CRITERIA FOR IMBEDDINGS OF SOBOLEV-POINCARÉ TYPE

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§1. Introduction

Our aim in this paper is to give geometrical characterizations of domains which support Sobolev-Poincaré type imbeddings. The classical Sobolev-Poincaré imbeddings for a "nice" bounded domain $\Omega \subset \mathbb{R}^n$ depend on whether the exponent p is less than, equal to, or greater than n (throughout this paper, $n \geq 2$). In the case $1 \leq p < n$, we get the Sobolev-Poincaré inequality

$$\left(\int_\Omega |u - u_\Omega|^{pn/(n-p)}\, dx\right)^{(n-p)/pn} \leq C \left(\int_\Omega |\nabla u|^p\, dx\right)^{1/p} \tag{1.1}$$

whenever u is smooth, $u_\Omega = |\Omega|^{-1} \int_\Omega u\, dx$ and $\Omega \subset \mathbb{R}^n$ is bounded and satisfies a uniform interior cone condition; by density of smooth functions (1.1) then holds for all functions in the Sobolev space $W^{1,p}(\Omega)$ consisting of all functions in $L^p(\Omega)$ whose distributional gradients belong to $L^p(\Omega)$. For $1 < p < n$, inequality (1.1) was proved by Sobolev ([So1], [So2]); for $p = 1$, it is due to Gagliardo [G] and Nirenberg [N] (also see [M1]).

When $p = n$, it is well known that the simple limiting version (i.e. $pn/(n-p) = \infty$) of the above inequality is false. In its place, one gets (with all other assumptions unchanged) the following Trudinger inequality [Tr]:

$$\|u - u_\Omega\|_{\phi(L)(\Omega)} \leq C \left(\int_\Omega |\nabla u|^n\, dx\right)^{1/n} \tag{1.2}$$

Here $\phi(x) = \exp(x^{n/(n-1)}) - 1$, and $\|\cdot\|_{\phi(L)(\Omega)}$ is the corresponding Orlicz norm on Ω defined by

$$\|f\|_{\phi(L)(\Omega)} = \inf\{s > 0 \mid \int_\Omega \phi(|f(x)|/s)\, dx \leq 1\}.$$

In both (1.1) and (1.2), the exact value of u_Ω is not crucial. In fact, it is easy to see that we may replace it by the average of u over some fixed ball $B \subset\subset \Omega$ (see, for instance, [M2], [HaK] for the case of the Sobolev-Poincaré inequality). This simple variation will be quite useful to us.

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Finally when $p > n$, one gets the following Hölder imbedding inequality (for certain domains Ω, including all balls):

$$|u(x) - u(y)| \le C|x-y|^{1-n/p} \left(\int_\Omega |\nabla u|^p \, dx \right)^{1/p} . \tag{1.3}$$

For bounded domains, this imbedding is equivalent (see [KR, Theorem 3.1]) to the imbedding $W^{1,p}(\Omega) \subset C^{0,1-n/p}(\overline{\Omega})$ where

$$||u||_{C^\alpha(\overline{\Omega})} = \sup_{x\in\Omega} |u(x)| + \sup_{x,y\in\Omega, x\ne y} \frac{|u(x)-u(y)|}{|x-y|^\alpha}$$

and

$$||u||_{W^{1,p}(\Omega)} = ||u||_{L^p(\Omega)} + ||\nabla u||_{L^p(\Omega)}.$$

This latter imbedding is more natural for unbounded domains which we shall consider in Section 5.

We shall use the term *Sobolev imbedding (of order p)* to refer to any of the three types of imbeddings above. The classical cone or ball assumptions are not at all necessary for Sobolev imbeddings. For instance, Bojarski [B] (also see [HaK]) has verified the Sobolev-Poincaré inequality (1.1) for John domains, and the authors showed in [BK] that for domains with a "separation property" (including all finitely-connected plane domains), the validity of (1.1) implies the domain is John. In this paper, we continue this work by giving necessary and sufficient conditions for certain classes of domains (including all finitely-connected plane domains) to support Sobolev imbeddings of each order $p \ge n$.

To give the flavor of these results, let us state a theorem which completely characterizes those simply-connected plane domains that support Sobolev imbeddings. In Section 2, we shall define and discuss all the geometric conditions involved in this theorem. For now, let us only mention that if $0 < \alpha < \beta < 1$, and we denote by F_α the family of weak α-cigar domains in $\mathbb{R}^n$, then F_α contains all uniform domains and F_α is a proper subset of F_β. The weak carrot condition has previously been studied, although it is usually called a quasihyperbolic boundary condition. For us it is an endpoint case ($\alpha = 0$) of the weak α-carrot condition which we shall define in Section 2.

