MIXED NORMS AND ANALYTIC FUNCTION SPACES

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Abstract

We define and investigate general mixed-norm type sequence spaces, and strengthen inequalities of Hardy-Littlewood, Hausdorff-Young, and others.

0. Introduction

Since the time of Hardy and Littlewood, mixed-norm and related spaces have been used to study Taylor coefficients in function spaces on the unit disk, and later to study multipliers between such spaces; for example, see [9], [2], [13], [3], [5], [6], and the many references in those papers. Here, we define very general mixed-norm spaces in Section 1, and investigate their interaction with multipliers in Section 2; in particular, we generalise a result of Kellogg. In Section 3, we prove mixed-norm containments involving H^p which strengthen results due to Hardy-Littlewood, Flett, Hausdorff-Young, and Kellogg. Our results have also found applications in the investigation of restricted solidity for function spaces [4].

1. Preliminaries

Throughout this paper, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $dA(z) = \pi^{-1}\, dx\, dy$ $(z = x+iy)$ is normalised area measure, $\mathcal{H}(\mathbb{D})$ is the algebra of holomorphic functions in $\mathbb{D}$. We identify any function $f \in \mathcal{H}(\mathbb{D})$ with its Taylor sequence $(a_n)_{n=0}^{\infty}$, and adopt the conventions $1/\infty = 0$, $1/0 = \infty$, $0 \cdot \infty = 0$, and $x^{1/\infty} = x^0 = 1$, for all $x \geq 0$. For any exponent $1 < p < \infty$, p' is the dual exponent $p/(p-1)$. In proofs, we use C to denote any constant that does not affect the argument; it can change from one instance to the next. If A and B are positive quantities, $A \lesssim B$ means that $A \leq CB$ for some such C, and $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

H^p is the well-known *Hardy space* H^p on the unit disk and, for $p, q \in (0, \infty]$, $t \in (0, \infty)$, we write $f \in H(p, q, t)$ if $f \in \mathcal{H}(\mathbb{D})$ and $\|f\|_{H(p,q,t)} < \infty$, where

$$\|f\|_{H(p,q,t)} \equiv \begin{cases} \left(\int_0^1 M_p(r,f)^q (1-r)^{tq-1}\, dr\right)^{1/q}, & q < \infty, \\ \sup_{0<r<1} (1-r)^t\, M_p(r,f), & q = \infty. \end{cases}$$

As usual, $M_p(r, f)$ denotes the L^p mean of f at radius $0 < r < 1$. In particular, we have the *Bergman spaces* $A^p \equiv H(p, p, 1/p)$, $0 < p < \infty$. Using fractional differentiation, we shall soon extend the definition of $H(p,q,t)$ to all $t \in \mathbb{R}$ (and all $0 < p,q \le \infty$); the above definitions are, however, valid only for $t > 0$.

X and Y will always be sequence spaces, and λ a sequence. If a is a sequence, a_n is its nth term; the same convention applies to all unsubscripted letters. We say that λ is a *multiplier* from X to Y, denoted $\lambda \in (X,Y)$, if $T_\lambda a \equiv (\lambda_n a_n) \in Y$ for all $a \in X$; we call T_λ a *multiplier operator* from X to Y. We often write λX in place of $T_\lambda X$, and $XY = \bigcup_{\lambda \in X} \lambda Y$. *Fractional differentiation* multipliers $D^t = ((n+1)^t)$, $t \in \mathbb{R}$, will be of particular interest; the associated operator will also be denoted D^t. Flett [8] showed that $D^t H(p,q,s) = H(p,q,s+t)$, $s,t > 0$. Thus we can define $H(p,q,t)$, $t \le 0$, by $H(p,q,t) = D^{-s} H(p,q,s+t)$ for any $s > -t$. Note that $H(p,\infty,0)$ and H^p are not the same, e.g. $f(z) = 1/(1-z) \in H(1,\infty,0) \setminus H^1$, since $f' \in H(1,\infty,1)$.

