mathsf{T} \succ 0$ as a direct consequence of the excitability assumption of the pair $(S_e, \omega_e(0))$. ∎

Remark 9: The term $F_{nl}(\widetilde{\Pi}, L_e)\Omega^{-1}$ in (26) is the projection of $f_{nl}(\widetilde{\Pi}\omega_e, L_e\omega_e)$ onto the function space spanned by the set F_{ω_e}.

Corollary 2 provides the approximating solution in a very specific form: the linear Sylvester equation (7) appears explicitly with a non-linear "correction" term.

Remark 10: If $f_{nl}(x, u) = 0$, then $F_{nl}(\widetilde{\Pi}, L_e) = 0$, and the solution of (26) can be straightforwardly computed as $\widetilde{\Pi} = \widetilde{\Pi}_l = [\widehat{\Pi} \ 0]$, with $\widehat{\Pi}$ the solution (provided it exists) of the Sylvester equation $A\widehat{\Pi} - \widehat{\Pi}S = -BL$. In other words, if system (3) is linear, then, the solution of the proposed method coincides with that of the classical approach (discussed in Section II-B).

Remark 11: State-of-the-art numerical routines, such as those described in [21], can be readily used to solve the homogeneous system of algebraic equations (26) in $\widetilde{\Pi}$.

Remark 12: The choice of an appropriate initial guess $\widetilde{\Pi}_0$ is crucial for the fast convergence of any numerical routine when solving (26). In most cases a sensible initial point is the solution of (26) with $f_{nl}(x, u) = 0$ in (3), i.e., we select $\widetilde{\Pi}_0 = \widetilde{\Pi}_l$, with $\widetilde{\Pi}_l$ as in Remark 10.

Remark 13: From the previous remark, we note that the use of the "phantom" input, which does not appear in the linear part (because of the block of zeros in $L_e = [L \ 0]$), is instrumental to interpolate the nonlinear term in (22).

To conclude this section, we illustrate the proposed method by means of the following example.

Example 1: Consider the forced Van der Pol oscillator with a non-linear output map described by the following differential equations:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\frac{3}{2}(1 - x_1^2)x_2 - x_1 + u, \quad y = x_2^2 \quad (28)$$

where $u(t) = A_u \cos(f_0 t)$, with $A_u \in \mathbb{R}$ and $f_0 \in \mathbb{R}^+ \setminus \{0\}$. We can write u using the signal generator

$$\dot{\omega} = S\omega, \quad u = L\omega \quad (29)$$

with

$$S = \begin{bmatrix} 0 & f_0 \\ -f_0 & 0 \end{bmatrix}, \quad L = [A_u \ 0], \quad \omega(0) = \varepsilon_2. \quad (30)$$

Note that, with this selection of matrices, the triple $(L, S, \omega(0))$ is minimal. For this example, the set describing the input signal is $F_\omega = \{\omega_1, \omega_2\} = \{\cos(f_0 t), -\sin(f_0 t)\}$. Following (18), we define the extended signal generator

$$\dot{\omega}_e = S_e\omega_e, \quad u = L_e\omega_e \quad (31)$$

with matrices

$$S_e = \bigoplus_{p=1}^{k} \begin{bmatrix} 0 & pf_0 \\ -pf_0 & 0 \end{bmatrix}, \quad L_e = [L \ 0], \quad \omega_e(0) = \varepsilon_\nu \quad (32)$$

where $N = 2k$, with $k \in \mathbb{N}$. Note that the pair $(S_e, \omega_e(0))$ is excitable and $F_\omega \subset F_{\omega_e}$. The parameters of u are selected as $A_u = 1$ and $f_0 = \frac{1}{2}$.