Theorem 1.5. *Let $1 \le p < \infty$ and let Ω be a bounded simply-connected planar domain. Then*

(i) *If $p < 2$, Ω supports the Sobolev-Poincaré inequality (1.1) if and only if Ω is a John domain.*
(ii) *If $p = 2$, Ω supports the Trudinger inequality (1.2) if and only if Ω is a weak carrot domain.*
(iii) *If $p > 2$, Ω supports the Hölder imbedding (1.3) if and only if Ω a weak α-cigar domain, where $\alpha = (p-2)/(p-1)$.*

Part (i) of this theorem is stated only for completeness: as already mentioned, necessity of the John condition is proved in [BK] and, sufficiency is due to Bojarski [B]. The sufficiency of the weak carrot condition (for general domains) in (ii) was shown by Smith and Stegenga [SS], but the necessity is new. Both the necessity and sufficiency in (iii) are new (for some partial results in both directions, see [KR]). We note that, as with sufficiency in (i) and (ii), sufficiency in (iii) is valid without the assumption that the domain is simply-connected.

We claim that none of the "only if" parts of Theorem 1.5 are valid for general open sets Ω: we need some additional assumption such as "simply connected and planar" or the slice property defined in Section 3. This claim was justified for (i) in [BK], to which we refer the reader. All the counterexamples given there are also counterexamples for (ii) and (iii).

The task of proving, when $p \geq n$ ($n = 2$ above), that a Sobolev imbedding for certain types of domains implies a certain geometric condition, bears some similarity to the task for $p < n$ considered in [BK] (for instance, we shall again employ quasiconformal mappings), but there are also significant differences. For instance, the above theorem indicates that, while the class of planar simply-connected domains satisfying a Sobolev-Poincaré inequality of exponent p is the same for all $p < 2$, the corresponding class is different for every value of $p \geq 2$. Furthermore, the $p \geq 2$ conditions, which all involve a bound on the length of the path with respect to some metric, are weak versions of the conditions for a John or uniform domain, giving "average" rather than pointwise bounds for the distance from a certain path to the boundary. Because of this, the proof of the necessity of such conditions is more intricate, involving cutting the domain into many disjoint pieces of suitable geometry (the so-called "slice property" defined in Section 3) rather than simply separating the domain into two pieces using a ball (the so-called "separation property" of [BK]).

When $p > 2$, classical results on a disk allow us to get *local* Hölder continuity with the correct exponent for an arbitrary domain. If the domain is a weak t-cigar domain, for $t = 1 - 2/p$, we can then deduce the required global Hölder continuity by a result of Gehring and Martio [GM1, Theorem 2.2]). However, t is strictly less than the index α in (iii) above, and so this simple argument does not give the sharp necessary condition. It may seem a little strange that on all weak α-cigar domains, Sobolev functions which satisfy a certain local Hölder estimate must satisfy the corresponding global Hölder estimate, while this is not true of all functions which are locally Hölder with the corresponding exponent. The explanation lies in the fact that a Sobolev condition is much stronger than the local Hölder condition which it implies. For example, [GM1] defines a function u on a general domain $\Omega \subset \mathbb{R}^n$ which is always locally Hölder of order t but, if Ω is not a weak t-cigar domain, u is not globally Hölder; however the gradient of u is far too large near the boundary to allow u to be in the corresponding Sobolev class. Similar comments apply to the main higher-dimensional result in Section 4.

In Section 2, we shall investigate the various carrot and cigar properties we have defined. In Section 3, we discuss domains in $\mathbb{R}^n$ with a slice property (these include all planar simply-connected domains); Theorem 1.5 is then a special case of the imbedding results that we prove for such domains in Section 4. In Section 5, we consider variations of the earlier imbedding results.

§2. Cigars and carrots

Let us first introduce some notation that we shall use throughout this paper. We denote the Euclidean distance between x and y by $|x - y|$. We shall also use δ to refer to Euclidean distance when sets are involved (in which case distances are defined as infima of point distances). Thus $\delta(x, A)$ is the distance between a point x and the set A. As we are often concerned with distance to the boundary, we also write $\delta_\Omega(x)$, or even $\delta(x)$ if the domain is understood, in place of $\delta(x,\partial\Omega)$. $B(x,r)$ denotes the Euclidean ball with center x and radius r. If $X,Y\subset\Omega$, we shall write $\delta_\Omega(X,Y)$ for the internal distance between X and Y, i.e. the infimum of the length of rectifiable curves lying in Ω which begin in X and end in Y (note that this notation is consistent with the notation $\delta(x)$). Whenever $B=B(x,r)$ is a ball and $t>0$, tB denotes the concentric ball $B(x,tr)$. Ω will always denote a proper subdomain of $\mathbb{R}^n$.

We say that Ω is a *John domain* if it is bounded, equipped with a distinguished point $x_0\in\Omega$, and if it satisfies the following "carrot" condition: there exists a constant $C>1$ such that for all $x\in\Omega$, there is a path $\gamma=\gamma_x:[0,l]\to\Omega$ parametrized by arclength such that $\gamma(0)=x$, $\gamma(l)=x_0$, and $\delta(\gamma(t))\ge t/C$. We call such a path a *John path for* x. We say that Ω is a *uniform domain* if it satisfies the following "cigar" condition: there exists a constant $C>1$ such that for all $x,y\in\Omega$, there is a path $\gamma=\gamma_x:[0,l]\to\Omega$ parametrized by arclength such that $\gamma(0)=x$, $\gamma(l)=y$, $\delta(\gamma(t))\ge\min\{t,l-t\}/C$, and $l\le C|x-y|$.