A vector space X is *sized* if it is equipped with a *size function* $\|\cdot\|_X : X \to [0,\infty]$ satisfying the equation $\|\alpha x\|_X = |\alpha| \|x\|_X$ for all $x \in X$, $\alpha \in \mathbb{C}$ (thus $\|0\|_X = 0$). Wherever we define a space X and a quantity $\|\cdot\|_X$ (above, and in the remainder of this paper), $\|\cdot\|_X$ is a size function, as the reader may verify. If X is sized, then $D^s X$ is sized by pulling back the size function. For the sake of a definitive size function on $H(p,q,t)$, $t \le 0$, let $\|f\|_{H(p,q,t)} = \|D^{1-t} f\|_{H(p,q,1)}$ in this case.

Let Δ_k, $k \ge 0$, be the multiplier given by the characteristic function of I_k, where I_k is the *kth dyadic block of integers*, i.e. $I_0 = \{0\}$ and $I_k = \{2^{k-1}, \dots, 2^k - 1\}$, $k \in \mathbb{N}$. Let S_k be the operator which selects and shifts to an initial position the kth dyadic block of a sequence; thus $S_k((a_n)_{n=0}^\infty)$ equals $(a_0, \dots)$ if $k = 0$, or $(a_{2^{k-1}}, \dots, a_{2^k - 1}, 0, 0, \dots)$ if $k > 0$. If A, B are sequence spaces and $\mathcal{B} = (B_k)$ is a sequence of sized spaces, the "mixed-norm" space $A[B]$ consists of all sequences λ such that each $S_k \lambda \in B_k$, $\|S_k \lambda\|_{B_k} < \infty$, and $(\|S_k \lambda\|_{B_k})_{k=0}^\infty \in A$. If A is also sized, then we give $A[\mathcal{B}]$ the size function $\|\,\lambda\,\|_{A[\mathcal{B}]} = \|\,(\|S_k \lambda\|_{B_k})_{k=0}^\infty\,\|_A$. We mainly consider the special case $B_k = B$ for all k, and then write $A[B]$ in place of $A[\mathcal{B}]$. When all spaces are sized, we can iterate this construction "from the inside outwards" to get for instance $A[B[C]]$, where $\|\,\lambda\,\|_{A[B[C]]} = \|\,(\|S_k \lambda\|_{B[C]})_{k=0}^\infty\,\|_A$. We shall only be interested in spaces $A[B]$ where $\|\cdot\|_A$, $\|\cdot\|_B$ happen to be shift-invariant, so we could have used Δ_k in place of S_k in the above definition; however, the natural iterated definition requires S_k. We shall not need to use the completeness of mixed-norm spaces; we refer the interested reader to [10].

An important example is the space $l^q[l^p]$; we often use the more common notations $l(p,q)$ and $\|\cdot\|_{p,q}$ instead of $l^q[l^p]$ and $\|\cdot\|_{l^q[l^p]}$; similarly, $l(p,q,r) \equiv l^r[l^q[l^p]]$, etc.

If $0 < p, q \leq \infty$, we write $p \ominus q = r$, where $1/r = \max\{1/p - 1/q, 0\} \in (0, \infty]$. The following is a result of Kellogg [12] when $a, b, c, d \geq 1$; it follows for all positive exponents, since $(\lambda_n) \in (l(a,b), l(c,d))$ if and only if $(\lambda_n^{1/t}) \in (l(at, bt), l(ct, dt))$.

Lemma 1.1. *If $0 < a, b, c, d \leq \infty$, then $(l(a,b), l(c,d)) = l(c \ominus a, d \ominus b)$.*

The following characterisation of $H(p,q,t)$ hints at its natural connection with mixed-norm spaces.