Given that, for this example, the origin of $\dot{x} = f(x, 0)$ is locally exponentially stable and the matrix S of the signal generator (31) has
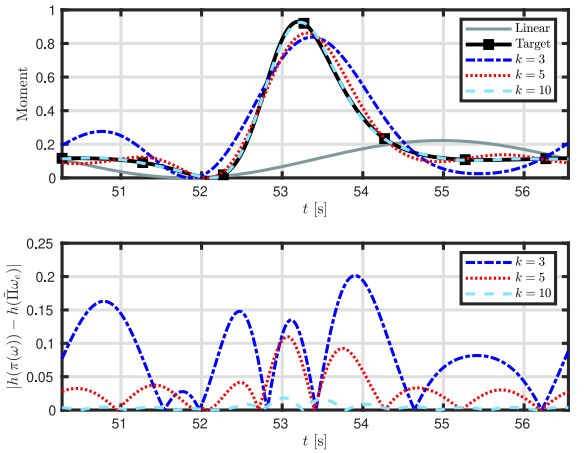

Fig. 1. Top: time histories of the approximated moment for the forced Van der Pol oscillator driven by the signal generator (31) for different values of k. Bottom: time histories of the corresponding absolute errors.

all simple eigenvalues with zero real part, the assumptions of Theorem 1 are fulfilled, and hence, the moment of (28) computed along a particular trajectory $\omega(t)$ coincides with the steady-state response of the output of the interconnected system. Thus, the moment of the system can be computed along $\omega(t)$ using a numerical integration method. This allows for a direct comparison with the approximated moment.

Note that, defining $\Xi = [t, t + T_0]$, with $T_0 = 2\pi/f_0$, implies that the elements of the set $F_{\omega_e} = \{\omega_{e_i}\}_{i=1}^{2k}$ are orthogonal on $L^2(\Xi)$ under the inner-product definition in (8) with weighting function $w(t) \equiv 1$. Furthermore, note that Assumption 5 holds for this example with $\mathscr{H} = L^2(\Xi)$, as discussed in the following. Given the nature of the signal generator defined in (29) the input u is always T_0-periodic. Moreover, since the zero equilibrium of $\dot{x} = f(x, 0)$ is locally exponentially stable, the (well-defined) steady-state solution of the interconnection between (28) and (29) is T_0-periodic [22, Sec. VI], i.e., $x_{ss}(t) = x_{ss}(t - T_0)$. Since, as discussed in the previous paragraph, Theorem 1 holds, i.e., $x_{ss}(t) = \pi(\omega(t))$, it is straightforward to conclude that each element of the mapping π belongs to $L^2(\Xi)$.

To illustrate the method proposed, we compute $\widetilde{\Pi}$ as in (26), for a different number of basis functions $\{\omega_{e_i}\}$, with ω_e as in (31). The algorithm used to solve (26) is based on the interior-reflective Newton method described in [21].

Fig. 1 (top) depicts the time histories of $h(\pi(\omega(t)))$ (i.e., the moment of system (28) along $\omega(t)$) computed using a Runge–Kutta method (solid-black line), and the corresponding approximated moment, i.e., $h(\widetilde{\Pi}\omega_e(t)) = ([0 \ 1]\widetilde{\Pi}\omega_e(t))^2$ for $k \in \{3, 5, 10\}$ (dot-dashed-blue, dotted-red, and dashed-cyan lines, respectively). Note that a clear improvement can be appreciated with an increasing number of basis functions. This is further confirmed in Fig. 1 (bottom), in which the time histories of the absolute value of the approximation error, i.e., $|h(\pi(\omega(t))) - h(\widetilde{\Pi}\omega_e(t))|$, is plotted for each value of k considered. Note that Fig. 1 (top) also shows the associated linear solution $\widetilde{\Pi}_l$ (solid-gray line), which is used as initial guess to solve (26) (see Remark 12).

IV. APPROXIMATION OF MOMENTS: EXPLICIT SIGNAL GENERATOR CASE

The mathematical formalism behind moments has been extended to the case in which system (3) is *linear* and the input is given in explicit form [12], [23]. This provides an extension of the moment-based framework for a very general class of inputs, including discontinuous

periodic signals. Motivated by this, in this section, we first provide a further extension of the notion of moment to *nonlinear* systems driven by explicit signal generators. Then, based and inspired by the methods developed in Section III, we propose a method to compute an approximation of such a moment.