Suppose $0<\alpha<1, 0<\beta\le\alpha$. A pair of points $x,y\in\Omega$ is said to satisfy a *weak* (α,β,C)-*cigar condition* if there exists a path $\gamma:[0,1]\to\Omega$ such that $\gamma(0)=x$, $\gamma(1)=y$, and

$$\int_\gamma \delta(z)^{\alpha-1}|dz|\le C|x-y|^\beta.$$

We say $x,y\in\Omega$ satisfy a *weak* $(0,0,C)$-*cigar condition* if there exists a path $\gamma:[0,1]\to\Omega$ such that $\gamma(0)=x$, $\gamma(1)=y$, and

$$\int_\gamma \delta(z)^{-1}|dz|\le C\log(1+|x-y|/\min\{\delta(x),\delta(y)\}). \tag{2.1}$$

We say that Ω is a *weak* (α,β)-*cigar domain* if for some $C>0$, all $x,y\in\Omega$ satisfy a weak (α,β,C)-cigar condition. We say that Ω is a *weak* α-*carrot domain* with respect to $x_0\in\Omega$ if there exists $C>0$ such that for every $x\in\Omega$, there exists a path $\gamma=\gamma_x:[0,1]\to\Omega$ such that $\gamma(0)=x_0$, $\gamma(1)=x$, and

$$\int_\gamma \delta(z)^{\alpha-1}|dz|<C \qquad \text{if } 0<\alpha<1,$$
$$\int_\gamma \delta(z)^{-1}|dz|<C\,(1+\log(1/\delta(x))) \qquad \text{if } \alpha=0.$$

In certain cases, we shall omit some of these parameters. Specifically, a weak α-cigar domain is a weak (α,α)-cigar domain, a weak cigar domain is a weak $(0,0)$-cigar domain, and a weak carrot domain is a weak 0-carrot domain.

It is easy to see that the weak α-carrot condition forces a domain to be bounded and is equivalent to assuming a weak (α,β,C')-cigar condition for all pairs $\{(x,x_0)\mid x\in\Omega\}$, where β is any fixed number in the interval $(0,\alpha]$ (or $\beta=\alpha=0$). To see this we need only note that the weak cigar condition is trivially satisfied within the ball $B(x_0,\delta(x_0))$.

Examples of weak (α,β)-cigar domains include all uniform domains. In fact, the class of weak $(0,0)$-cigar domains coincides with the class of uniform domains, as follows from [J] (in the simply-connected planar case) and [GO] (in general). It is shown in [GM1] that when $\alpha > 0$, the class of weak (α,β)-cigar domains is strictly larger than the class of uniform domains. Weak (α,β)-cigar domains are called $\mathrm{Lip}_{\alpha,\beta}$ extension domains by Lappalainen [L], who proves that they are precisely the class of domains Ω for which all functions on Ω which are locally Lipschitz of order α must be globally Lipschitz of order β (the term "Lipschitz" means the same as "Hölder" for us; "locally Lipschitz" means Lipschitz with a uniform constant on all balls with double dilates contained in Ω); it follows that locally Lipschitz functions on Ω can even be extended to functions Lipschitz on all of $\mathbb{R}^n$ (with comparable norm).

The weak carrot condition is more commonly known in the literature as a quasihyperbolic boundary condition; it clearly implies that the domain is bounded. It is easy to verify that John domains are weak α-carrot domains for all $0 \le \alpha < 1$, but it is not hard to produce an example of a weak carrot domain which is not John (see [GM1] and Section 2; see also [K] for an exposition of research related to such domains).

We now give a slightly different characterization of weak cigar domains, which sheds some light on the terminology (intuitively, to change a "weak cigar"—the envelope around the curve linking a pair of points—into some sort of "cigar" we must slice it up and rearrange the pieces). This lemma, although elementary, is useful when investigating the relationship between the various weak cigar conditions.