Lemma 1.2. *Let $t \in \mathbb{R}$, $0 < q \leq \infty$. If $1 < p < \infty$ then*

$$\|f\|_{H(p,q,t)} \approx \| \, (2^{-kt} \|\Delta_k f\|_{H^p}) \, \|_{l^q}, \tag{1.1}$$

$$H(p,q,t) = D^t l^q [H^p]. \tag{1.2}$$

(1.1) is due to Mateljević and Pavlović [13] for $t > 0$; the case $t \leq 0$ then follows immediately. Equation (1.2), intuitively very similar to (1.1), will be the key to proving Theorem 3.1. A version of (1.2) was proved in the case of $A^p = H(p,p,1/p)$ in [6], and our more general proof is very similar.

Proof of Lemma 1.2. In view of the above remarks, it suffices to prove (1.2). Proposition 3.7 of [5] says that for $1 < p < \infty$, BV (the space of bounded variation sequences) is a subset of (H^p, H^p), and thus also $H^p \subset (BV, H^p)$. Applying the Closed Graph Theorem to BV multipliers from H^p to itself, and to H^p multipliers from BV to H^p, we see that all of these multipliers are bounded. Applying the Uniform Boundedness Principle to the family $\{T_\lambda : \|\lambda\|_{BV} \leq 1\}$ of multipliers from H^p to itself, it follows that $\|(a_n b_n)\|_{H^p} \leq C \|(a_n)\|_{BV} \cdot \|(b_n)\|_{H^p}$.

To prove the result, we need to show that the quantities $2^{-k}\|\Delta_k f\|_{H^p}$ and $\|\Delta_k D^{-1/p} f\|_{H^p}$ are uniformly comparable in size. To change one of these expressions to the other, we apply to f the multiplier $\Delta_k D^{-1/p}(2^{k/p})$, or its "inverse" $\Delta_k D^{1/p}(2^{-k/p})$. Both of these are in BV, with total variation at most $2^{1+1/p}$. $\square$

We end this section with an easy corollary of Lemma 1.2, which characterises the monotonic sequences in $H(p,q,t)$. This was done for A^p in [5]; we include a proof for this more general situation for the sake of completeness.

Corollary 1.3. *Suppose that $a_n \geq 0$ for all $n \geq 0$, and that $t \in \mathbb{R}$, $1 < p < \infty$, $0 < C < \infty$, $0 < q \leq \infty$. If either*

(i) *(a_n) is monotonic, or*

(ii) *$(a_n)_{n \in I_k}$ is monotonic and, for all $n, m \in I_k$, $a_m \leq C a_n$,* *then*

$$(a_n) \in H(p,q,t) \Longleftrightarrow \begin{cases} (a_n) \in D^{t+1/p+1/q-1}\, l^q, & q < \infty \\ (a_n) \in D^{t+1/p-1}\, l^\infty, & q = \infty \end{cases}$$

Proof. It is not hard to show that for all $n \in I_k$, $\| \sum_{n \le m \in I_k} z^m \|_{H^p} \lesssim 2^{k(p-1)/p}$. One approach to justifying this is to write the sum as $(z^{2^k} - z^n)/(z-1)$ and find upper bounds for this expression for $z = e^{i\theta}$ in the ranges $|\theta| \le 2^{-k}$ and $2^{-j-1} < |\theta| \le 2^{-j}$, $j = 0, \ldots, k$; we leave the details to the reader. Using monotonicity and the equation

$$\sum_{n \in I_k} a_n z^n = a_{2^{k-1}} \sum_{n \in I_k} z^n + \sum_{2^{k-1} < n \in I_k} (a_n - a_{n-1}) \sum_{n \le m \in I_k} z^m$$

we see that

$$\| \sum_{n \in I_k} a_n z^n \|_{H^p} \lesssim 2^{k(p-1)/p} \left(a_{2^{k-1}}^p + a_{2^k-1}^p \right)^{1/p} \lesssim 2^{k(p-1)/p} \left(a_{2^{k-1}} + a_{2^k-1} \right).$$

We now get one half of the equivalence by using Lemma 1.2 to deduce that

$$\|f\|_{H(p,q,t)}^q \lesssim \| \, (2^{k(1-1/p-t)} [a_{2^{k-1}} + a_{2^k-1}])_{k=1}^\infty \, \|_{l^q}.$$