A. Moment for Nonlinear Systems at Explicit Signal Generators

From now on, we focus our interest on signals described by a T-periodic explicit form signal generator as

$$\omega_\Lambda(t) = \Lambda(t)\omega_\Lambda(0), \ \omega_\Lambda(t) = \omega_\Lambda(t - T), \ u(t) = L\omega_\Lambda(t) \quad (33)$$

$t \geq T$, where the matrix $\Lambda(t) \in \mathbb{R}^{\nu \times \nu}$ is nonsingular for all $t \in \mathbb{R}^+$. Analogously to the implicit signal generator case of Section II-A, we now introduce a set of assumptions required to formalize the definition of moment.

Assumption 6: There exists a unique mapping $\pi(t, \omega_\Lambda)$, defined in a neighborhood of $(x, \omega_\Lambda) = (0, 0)$, which is the solution of the following integral equation

$$\pi(t, \omega_\Lambda) = \lim_{\bar{t} \to -\infty} \int_{\bar{t}}^{t} e^{A(t-\tau)} BL\omega_\Lambda(\tau)d\tau$$
$$+ \lim_{\bar{t} \to -\infty} \int_{\bar{t}}^{t} e^{A(t-\tau)} f_{nl}(\pi(\tau, \omega_\Lambda(\tau)), L\omega_\Lambda(\tau))d\tau. \quad (34)$$

Assumption 7: The vectors L and $\omega_\Lambda(0)$ in (33) are such that $\sigma(\mathcal{L}\{L\Lambda(t)\omega_\Lambda(0)\}) = \sigma(\mathcal{L}\{\Lambda(t)\})$.

Definition 4: Consider system (3) and the signal generator (33). Suppose Assumptions 6 and 7 hold. Then, we call the function $h \circ \pi$ the moment of system (3) at (Λ, L).

Analogously to the linear case discussed in [12] and [23], defining the moment of system (3) according to Definition 4 is justified by the equivalence, when an implicit model of (33) is available, between the new and the classical definition (see Definition 2).

Assumption 8: $\omega_\Lambda(t)$ in (33) is Laplace transformable and such that $\sigma(\mathcal{L}\{\omega_\Lambda(t)\}) \subset \mathbb{C}^0$.

Assumption 7 plays the role of the minimality condition of Assumption 2 for the implicit signal generator case, while Assumption 8 corresponds to the persistence of excitation condition of Assumption 3. We now formulate a proposition, which relates the moment as in Definition 4 with the steady-state output response of system (3) driven by (33).

Proposition 3: Consider system (3) and the signal generator (33). Suppose Assumptions 7 and 8 hold and that the zero equilibrium of system $\dot{x} = f(x, 0)$ is locally exponentially stable. Then, Assumption 6 holds and the moment of (3) at (Λ, L) computed along the trajectory $\omega(t)$ coincides with the (locally well-defined) steady-state output response of system (3) driven by (33), i.e., $y_{ss}(t) = h(\pi(t, \omega_\Lambda))$.

Proof: We begin by noting that the input u is bounded and periodic by Assumption 8. Then, since the zero equilibrium of $\dot{x} = f(x, 0)$ is locally exponentially stable, the steady-state response x_{ss} is locally well-defined [22, Sec. VI] and, using the well-known "variation of parameters formula," can be written as

$$x_{ss}(t) = \lim_{\bar{t} \to -\infty} \int_{\bar{t}}^{t} e^{A(t-\tau)} BL\omega_\Lambda(\tau)d\tau$$
$$+ \lim_{\bar{t} \to -\infty} \int_{\bar{t}}^{t} e^{A(t-\tau)} f_{nl}(x_{ss}(\tau), L\omega_\Lambda(\tau))d\tau. \quad (35)$$

The proof follows by noting that $x_{ss}(t) = \pi(t, \omega_\Lambda)$ in (34) and, hence, $y_{ss}(t) = h(\pi(t, \omega_\Lambda))$. ∎

In the following, we explicitly exploit the T-periodicity of ω_Λ to simplify the computation of the mapping π for the case in which the zero equilibrium of $\dot{x} = f(x, 0)$ is locally exponentially stable.