Lemma 2.2. *Suppose Ω is a bounded domain and that $0 < \alpha < 1$, $0 < \beta \le \alpha$. Then Ω is a weak (α,β)-cigar domain if and only if there exists a constant $C = C(\alpha,\beta,n)$ such that for every distinct pair of points $x, y \in \Omega$ there is a path $\gamma_{x,y} : [0,l] \to \Omega$ connecting them which is parametrized by arclength and satisfies the following conditions:*

(i) *$l \le C|x-y|^{\beta/\alpha}$.*

(ii) *For all $\gamma = \gamma_{x,y}$, $\int_0^{l'} r(s)^{\alpha-1} ds \le C|x-y|^\beta$, where $r : [0,l] \to \mathbb{R}$ is the non-decreasing rearrangement of $t \mapsto \delta(\gamma(t))$ and $l' = \min\{l, |x-y|^{\beta/\alpha}\}$.*

Furthermore, (ii) implies that $r(t) \ge (Ct|x-y|^{-\beta})^{1/(1-\alpha)}$ for all $t \le l'$ and, for any $\alpha' < \alpha < 1$, (ii) is implied by the condition

$$r(t) \ge C' \left(t|x-y|^{-\beta\alpha'/\alpha}\right)^{1/(1-\alpha')} \qquad \forall\, t \le l', \tag{2.3}$$

where $C' = C'(\alpha,\beta,n,C)$ is a positive constant.

Proof. The weak cigar condition is clearly equivalent to $\int_0^l r(s)^{\alpha-1} ds \le C|x-y|^\beta$, where r is as in (ii). It is then follows easily that (i) and (ii) imply the weak cigar condition, and that the weak cigar condition implies (ii).

We next show that the weak cigar condition implies (i). We fix distinct points $x, y \in \Omega$ and write $\gamma = \gamma_{x,y}$. We may assume $|x-y| > \delta(x)/2$, since otherwise the line segment joining x and y trivially satisfies (i) and (ii). For any real numbers satisfying $0 < 2a < b$, we have $\int_a^b t^{\alpha-1} dt \ge cb^\alpha$, where $c = c(\alpha)$. It follows that $\delta(\gamma(t))$ is bounded above by a constant times $|x-y|^{\beta/\alpha}$. Thus the weak cigar condition implies that the length of γ is at most a constant times $|x-y|^{\beta(1-\alpha)/\alpha}|x-y|^\beta$ from which (i) follows immediately.

Since $r(s) \le r(t)$ if $0 < s \le t$, (ii) implies that

$$t(r(t))^{\alpha-1} \le \int_0^t r(s)^{\alpha-1} ds \le C|x-y|^\beta.$$

It follows that $r(t) \ge (Ct|x-y|^{-\beta})^{1/(1-\alpha)}$, as required. Conversely if (2.3) holds, we recover (ii) immediately by integration. □

Let us denote by $\mathrm{Cig}(\alpha,\beta)$ the class of all bounded weak (α,β)-cigar domains in $\mathbb{R}^n$ (here and later, pairs such as α, β are implicitly assumed to be numbers for which these concepts have been defined, i.e. either $\alpha = \beta = 0$ or $0 < \beta \le \alpha < 1$). In the following proof, and all subsequent ones, we write $P \lesssim Q$ if the quantity P is less than a constant times Q, where the exact value of the constant is of no importance to us (which here means that it depends only on α, α', β, β' and n). We also write $P \approx Q$ if $P \lesssim Q \lesssim P$.

Proposition 2.4. $\mathrm{Cig}(\alpha,\beta) \subseteq \mathrm{Cig}(\alpha',\beta')$ *if and only if* $\alpha \le \alpha'$ *and* $\beta'\alpha \le \beta\alpha'$*.*

Proof. If $\alpha = 0$ (and so $\beta = 0$), $\mathrm{Cig}(\alpha,\beta)$ is the class of uniform domains (see [GO]), and it readily follows that any such domain satisfies all weak (α,β)-cigar conditions. Since the inequality $\beta'\alpha \le \beta\alpha'$ is also trivially true, we are done with the case $\alpha = 0$.

Suppose therefore that $\alpha > 0$. Then $\mathrm{Cig}(\alpha,\beta) \supset \mathrm{Cig}(\alpha,\alpha)$ and, since $\mathrm{Cig}(\alpha,\alpha)$ already includes domains which are not uniform (see [GM1] and [L]), we may assume $\alpha' > 0$ also. Lappalainen [L, Section 6] shows that $\mathrm{Cig}(\alpha',\alpha')$ is a proper subset of $\mathrm{Cig}(\alpha,\alpha)$ if $\alpha' < \alpha$. In fact, he gives an explicit example of a domain D^*, which is bounded and in $\mathrm{Cig}(\alpha,\alpha)$ but not in $\mathrm{Cig}(\alpha',\alpha')$. If one examines the proof, it is clear that D^* is not in $\mathrm{Cig}(\alpha',\beta')$ for any $0 < \beta' \le \alpha'$. Thus $\mathrm{Cig}(\alpha,\beta) \subseteq \mathrm{Cig}(\alpha',\beta')$ implies $\alpha \le \alpha'$.