The converse direction follows easily from Lemma 1.2 and the estimate

$$\| \sum_{n \in I_k} a_n z^n \|_{H^p} \ge C 2^{k(p-1)/p} \min_{n \in I_k} a_n.$$

This last inequality is immediate if one considers the real part of the left-hand sum on the arc $\{z = e^{i\theta} : |\theta| < 2^{-k}\}$. $\square$

2. Mixed norms and multipliers

For this section, it is useful to define the families $l^* = \{l^p : 0 < p \le \infty\}$ and $Hl^* = l^* \cup \{H^p : 1 < p < \infty\}$. We say that a sequence space X is *solidly sized* and that $\| \cdot \|_X$ is a *solid size function* if X is both solid and sized, and $\|(x_n)\|_X \le \|(y_n)\|_X$ whenever $(x_n), (y_n) \in X$, and $|x_n| \le |y_n|$ for all $n \in \mathbb{N}$. If Y, W are sized spaces, then we associate with (Y,W) and YW the size functions

$$\begin{aligned} \|\lambda\|_{(Y,W)} &= \sup \{ \|(\lambda_n y_n)\|_W \, : \, \|y\|_Y \le 1 \}, \\ \|x\|_{YW} &= \inf \, \{ \|y\|_Y \|w\|_W \, : \, (x_n) = (y_n w_n), \, y \in Y, w \in W \}. \end{aligned}$$

Using these size functions, we define the spaces $(X, Z)[(Y, W)]$ and $XZ[YW]$, whenever Y, W are sized (and these new spaces are sized if X, Z are also sized).

In general, the size function $\|\cdot\|_{(Y,W)}$ might take on infinite values even if $\|\cdot\|_W$ takes on only finite values. However if Y, W are complete quasi-normed spaces (with their quasi-norms as size functions) on which the point evaluation functionals are continuous, then the Closed Graph Theorem tells us that $\|\cdot\|_{(Y,W)}$ takes on only finite values. In particular, this is the case when $Y, W \in Hl^*$.

We are now ready to state the main result of this section. As it is rather abstract, we give some applications before the proof. Note that each of the assumptions in this theorem is *hereditary*, i.e. $X[Y]$ satisfies the assumption if both X and Y do; this allows us to use the theorem iteratively.

Theorem 2.1. *Suppose that X, Z are solidly sized, that Y, W are sized, and that*

$$\exists\, C\ \forall\, k \geq 0 : \qquad \|T_k y\|_Y \leq C\|y\|_Y, \tag{2.1}$$

where T_k is the multiplier operator given by $\sum_{j=0}^{k} \Delta_k$. Then $(X[Y], Z[W]) = (X, Z)[(Y, W)]$, $X[Y]Z[W] = XZ[YW]$, and the associated size functions are also comparable in both cases.

All l^p spaces are solidly sized, and satisfy (2.1) with $C = 1$. Thus Theorem 2.1 allows us to reduce Lemma 1.1 to the unmixed case $a = b$, $c = d$ (in which case, it is easy to prove), and to extend it to the case of an arbitrary number of nestings. In fact, writing $p \oplus q = s$, with $1/s = 1/q + 1/p$, we have the following theorem.

Theorem 2.2. *Suppose that $j \in \mathbb{N}$ and that $p_i, q_i \in (0, \infty]$ for all $1 \leq i \leq j$. Then*

$$(l(p_1, \ldots, p_j), l(q_1, \ldots, q_j)) = l(q_1 \ominus p_1, \ldots, q_j \ominus p_j), \tag{2.2}$$

$$l(p_1, \ldots, p_j)\, l(q_1, \ldots, q_j) \ = l(q_1 \oplus p_1, \ldots, q_j \oplus p_j), \tag{2.3}$$

and furthermore the associated size functions are comparable.