Corollary 3: Suppose Assumptions 7 and 8 hold and the zero equilibrium of system $\dot{x} = f(x, 0)$ is locally exponentially stable. Then, given that (33) is T-periodic, (34) becomes

$$\pi(t, \omega_\Lambda) = P \left[\int_{t-T}^{t} e^{A(t-\tau)} BL\omega_\Lambda(\tau)d\tau \right.$$
$$\left. + \int_{t-T}^{t} e^{A(t-\tau)} f_{nl}(\pi(t, \omega_\Lambda(\tau)), L\omega_\Lambda(\tau)) d\tau \right] \quad (36)$$

with $P \in \mathbb{R}^{n \times n}$ a constant matrix defined as $P = (\mathbb{I}_n - e^{AT})^{-1}$.

Proof: Under the abovementioned assumptions, if the input is T-periodic, the (locally well-defined) steady-state solution is also T-periodic [22, Sec. VI], i.e., $x_{ss}(t) = x_{ss}(t - T)$. Since under the same set of assumptions the equality $x_{ss}(t) = \pi(t, \omega_\Lambda)$ holds (see Proposition 3), it is straightforward to conclude that $\pi(t, \omega_\Lambda)$ is T-periodic. Finally, (36) follows analogously to the proof of [2, Corollary 5.5]. ∎

In other words, the result of Corollary 3 indicates that (under the above assumptions) the moment of (3) at (Λ, L) can be fully described by computing (34) over only one time period T.

Remark 14: The mapping π, solution of (34), can always be expressed as $\pi(t, \omega_\Lambda(t)) = \Pi(t)\omega_\Lambda(t)$.

Remark 15: If system (3) is linear (i.e., $f_{nl}(x, u) = 0$ and $h_{nl}(x) = 0$) then, the solution $\pi_l(t, \omega_\Lambda) = \Pi_l(t)\omega_\Lambda(t)$ of (34) with

$$\Pi_l(t) = \lim_{\bar{t} \to -\infty} \int_{\bar{t}}^{t} e^{A(t-\tau)} BL\omega_\Lambda(\tau)d\tau \, \Lambda(t)^{-1} \quad (37)$$

coincides with the solution obtained in [12] and [24].

B. Approximation of $\Pi(t)$ Driven by (Λ, L)

In this section, we present a method to approximate the moment of a nonlinear system (3) driven by an explicit signal generator (33) (as formalized in Definition 4). To achieve this, we introduce the following assumption.

Assumption 9: The set $F_{\psi_M} = \{\psi_k(t)\}_{k=1}^{M} \subset \mathcal{H}^*$, with $\psi_k(t) \in \mathbb{R}$ and where $\mathcal{H}^*$ is complete inner-product space with (closed) domain $\Xi \subset \mathbb{R}$, is a complete orthogonal set in Ξ and each component of the mapping $\Pi(t) = [\Pi(t)_{ij}]^{n, \nu}$ belongs to $\mathcal{H}^*$, i.e., it can be expressed as a unique linear combination of the set F_{ψ_M} as

$$\Pi(t)_{ij} = \sum_{k=1}^{M} \gamma_{ij_k} \psi_k(t) = \Gamma_{ij} \Psi_M(t) \quad (38)$$

where $\gamma_{ij_k} \in \mathbb{R}$ for $k \in \mathbb{N}_M$, $\Psi_M(t) = [\psi_1(t), \ldots, \psi_M(t)]^\mathsf{T}$, and $\Gamma_{ij} = [\gamma_{ij_1}, \ldots, \gamma_{ij_M}]$.

Note that, for this explicit signal generator case, we propose to approximate the matrix valued function Π directly, and then reconstruct the moment as $\pi(t, \omega_\Lambda(t)) = \Pi(t)\omega_\Lambda(t)$.

Remark 16: Under Assumption 9, $\Pi(t)$ can be written as $\Pi(t) = \Gamma^\Pi(\mathbb{I}_\nu \otimes \Psi_M(t))$ where $\Gamma^\Pi \in \mathbb{R}^{n \times N\nu}$ is given by

$$\Gamma^\Pi = [\Gamma_{ij}]^{n, \nu}. \quad (39)$$

Moreover, defining the vectors $\Psi(t) = [\psi_1(t), \ldots, \psi_N(t)]^\mathsf{T}$ with $N \geq M$ and $\Psi_M^*(t) = [\psi_{N+1}(t), \ldots, \psi_M(t)]^\mathsf{T}$, we can always expand $\Pi(t)$ as