Suppose from now on that $\alpha \le \alpha'$. If $\beta'\alpha \le \beta\alpha'$ and $\Omega \in \mathrm{Cig}(\alpha,\beta)$, then by Hölder's inequality we get

$$\begin{aligned}\int_0^{l'} r(s)^{\alpha'-1} ds &\le \left(\int_0^{l'} r(s)^{\alpha-1} ds\right)^{(1-\alpha')/(1-\alpha)} \left(\int_0^{l'} ds\right)^{(\alpha'-\alpha)/(1-\alpha)} \\ &\lesssim |x-y|^{\beta(1-\alpha')/(1-\alpha)} l'^{(\alpha'-\alpha)/(1-\alpha)}.\end{aligned}$$

Now $l' \le |x-y|^{\beta/\alpha}$ and

$$\frac{\beta(1-\alpha')}{(1-\alpha)} + \frac{\beta(\alpha'-\alpha)}{\alpha(1-\alpha)} = \frac{\beta\alpha'}{\alpha} \ge \beta'.$$

The required containment follows immediately.

To construct examples to show that the inequality $\beta'\alpha \le \beta\alpha'$ is necessary for containment, let us fix $t > 1$ and use the notation $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$ as before. Suppose $t > 1$ is fixed but arbitrary. For all $0 < r < 1$, define

$$\Omega_r = B(0,1) \setminus \{x \in \mathbb{R}^n : |x_1| > 1-r, |x'| < r^t/4\}.$$

It is not hard to show that Ω_r is uniformly (in r) a weak (α,β)-cigar domain if and only if $0<\alpha<1$, $\beta\le\alpha/t$ (hint: in the planar case, consider the points $x=(1-r/2,r^t/2)$ and $y=(1-r/2,-r^t/2)$). Taking $t=\beta/\alpha$, Ω_r satisfies a uniform weak (α,β)-cigar condition but does not satisfy a uniform weak (α',β')-cigar condition whenever $\alpha\le\alpha'<1$ and $\beta'>\beta\alpha'/\alpha$. As is often the case in analysis, this counterexample for uniform containment easily leads to a counterexample for containment—one simply considers a ball with an appropriate sequence of notches removed. We leave the details to the reader. □

Whenever $0\le\alpha<1$, let us denote by $\mathrm{Car}(\alpha)$ the class of all bounded weak α-carrot domains in $\mathbb{R}^n$.

Proposition 2.5. $\mathrm{Car}(\alpha)\subseteq\mathrm{Car}(\alpha')$ *if and only if* $\alpha\le\alpha'$.

Proof. To see that the spaces are all distinct, simply consider the domain $\{x\in\mathbb{R}^n : 0<x_1<1, |x'|<x_1^t\}$, where $t>1$. This domain is never in $\mathrm{Car}(1)$ and lies in $\mathrm{Car}(\alpha)$ exactly when $\alpha t<1$.

It is left to prove that $\mathrm{Car}(\alpha)\subseteq\mathrm{Car}(\alpha')$ if $\alpha<\alpha'$. If $0<\alpha$, Hölder's inequality alone implies the required containment. Suppose therefore that $\alpha=0$. Let us fix a point $x\ne x_0$ and let γ be the carrot joining them. Suppose for the moment that the initial segments of γ satisfy a uniform weak carrot condition, i.e. that

$$\int_{\gamma|_{[0,t]}}\delta(z)^{-1}|dz|\le 2C\log(1+1/\delta(\gamma(t))) \tag{2.6}$$

for all $0<t<1$. Letting $\gamma^k=\{\gamma(t)\,|\,0\le t\le 1, 2^{-k}\le\delta(y)<2^{-k+1}\}$, we get from (2.6) that $\int_{\gamma^k}|dz|\lesssim 2^{-k}(k-k_0)$ for all $k>k_0=\log_2\operatorname{diam}(\Omega)$ (and γ^k is empty if $k\le k_0$). Therefore

$$\int_\gamma\delta(z)^{\alpha'-1}|dz|\lesssim\sum_{k=k_0+1}^{\infty}2^{-k\alpha'}(k-k_0)=C',$$

as required.

We assumed above that γ satisfied (2.6). This does not have to true in general but, given a domain satisfying the weak carrot condition, we can always construct such nicer carrots. To see this, note first that it is certainly true if $\delta(\gamma(t))<2\delta(x)$, so suppose $\delta(y)\ge2\delta(\gamma(x))$ for some $y=\gamma(t)$. Letting $y_0=\gamma(t_0)$, where t_0 is the largest value of t for which $\delta(\gamma(t))\ge2\delta(x)$, we can replace the initial part of the γ with γ_{y_0}, the carrot for y_0 (adjusting the parametrization in the obvious way). Repeating this process a finite number of times, we get a path γ' whose quasihyperbolic length is less than or equal to that of γ and such that (2.6) is true. □

§3. The slice property

Before we define the slice property in general, it is instructive to consider a simple domain with a slice property. We choose to use rather different slices than we shall use when considering more general cases later to illustrate how varied the choice of valid slices can be.