Proof. Since l^p-type spaces satisfy all the assumptions of Theorem 2.1, (2.2) follows inductively from Lemma 1.1 and Theorem 2.1. We similarly deduce (2.3) once we establish it in the case $j = 1$. In this case, it is an easy application of Hölder's inequality for l^p spaces (including the condition for sharpness of this inequality). The comparability of size functions follows from case $j = 1$ in which case it is easy to establish (or follows from the Closed Graph Theorem). $\square$

A special case of (2.2) (where all indices are at least 1, and $p_i = p$, $q_i = q$ for all $1 < i < j$) is proved in [14], and the general case is conjectured to be true.

H^p spaces satisfy (2.1) for $1 < p < \infty$, since $BV \subset (H^p, H^p)$; see [5]. Thus, using Theorem 2.1 and (1.2), we get (for any $b, d \in (0, \infty]$, $s, t \in \mathbb{R}$):

$$(2.4) \qquad (H(a,b,s), H(c,d,t)) = D^{t-s} l^{d \ominus b}[(H^a, H^c)], \quad 1 < a, c < \infty,$$

This identity connects some known results on A^p and H^p multipliers, e.g. the results on (A^p, A^q) and (H^p, H^q) in [11]. Also, (2.4) gives the simple relation $(A^p, A^p) = l^\infty[(H^p, H^p)]$ when $1 < p < \infty$; these self-multipliers appear to be known only for $p = 1, 2$ (see [6] for $p = 1$).

If D is a fixed sequence space, the D-dual X^D of a sequence space X is defined, as in [1], to be (X, D). In particular, the Köthe dual X^K of X is (X, l^1). Using Theorem 2.1 inductively, we see that

$$(2.5) \qquad X_1[\ldots[X_n]\ldots]^K = X_1^K[\ldots[X_n^K]\ldots],$$

as long as these spaces satisfy the assumptions of Theorem 2.1. For instance, (2.5) is true if $X_n \in Hl^*$ and $X_i \in l^*$, $1 \le i < n$ (the purely l^p version of (2.5) is proved in [14]). Note that the Köthe dual of $X = H^p$ (or l^p), $1 < p < \infty$, equals $H^{p'}$ (or $l^{p'}$, respectively). Also we can replace the duals in (2.5) by D-duals, for any sized space D satisfying $D = D[D]$ and (2.1) (e.g. $D = l^p$, $p > 0$).

Proof of Theorem 2.1. For both of our conclusions, showing that the spaces are equal is similar to, but somewhat easier than, showing that the size functions are comparable, so we prove only the size function inequalities.

We write X_1, Y_1 and $X[Y]_1$ for the closed unit balls of X, Y, Z, respectively, i.e. the set of all sequences in those spaces whose size function value is at most 1. We also write Y_1^k for the collection of all $y \in Y_1$ such that $y_n = 0$ for all $n \ge 2^{k-1}$ (all $n > 0$ if $k = 0$). Then $X[Y]_1$ consists precisely of the set of sequences a such that $S_k(a) = x_k y^k$, where $(x_j) \in X_1$, and $y^k \in Y_1^k$ for all $k \ge 0$ (we could also assume that $\|y_k\|_Y \in \{0, 1\}$, and $x_k \ge 0$, but we do not need to do so). It follows that if $\lambda \in (X, Z)[(Y, W)]$, then

$$\begin{aligned} \|\lambda\|_{(X[Y], Z[W])} &= \sup_{a \in X[Y]_1} \|\,(\|S_k(\lambda a)\|_W)\,\|_Z \\ &= \sup_{x \in X_1, y^k \in Y_1^k} \|\,(|x_k| \cdot \|S_k(\lambda) y^k\|_W)\,\|_Z \\ &\le \sup_{x \in X_1} \|\,(|x_k| \cdot \|S_k(\lambda)\|_{(Y,W)})\,\|_Z = \|\lambda\|_{(X,Z)[(Y,W)]}. \end{aligned}$$

For the reverse direction, we need to reverse the one inequality above. Since $S_k\lambda$ has zero coefficients beyond position 2^{k-1}, (2.1) enables us to do precisely this (of course, the inequality inherits C as a constant of comparability).