$$\Pi(t) = \widetilde{\Gamma}^\Pi(\mathbb{I}_\nu \otimes \Psi(t)) + E(\mathbb{I}_\nu \otimes \Psi_M^*(t)) \quad (40)$$

where $\widetilde{\Gamma}^\Pi = [\widetilde{\Gamma}_{ij}]^{n, \nu}$, with $\widetilde{\Gamma}_{ij} = [\gamma_{ij_1}, \ldots, \gamma_{ij_N}]$, and $E = [\bar{\Gamma}_{ij}]^{n, \nu}$, with $\bar{\Gamma}_{ij} = [\gamma_{ij_{N+1}}, \ldots, \gamma_{ij_M}]$.

As in the implicit signal generator case, we now propose a method to compute an N-dimensional approximation $\widetilde{\Pi}(t)$ of $\Pi(t)$ based on a residual equation. This is considered in the following proposition.

Proposition 4: Consider the nonlinear system (3) driven by the explicit signal generator (33). Suppose that Assumptions 6, 7, and 9

hold. Then, the moment of system (3) at (Λ, L) can be approximated as $h(\widetilde{\Pi}(t)\omega_\Lambda(t))$, where $\widetilde{\Pi}(t) = \widetilde{\Gamma}^\Pi(\mathbb{I}_\nu \otimes \Psi(t))$ and $\widetilde{\Gamma}^\Pi$ is the solution of the system of algebraic equations

$$\left\langle \widetilde{\Pi}(t)\omega_\Lambda(t) - \Pi_l(t)\omega_\Lambda(t) - \widetilde{\Pi}_{nl}(t), \Psi_\Lambda(t)^\intercal \right\rangle = 0 \quad (41)$$

where $\Psi_\Lambda(t) = (\mathbb{I}_\nu \otimes \Psi(t))\omega_\Lambda(t)$ and

$$\Pi_l(t) = \int_0^t e^{A(t-\tau)} BL\omega_\Lambda(\tau)d\tau\, \Lambda(t)^{-1}$$

$$\widetilde{\Pi}_{nl}(t) = \int_0^t e^{A(t-\tau)} f_{nl}(\widetilde{\Pi}(\tau)\omega_\Lambda(\tau), L\omega_\Lambda(\tau))d\tau. \quad (42)$$

Proof: We omit the proof since it is analogous to that of Proposition 2, using the integral equation (34) and Remark 14. ∎

Remark 17: If, additionally, Assumption 8 holds and the zero equilibrium of $\dot{x} = f(x, 0)$ is locally exponentially stable, then $h(\widetilde{\Pi}(t)\omega_\Lambda(t))$, with $\widetilde{\Pi}(t)$ computed as in Proposition 4, approximates the steady-state output response of system (3) driven by (33). Additionally, exploiting the periodicity of the steady state and following the result of Corollary 3, it is possible to compute Π_l and $\widetilde{\Pi}_{nl}$ in Proposition 4 as

$$\Pi_l(t) = P \int_{t-T}^T e^{A(t-\tau)} BL\omega_\Lambda(\tau)d\tau\Lambda(t)^{-1}$$

$$\widetilde{\Pi}_{nl}(t) = P \int_{t-T}^T e^{A(t-\tau)} f_{nl}(\widetilde{\Pi}(\tau)\omega_\Lambda(\tau), L\omega_\Lambda(\tau))d\tau \quad (43)$$

and, hence, $\widetilde{\Pi}$ can be fully described using only information over one period.

Corollary 4: Let Assumptions 6, 7, and 9 be satisfied. Then, the system of algebraic equations (41) can be equivalently written in matrix form as

$$\left(\widetilde{\Gamma}^\Pi - \Pi_l^\Psi \Delta^{-1}\right) - \widetilde{\Pi}_{nl}^\Psi \Delta^{-1} = 0 \quad (44)$$

where the matrices $\Delta \in \mathbb{R}^{\nu N \times \nu N}$ and $\Pi_l^\Psi, \widetilde{\Pi}_{nl}^\Psi \in \mathbb{R}^{n \times \nu N}$ are defined as

$$\Delta = \langle \Psi_\Lambda(t), \Psi_\Lambda(t)^\intercal \rangle$$
$$\Pi_l^\Psi = \langle \Pi_l(t)\omega_\Lambda(t), \Psi_\Lambda(t)^\intercal \rangle$$
$$\widetilde{\Pi}_{nl}^\Psi = \langle \widetilde{\Pi}_{nl}(t), \Psi_\Lambda(t)^\intercal \rangle. \quad (45)$$