Let B be the open unit ball in $\mathbb{R}^n$ and, if $x \in B$, we write $x = (x_1, x')$ where $x_1 \in \mathbb{R}$ and $x' \in \mathbb{R}^{n-1}$. Let $\gamma = \gamma_x : [0,1] \to B$ be the path connecting the origin and a fixed point $x \in B$ defined by $\gamma(t) = tx$. We also define the "slices"

$$S_i = \{y \in B \,:\, |y'| + a_{i-1} < y_1 < |y'| + a_i\},$$

where $a_0 = -\epsilon$, $a_i = 1 - (1-\epsilon)^i$ for $i > 0$, and $0 < \epsilon < 1$ is fixed. Let γ_i be the intersection of $\gamma([0,1])$ and the closure of S_i, and let us call the distance between the two components of $B \setminus S_i$ the "width" of S_i. Clearly the diameter and width of S_i are comparable. All slices S_i, except those containing 0 and x, split $\gamma([0,1])$ into three pieces: γ_i, whose distance to the boundary is comparable with the diameter of S_i, and two "tails" which lie in different components of $B \setminus S_i$.

As we shall see, simply-connected planar domains Ω have a similar property. We can associate to any point x in Ω a path $\gamma : [0,1] \to \Omega$ linking it with a fixed point $x_0 \in \Omega$, together with a finite number of bounded slices S_i, most of which split $\gamma([0,1])$ into three pieces as above, and have some other nice properties like those of the slices for a ball. It is this slice property that makes possible the proof of results like Theorem 1.5.

It is convenient to denote the image of a path by the same symbol as the path itself, allowing us to write for instance $\gamma \subset G$ if the image of a path γ lies in G. The quasihyperbolic length of a rectifiable path $\gamma \subset G$, is defined by

$$k(\gamma) \equiv k_G(\gamma) = \int_\gamma \delta(x)^{-1} ds.$$

The quasihyperbolic distance between $x, y \in G$, $k_G(x,y)$, is the infimum of $k_G(\gamma)$ as γ ranges over all rectifiable paths in G that link x and y. Whenever S is a closed set, and γ is a path, $k_G(\gamma; S)$ denotes the sum of the quasihyperbolic lengths of the path segments of γ that are contained in S.

The definition of a slice property is designed to allow us to prove the necessity of cigar and carrot type geometric conditions for Sobolev imbeddings. The definition itself is somewhat complicated, but we shall prove that quasiconformal images of uniform domains (in particular,all simply-connected planar domains) have a slice property (the new path will simply be the quasiconformal image of the cigar core; we need to be more careful with the slices as the image of a slice may be unbounded whereas a slice must be bounded). After reading the following definition, the reader is invited to verify, using the path γ and slices S_i given previously, that the unit ball B (or any Euclidean ball) has a slice property with $C = C(n, \epsilon)$ (the two slices with largest indices might have to be welded together if x is too near a slice boundary).[1]

Definition 3.1. *Suppose $G \subset \mathbb{R}^n$ is a domain with a distinguished point x_0 and $C > 1$. We say G has the C-*slice property *with respect to x_0 if, for each $x \in G$, there is a path $\gamma : [0,1] \to G$, $\gamma(0) = x_0$, $\gamma(1) = x$, and a pairwise disjoint collection of open subsets $\{S_i\}_{i=0}^j$, $j \geq 0$, of G such that:*

(a) *$x_0 \in S_0$, $x \in S_j$, and x_0 and x are in different components of $G \setminus \overline{S_i}$ for all $0 < i < j$.*

[1] We would like to thank Chris Bishop for pointing out a defect in an earlier version of our slice definition.

(b) *If* $F \subset\subset G$ *is a curve containing both* x *and* x_0, *and* $0 < i < j$, *then* $\operatorname{diam}(S_i) \le C \operatorname{len}(F \cap S_i)$.

(c) *For* $0 \le t \le 1$, $B(\gamma(t), C^{-1}\delta(\gamma(t))) \subset \bigcup_{i=0}^{j} \overline{S_i}$.

(d) *If* $0 \le i \le j$, *then* $\operatorname{diam}(S_i) \le C\delta(x)$ *for all* $x \in \gamma_i \equiv \gamma([0,1]) \cap S_i$. *Also, there exists* $x_i \in \gamma_i$ *such that* x_0 *is as previously defined,* $x_j = x$, *and* $B(x_i, C^{-1}\delta(x_i)) \subset S_i$.

We say that G *has a slice property if it has the* C*-slice property for some* C.

Condition (b) says that S_i is about as thick as its diameter. By replacing γ with an appropriate polygonal path if necessary, we obtain from (c) and (d) that $k_G(\gamma; \overline{S_i}) \le C$ for all i, and so $j + 1 \ge k_G(x_0, x)/C$.

We say that a mapping f is a K-quasiconformal mapping from $G \subset \mathbb{R}^n$ onto $G' \subset \mathbb{R}^n$ if f is a homeomorphism that belongs to the local Sobolev class $W^{1,n}_{\mathrm{loc}}(G)$, and $|Df(x)|^n \le K J_f(x)$ for almost every $x \in G$, where $|Df|$ is the operator norm of the formal derivative Df of f, J_f is the Jacobian determinant of Df, and $K \ge 1$ is a fixed constant (referred to as the *dilatation* of f). The class of 1-quasiconformal planar mappings is the class of conformal mappings.