We next prove that $X[Y]Z[W]$ and $XZ[YW]$ have comparable size functions. Let $a \in XZ[YW]$ and assume that $N = \|a\|_{XZ[YW]} < \infty$. Then $S_k a = b_k c^k$, where $\|(b_k)\|_{XZ} = N$, and each c^k is in the closed unit ball of YW. Letting $\epsilon > 0$ be arbitrary, we can, by suitable scaling, write $b = xz$ where $\|x\|_X, \|z\|_Z \le (1+\epsilon)N^{1/2}$, and we can write each c_k as $y_k w_k$, where $\|y_k\|_Y, \|w_k\|_W \le 1+\epsilon$. Thus $a = de$, where $S_k d = x_k y^k$, $S_k e = z_k w^k$. Since $\epsilon > 0$ is arbitrary, it is easy to deduce one half of the result. The reverse inequality is easier, so we omit the proof. $\square$

As the reader may check from the proof, Theorem 2.1 remains valid if we replace Y and W by $\mathcal{Y} = (Y_k)$ and $\mathcal{W} = (W_k)$, where each of the spaces Y_k and W_k are sized and $Y = Y_k$ satisfies (2.1) with a constant independent of k. Naturally, the spaces (Y, W) and YW in the statement of this more general result are to be replaced by sequences of spaces whose kth members are (Y_k, W_k) and $Y_k W_k$, respectively.

3. Containments involving H^p and mixed norm spaces

The following is the main result of this section; it improves inequalities of Hardy-Littlewood, Flett, Hausdorff-Young, and Kellogg.

Theorem 3.1. *Let $1 < p \le 2$, and define $X_0 = Y_0 = H^p$, $Z_0 = l^{p'}$, $X_m = l^p[X_{m-1}]$, $Y_m = l^2[Y_{m-1}]$, $Z_m = l^2[Z_{m-1}]$, for all $m > 0$. Then*

$$\begin{array}{ccccccccccccccc}
\ldots \subset X_m \subset X_{m-1} \subset \ldots \subset X_1 \subset & H^p & \subset & Y_1 & \subset & \ldots & \subset & Y_{m-1} & \subset & Y_m & \subset \ldots \\
& \cap & & \cap & & & & \cap & & \cap & \\
& l^{p'} & \supset & Z_1 & \supset & \ldots & \supset & Z_{m-1} & \supset & Z_m & \supset \ldots
\end{array}$$

Furthermore, every containment is proper if $p < 2$.

We first discuss the containments above that can be found in the literature once we recast them in the light of (1.2). This recasting is really the crucial step in the proof of the containments, which then follow by easy induction arguments. First, the containment $X_1 \subset H^p$ is equivalent by (1.2) to the statement $D^{-t}H(p,p,t) \subset H^p$, for any $t > 0$, which in turn follows from a result of Flett [8, Theorem 5]. Similarly, (1.2) allows us to rewrite $H^p \subset Y_1$ as $H^p \subset D^{-1}H(p,2,1)$; this containment is due to Hardy and Littlewood (see [3, Lemma D]). The imbedding $H^p \subset l^{p'}$ is the well-known Hausdorff-Young Theorem [7, Theorem 6.1]); it was strengthened by Kellogg [12] who showed that $H^p \subset Z_1 = l(p',2)$. We recently discovered that Ramanujan and Tanović-Miller [14] proved that $H^p \subset Z_m$, $m > 0$, but our proof is much shorter and very different. The other H^p imbeddings appear to be new.

The fact that $Z_\infty \equiv \bigcap_m Z_m$ is not equal to H^p, $p < 2$, is proved in a non-constructive manner in [14]. By our theorem, Z_∞ contains $\bigcup_m Y_m$, and so the example of a sequence in $Y_1 \setminus H^p$ in our proof suffices to show that $Z_\infty \neq H^p$.