Proof: Note that, considering the orthogonality of the set $F_\Psi = \{\psi_i\}_{i=1}^N$ under Assumption 9, the matrix Δ can be explicitly written as $\Delta = \oplus_{i=1}^\nu \oplus_{j=1}^N \|\psi_j\omega_{\Lambda_i}\|^2$, where $\|\cdot\|$ denotes the norm induced by the inner-product defined in (8). Then, it follows from the minimality condition of Assumption 7 and the invertibility of $\Lambda(t)$ for $t \geq 0$, that $\omega_{\Lambda_i} \neq 0$ for all $i \in \mathbb{N}_\nu$ and then $\|\psi_j\omega_{\Lambda_i}\|^2 > 0$ for all $j \in \mathbb{N}_N$ and $i \in \mathbb{N}_\nu$. Hence, $0 \notin \sigma(\Delta)$ since $\Delta = \Delta^\intercal \succ 0$ and the proof follows. ∎

Analogously to the implicit case presented in Corollary 2, (44) is decomposed in a specific form, in which the contribution of the linear solution Π_l appears explicitly, together with a nonlinear "correction" term.

Remark 18: As in the case of (26), a sensible choice for the initial guess $\widetilde{\Gamma}_0^\Pi$ secures a fast convergence rate when solving (44). This can be chosen in terms of the linear solution of (44), i.e., $\widetilde{\Gamma}_0^\Pi = \Pi_l^\Psi \Delta^{-1}$.

To conclude this section, we provide an example that illustrates the applicability of the proposed method.

Example 2: Consider the nonlinear resonant inverter circuit depicted in Fig. 2, with dynamics described by the differential equations

$$C\frac{dv_c}{dt} = i_l, \quad L\frac{di_l}{dt} = -v_c - N(i_l) + u, \quad y = i_l \quad (46)$$
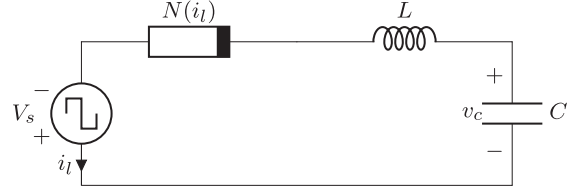

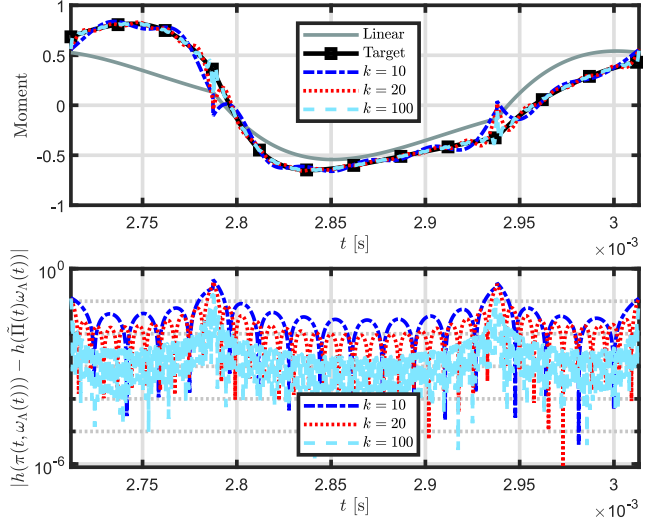Fig. 2.   Resonant converter.