We next state a theorem which tells us that there are many domains satisfying a slice property uniformly with respect to all of their points. This theorem should be compared with Lemma 3.3 in [BK], the corresponding result for the separation property. A domain which has either property has no "flat" tentacles (e.g. the product of a cusp and an interval) and there are no small pieces of boundary floating around.

Theorem 3.2. *If* f *is a* K*-quasiconformal mapping from a uniform domain* $G \subset \mathbb{R}^n$ *onto* Ω, *then* Ω *has the* C'*-slice property with respect to all of its points, for some constant* $C' = C'(C, n, K, G)$.

Let us note the few basic properties of quasiconformal mappings that we shall use. Suppose that G, Ω are domains in $\mathbb{R}^n$, and that f is a K-quasiconformal mapping from G onto Ω. Then f^{-1} is K'-quasiconformal, where $K' = K'(K, n)$. If $B = B(x, r) \subset G$ with $\delta(B, \partial G) = Cr$ for some $C > 0$, then for any $y \in fB$, we have $\delta_\Omega(y)/C' \le \operatorname{diam} fB \le C'\delta_\Omega(y)$ and $B(f(x), \delta_\Omega(f(x))/C') \subset fB$, where $C' = C'(C, K, n)$; in particular the inradius and diameter of fB are comparable (briefly, quasiconformal mappings send Whitney balls to Whitney type objects). Let us define the *conformal capacity*, $\operatorname{cap}(E, F; \Omega)$, of the disjoint compact subsets $E, F \subset \overline{\Omega}$ relative to Ω to be the infimum of $\int_\Omega |\nabla u|^n$, as u ranges over the class of functions which equal 1 on E and 0 on F, are continuous in $\Omega \cup E \cup F$, and are locally Lipschitz in Ω. Quasiconformal mappings quasipreserve conformal capacity (i.e. they preserve it up to a multiplicative constant dependent on the dilatation and the dimension). For details of these and other properties of quasiconformal mappings, we refer the reader to [V1], [V2], [GP], [GO]. The concept of variational p-capacity, which in the case $p = n$ gives the above-mentioned conformal capacity, is naturally associated with Sobolev space imbedding results; for a detailed exposition, see Maz'ya [M2].

As a first step towards the proof of Theorem 3.2, we have the following lemma.

Lemma 3.3.

(a) *If* Ω *is a uniform domain, then* Ω *has a* C*-slice property with respect to all* $x_0 \in \Omega$, *for some* C *dependent only on the uniformity constant of* Ω.

(b) *If* Ω *is a John domain with distinguished point* x_0, *then* Ω *has a* C*-slice property with respect to* x_0, *for some* C *dependent only on the John constant of* Ω.

Proof. Suppose that Ω is uniform and that $x_0, x \in \Omega$. Since the hypotheses and conclusions in (a) are symmetric with respect to the roles of x, x_0, we may as well assume that $\delta(x) \le \delta(x_0)$. Let $\gamma : [0, l] \to \Omega$ be the cigar path, parametrized by arclength, from x_0 to x. Then γ will also be our slice path, so let us now define the slices S_i. We write $\gamma' = \gamma([0, l/2])$ and $\gamma'' = \gamma([l/2, l])$. We define A_0 and B_0 to be the empty set and, for $i \ge 1$, $A_i = B(x_0, t^{i-1}\delta(x_0)/10)$ and $B_i = B(x, t^{i-1}\delta(x)/10)$. Here $t \in (1, 11/10)$ is a fixed number, so close to 1 that distance to the boundary is approximately constant on any non-empty set of the form $\gamma'_i \equiv \gamma' \cap (A_i \setminus A_{i-1})$ or $\gamma''_i \equiv \gamma'' \cap (B_i \setminus B_{i-1})$; this is possible by the uniformity assumption.

We may assume that $|x - x_0| \ge \delta(x_0)/2$, since otherwise the single slice $S_0 = B(x_0, \delta(x_0)/2)$, together with a line segment as the slice path, satisfy a slice property. For each $i \ge 1$, there is a positive integer $g(i)$ such that A_i intersects B_j precisely when $j \ge g(i)$. Writing $r(B)$ for the radius of any ball B, it is clear that $r(A_1)/r(B_{g(1)}) < 1/4$, while $r(A_i)/r(B_{g(i)}) < 1$ for all sufficiently large i. Also, $r(A_{i+1})/r(A_i) = t < 5/4$ and $r(B_{g(i+1)})/r(B_{g(i)}) \ge t^2$ whenever $r(A_i)/r(B_{g(i)}) \le 1$. Thus there exists an integer $i_0 > 2$ for which $r(A_{i_0})/r(B_{g(i_0)}) \in [t^{-3}, t^3]$. Let $j = i_0 + g(i_0) - 4$ and define the slices

$$
\begin{aligned}
S_i &= (A_{i+1} \setminus A_i) \cap \Omega, && 0 \le i \le i_0 - 3, \\
S_{j-i} &= (B_{i+1} \setminus B_i) \cap \Omega, && 0 \le i \le g(i_0) - 3 \\
S_{i_0-2} &= \big((A_{i_0} \cup B_{g(i_0)+1}) \setminus (A_{i_0-2} \cup B_{g(i_0)-2})\big) \cap \Omega
\end{aligned}
$$