Proof. The containments $X_1 \subset H^p \subset Y_1$ were proved above; induction readily gives $X_m \subset X_{m-1}$ and $Y_{m-1} \subset Y_m$, for all $m \in \mathbb{N}$. The Hausdorff-Young Theorem implies that $Y_m \subset Z_m$. The containment $Z_m \subset Z_{m-1}$ is elementary.

It is left to show that these containments are proper if $1 < p < 2$. We first consider $H^p \subset Y_1$. By Theorem 5.11 of [7], $H^p \subset H(2,p,1/p - 1/2) \equiv D^{1/p-1/2} l(2,p)$. By direct calculation, $\lambda \equiv (n^{1/p-1}(\log n)^{-1/p})_{n=2}^\infty \notin D^{1/p-1/2} l(2,p)$, and so $\lambda \notin H^p$. However $\lambda \in H(p,2,0) = l^2[H^p]$, according to Corollary 1.3, and so $Y_1 \neq H^p$.

To deduce inductively that $Y_m \setminus Y_{m-1}$ is non-empty, first note that the sequence of Taylor polynomials of every $f \in X_m$ is norm-convergent to f; this was proved by Zhu [15] for $m = 0$, and follows inductively for all $m \in \mathbb{N}$. Suppose that the containment $Y_{m-1} \subset Y_m$ is proper for $m = i$. By density of the Taylor polynomials, we can find for each $j \in \mathbb{N}$, polynomials f_j of degree n_j such that $\|f_j\|_{X_i} = 1$ but $\|f_j\|_{X_{i-1}} \geq 2^j$. We choose integers $0 < k_1 < k_2 < \ldots$, so large that $2^{k_j} > n_j$ for each j, and let $f(z) = \sum_{j=1}^\infty j^{-1} z^{2^{k_j}} f_j(z)$. Then $f \in X_i \setminus X_{i+1}$, as required.

Next, $f(z) \equiv \sum_{n=1}^\infty n^{-1/p} z^{2^n} \in H^2$, and so $f \in H^p$, but clearly $f \notin l^p[H^p]$. The proof that the containment $X_m \subset X_{m-1}$ is proper for all m follows as before.

It is well-known that the containment $H^p \subset l^{p'}$ is proper (it follows for instance from the fact that lacunary sequences are in H^p if and only if they in H^2), and it is easy to deduce that $Y_m \setminus Z_m$ is non-empty (again using the norm-convergence of Taylor polynomials). Similarly, the fact that each $Z_m \setminus Z_{m-1}$ is non-empty follows from the case $m = 1$. To prove this case, note that $\sum_{k=2}^\infty k^{-1/2} z^{2^k} \in l^{p'} \setminus l(p',2)$. $\square$

Applying duality with respect to the bilinear form $< x, y >= \sum_{n=0}^\infty x_n \overline{y_n}$ (or equivalently, Köthe duality) in the above theorem, we get that

$$l(p',2,\ldots,2) \subset l^2\,[\,l^2\,[\cdots l^2\,[H^p]\cdots]\,] \subset H^p \subset l^p\,[\,l^p\,[\cdots l^p\,[H^p]\cdots]\,], \quad 2 \leq p < \infty.$$

The above containments and those in the theorem all imply corresponding norm inequalities by the Closed Graph Theorem, but we need large constants of comparability if the mixed norms have many levels.

Finally, we discuss some spaces that can be rewritten as spaces of the form $D^t A[B]$. If $0 < p \leq \infty$, $0 < q < \infty$, $a > 1/p$, and $b \in \mathbb{R}$, let $S(p,q,a,b)$ be the space of all sequences a for which

$$\|a\|_{S(p,q,a,b)} \equiv |a_0| + \| \, (n^{-a}(\sum_{j=1}^{n}(j^b|a_j|)^q)^{1/q}) \, \|_{l^p} < \infty.$$

This definition also makes sense for $q = \infty$ when we make the obvious adjustment, and for $a = 0$ if $p = \infty$. As a sample use of such spaces, we mention [9], where Holland and Twomey showed that *areally mean s-valent* functions lie in $X^p = S(p,2,2/p,1/2)$ if and only if they lie in H^p, and one gets one-way containments for more general functions. The following theorem indicates in particular, that X^p is just a disguised form of the space $D^{1/p-1/2}l(2,p)$.