Fig. 3.   Top: time histories of the approximated moment for the nonlinear resonant converter driven by the signal generator (47), for different values of k. Bottom: time histories of the corresponding absolute errors (logarithmic scale).

where the voltage at the nonlinear resistor is given by $N(i_l) = R(i_l + \alpha i_l^2)$, with $\alpha > 0$. The input u is a switching function, which is given by a square wave with angular frequency $\omega_\sqcap$, described in explicit form as [23]

$$\omega_\Lambda(t) = \begin{bmatrix} \sqcap(\omega_\sqcap t + \frac{\pi}{2}) & -\sqcap(\omega_\sqcap t) \\ \sqcap(\omega_\sqcap t) & \sqcap(\omega_\sqcap t + \frac{\pi}{2}) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$u(t) = \begin{bmatrix} V_s & 0 \end{bmatrix} \omega_\Lambda(t) \quad (47)$$

where $\sqcap(t) = \text{sign}(\sin(t))$, with $\text{sign}(0) = 0$. We select the parameters L and C of the system (46) as in [23], i.e., $L = 229.3\,[\mu\text{H}]$, $C = 10\,[\mu\text{F}]$, while the resistance is set to $R = 5\,[\Omega]$. The characteristic of the source is set to $V_s = 2$, $\omega_\sqcap = \frac{1}{\sqrt{LC}}$ and the nonlinear resistor coefficient is set to $\alpha = 1.5$. Note that the signal generator (47) satisfies all the assumptions of Definition 4. In addition, (47) satisfies Assumption 8 and the zero equilibrium of $\dot{x} = f(x, 0)$ is locally exponentially stable, hence, the result of Proposition 3 holds, and the moment of system (46) driven by (47) coincides with the steady-state output response of such an interconnected system.

Given the periodic nature of the input, we select trigonometric polynomials as orthogonal set of basis functions in $\Xi = [t; t + T_\sqcap]$, with $T_\sqcap = 2\pi/\omega_\sqcap$, i.e., the set $\{\psi_i\}$ is chosen as $F_\Psi = 1 \cup \{\cos(p\omega_\sqcap t), \sin(p\omega_\sqcap t)\}_{p=1}^k$. Fig. 3 shows the time history of $h(\pi(t, \omega_\Lambda(t)))$ computed using a Runge–Kutta method, and the time history of $h(\widetilde{\Pi}(t)\omega_\Lambda(t))$ considering a different number of basis functions in the set F_Ψ, i.e., $k = \{10, 20\,100\}$. The large number of components to successfully approximate the moment relates to the

discontinuous nature of the problem. In fact, it can be seen in Fig. 3 (bottom) that the absolute value of the approximation error becomes higher at the points where $\omega_\Lambda(t)$ is discontinuous, though improving with increasing k.

V. Conclusion

This article presents a framework to approximate the moment of nonlinear systems driven by signal generators. The methods proposed are inspired by a connection between moment-based theory and the family of WRMs. This article formalizes and exploits this connection to propose a set of methods to approximate the moment of a nonlinear system driven by an implicit signal generator. Furthermore, we present the formal definition of moments for nonlinear systems driven by explicit signal generators and we propose a method to compute such moments, extending the applicability of the framework to a very general class of inputs, including periodic discontinuous sources. While our contributions are technical and strictly related to the definition and computation of moments, these contributions allow the computation and definition of new classes of reduced order models, following the system-theoretic approach to nonlinear model reduction by moment-matching presented in, e.g., [2]. For instance, using the contributions of this article, one can define and compute a family of reduced models achieving moment matching at an explicit signal generator for a general class of nonlinear systems, extending the framework presented in [2] for linear dynamical systems driven by this class of input signals. Finally, we article that the methods proposed in this article are illustrated by means of simple examples.

References

[1] A. C. Antoulas, "Approximation of large-scale dynamical systems," *SIAM Advances in Design and Control*, Philadelphia, PA, 2005. [Online]. Available: https://epubs.siam.org/doi/book/10.1137/1.9780898718713

[2] G. Scarciotti and A. Astolfi, "Nonlinear model reduction by moment matching," *Found. Trends Syst. Control*, vol. 4, no. 3/4, pp. 224–409, 2017.

[3] A. Astolfi, "Model reduction by moment matching for linear and nonlinear systems," *IEEE Trans. Autom. Control*, vol. 55, no. 10, pp. 2321–2336, Oct. 2010.

[4] G. Scarciotti, "Approximation, analysis and control of large-scale systems," Ph.D. dissertation, Control and Power Research Group, Imperial College London, London, U.K., 2016.

[5] J. B. Lasserre, D. Henrion, C. Prieur, and E. Trélat, "Nonlinear optimal control via occupation measures and LMI-relaxations," *SIAM J. Control Optim.*, vol. 47, no. 4, pp. 1643–1666, 2008.

[6] J. B.Lasserre, Moments, Positive Polynomials and their Applications, vol. 1. Singapore: *World Scientific*, 2010.

[7] A. Padoan and A. Astolfi, "Moments of random variables: A systems-theoretic interpretation," *IEEE Trans. Autom. Control*, vol. 64, no. 11, pp. 4407–4422, Nov. 2019.

[8] J. P. Boyd, *Chebyshev and Fourier Spectral Methods. Chelmsford, MA, USA: Courier Corporation*, 2001.

[9] B. Fornberg, *Practical Guide to Pseudospectral Methods (ser. Cambridge monographs on applied and computational mathematics)*. Cambridge, U.K.: Cambridge Univ. Press, 1998.

[10] C. Canuto *et al. Spectral Methods in Fluid Dynamics*. Berlin, Germany: Springer Science and Business Media, 2012.

[11] G. Scarciotti and A. Astolfi, "Data-driven model reduction by moment matching for linear and nonlinear systems," *Automatica*, vol. 79, pp. 340–351, 2017.

[12] G. Scarciotti and A. Astolfi, "Model reduction by matching the steady-state response of explicit signal generators," *IEEE Trans. Autom. Control*, vol. 61, no. 7, pp. 1995–2000, Jul. 2016.

[13] N. Faedo, G. Scarciotti, A. Astolfi, and J. V. Ringwood, "Energy-maximising control of wave energy converters using a moment-domain representation," *Control Eng. Pract.*, vol. 81, pp. 85–96, 2018.

[14] N. Faedo, G. Scarciotti, A. Astolfi, and J. V. Ringwood, "Moment-based constrained optimal control of an array of wave energy converters," in *Proc. Amer. Control Conf.*, 2019, pp. 4797–4802.

[15] E. D. Sontag, *Mathematical Control Theory: Deterministic Finite Dimensional Systems*, vol. 6, Berlin, Germany: Springer Science and Business Media, 2013.

[16] A. Isidori, *Nonlinear Control Systems*. Berlin, Germany: Springer Science and Business Media, 2013.

[17] A. Padoan, G. Scarciotti, and A. Astolfi, "A geometric characterization of the persistence of excitation condition for the solutions of autonomous systems," *IEEE Trans. Autom. Control*, vol. 62, no. 11, pp. 5666–5677, Nov. 2017.

[18] A. Isidori and C. I. Byrnes, "Steady-state behaviors in nonlinear systems with an application to robust disturbance rejection," *Annu. Rev. Control*, vol. 32, no. 1, pp. 1–16, 2008.

[19] W. Rudin, *Functional Analysis (ser. International series in pure and applied mathematics)*. New York, NY, USA: McGraw-Hill, 1991.

[20] P. R. Halmos, *Introduction to Hilbert Space and the Theory of Spectral Multiplicity*. New York, NY, USA: Dover Publications, 2017.

[21] T. F. Coleman and Y. Li, "An interior trust region approach for nonlinear minimization subject to bounds," *SIAM J. Optim.*, vol. 6, no. 2, pp. 418–445, 1996.

[22] A. Isidori and A. Astolfi, "Disturbance attenuation and H_∞-control via measurement feedback in nonlinear systems," *IEEE Trans. Autom. Control*, vol. 37, no. 9, pp. 1283–1293, Sep. 1992.

[23] G. Scarciotti and A. Astolfi, "Moment-based discontinuous phasor transform and its application to the steady-state analysis of inverters and wireless power transfer systems," *IEEE Trans. Power Electron.*, vol. 31, no. 12, pp. 8448–8460, Dec. 2016.

[24] G. Scarciotti and A. Astolfi, "Characterization of the moments of a linear system driven by explicit signal generators," in *Proc. IEEE Amer. Control Conf.*, 2015, pp. 589–594.