Proposition 3.2. *Suppose that $0 < p, q \leq \infty$, $b \in \mathbb{R}$, and that $a > 1/p$ (or $a \geq 0$ if $p = \infty$). Then $S(p,q,a,b) = D^r l(q,p)$, where $r = a - b - 1/p$. Furthermore, there exists a constant C dependent only on p, q, and a such that*

$$1/C \leq \frac{\|x\|_{S(p,q,a,b)}}{\|x\|_{D^r l(q,p)}} \leq C.$$

Proof. Without loss of generality, we may assume that $b = 0$ (since we may replace (x_n) by $(y_n) = (n^b y_n)$, if necessary). We prove the result for $q < \infty$; the case $q = \infty$ requires only minor adjustments. Let (x_n) be a sequence, $t_n = (\sum_{j=1}^n |x_j|^q)^{1/q}$, and $s_k = (\sum_{j \in I_k} |x_j|^q)^{1/q}$. We first show that $S(p,q,a,b) \subset D^r l(q,p)$, so let $(x_n) \in S(p,q,a,0)$. If $n \in I_{k+1}$, then $s_k \leq t_n$ and so, assuming for now that $p < \infty$, we have

$$\sum_{k=1}^{\infty} 2^{k(-ap+1)} s_k^p \lesssim \sum_{n=1}^{\infty} n^{-ap} t_n^p,$$

Thus $(a_n) \in D^{a-1/p} l(q,p)$, as required. In the case $p = \infty$, we similarly have

$$\|(2^{-ak} s_k)\|_{l^\infty} \leq \|(n^{-a} t_n)\|_{l^\infty}.$$

Conversely, suppose that $(a_n) \in D^r l(q,p)$. Now $t_n^q \le \sum_{j=1}^k s_j^q$ for all $n \in I_k$. Assuming for now that $p < \infty$, and fixing $\epsilon \in (0, ap-1)$, we have

$$\begin{aligned} S \equiv \sum_{n=1}^\infty n^{-ap} t_n^p &\lesssim \sum_{k=1}^\infty 2^{-kap} \sum_{n \in I_k} \left(\sum_{j=1}^k s_j^q \right)^{p/q} \le \sum_{k=1}^\infty 2^{k(1-ap)} \left(\sum_{j=1}^k s_j^q \right)^{p/q} \\ &\lesssim \sum_{k=1}^\infty 2^{k(1-ap)} \sum_{j=1}^k 2^{\epsilon(k-j)} s_j^p \quad \text{(by Holder's inequality)} \\ &= \sum_{j=1}^\infty 2^{-j\epsilon} s_j^p \sum_{k=j}^\infty 2^{k(1-ap+\epsilon)} \lesssim \sum_{j=1}^\infty 2^{j(1-ap)} s_j^p, \end{aligned}$$

as required. For the case $p = \infty$, let us assume that $\|(a_n)\|_{D^a l(q,p)} = N < \infty$. Then $\|2^{-ka} s_k\|_{l^\infty} \le N$, and so $\|(n^{-a} t_n)\|_{l^\infty}^q \lesssim \|(2^{-kaq} \sum_{j=1}^k s_j^q)\|_{l^\infty}$. It is not hard to see that this last quantity is at most CN^q, as required.

The last statement of the theorem follows from the above estimates. □

A related family of spaces of the form $D^t A[B]$ (discussed, for example, in [10]) are the spaces w_q^0 of sequences strongly $(C,1)$-summable to zero with index $q > 0$, i.e. $(a_n) \in w_q^0$ if and only if $\lim_{n\to\infty} n^{-1} \sum_{i=1}^n |a_i|^q = 0$. We can show, as in the above proof, that $w_q^0 = D^{1/q} c_0[l^q]$.

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