Although not needed here, it will be useful for the proof of Theorem 3.2 if we at this point define a *supercomponent* of $\Omega \setminus \overline{S_i}$ (when $0 < i < j$) to be the intersection with Ω of a component of $\mathbb{R}^n \setminus \overline{S_i}$; we also say that two points in Ω are *superseparated* by S_i if they are in different supercomponents of $\Omega \setminus \overline{S_i}$. Note that a supercomponent of $\Omega \setminus \overline{S_i}$ is a union of (possibly many) components of $\Omega \setminus \overline{S_i}$. By construction, there are always either two or three supercomponents: an *inner* one that contains x_0, the *outer* one that contains x and perhaps one other (which is only possible if $i = i_0 - 2$).

It is easy to verify that γ and $\{S_i\}_{i=0}^j$ satisfy the conditions of the slice property—in fact we can strengthen part (b) to say that any curve containing two superseparated points must have a subcurve contained in S_i of length greater that $C^{-1} \operatorname{diam}(S_i)$ (a fact that will be useful in the proof of Theorem 3.2). Note that the gap of 2 in the subscripts of the balls used in the definition of S_{i_0-2} is designed to ensure that this central slice has at least a certain thickness.

The proof of part (b) is similar but easier: one simply considers a single set of annuli centered at x. We leave the details to the reader. $\square$

We are now ready to prove Theorem 3.2. We use the notation of Definition 3.1 without comment for the domain G, and denote many of the corresponding concepts in Ω (which are defined where necessary) with primes.

Proof of Theorem 3.2. Given $y, y_0 \in \Omega$, let γ and $\{S_i\}_{i=0}^j$ be the path and slices for $x \equiv f^{-1}(y)$ with respect to $x_0 \equiv f^{-1}(y)$. The slice path for y with respect to y_0 will be $\gamma' = f \circ \gamma$, but we must still define suitable slices (morally the slices will also be the f-images of the original slices, but we must modify them to satisfy (d)).

Let $x_i \in \gamma_i$ be as in (d). For $0 \le i \le j$, we write $y_i = f(x_i)$ (these will be the points for Ω whose existence is hypothesised in (d)), and define S_i' to be the component of $f(S_i) \cap B'$ containing y_i, where $B' = B(y_i, C_1\delta_\Omega(y_i))$ and $C_1 > 1$ is a constant to be chosen later. We claim that if $C_1 < C'$ are both suitably large, then γ' and $\{S_i'\}$ satisfy (a)–(d) of Definition 3.1 with constant C', for the point y with respect to y_0.

Condition (d) for G implies that γ_i is contained in a bounded number of Whitney balls. Since f sends each Whitney ball to a Whitney type object, an elementary chaining argument shows that the distances to the boundary of any two points in $\gamma_i' \equiv \gamma'([0,1]) \cap S_i'$ are comparable (with comparability constant dependent only on C, K, and n). Obviously $\operatorname{diam}(S_i') \le 2C_1\delta_\Omega(y_i)$, and so the first part of (d) for Ω follows with a constant C' somewhat larger than $2C_1$. The second part is trivially true for similar reasons.

Condition (c) is also easy: since γ_i lies in a bounded number of Whitney balls, the quasiconformality of f guarantees that γ_i' lies in $B(y_i, C'\delta_\Omega(y_i))$ for suitably large C'. As f^{-1} sends Whitney balls to Whitney type objects, we also have $B(\gamma'(t), C'^{-1}\delta_\Omega(\gamma'(t))) \subset \bigcup_{i=0}^{j} \overline{S_i'}$.

We next prove the following stronger version of (a) (as it requires no extra effort): if $u', v' \in \Omega \setminus \overline{S_i'}$ are the images of a pair of points superseparated by S_i (as defined in the proof of Lemma 3.3), then they lie in different components of $\Omega \setminus \overline{S_i'}$. Suppose, for the purposes of contradiction, that this is false. By (b) for G (and the proof of Lemma 3.3), there exists a closed curve $F \subset S_i \setminus f^{-1}B'$ with $\operatorname{len}(F) \approx \delta_G(x_i)$. Let $E = B(x_i, \delta_G(x_i)/4C)$, $E' = fE$ and $F' = fF$. Then $E \subset S_i$ and E' is contained in a ball B_0' with center y_i and radius $c\delta_\Omega(y_i)$, where $c = c(C, n, K) > 0$ is fixed (and so $E' \subset S_i'$ if we choose $C_1 > c$). The diameter of E is comparable with $\delta_G(x_i)$ and the distance between E and F is at most a constant times $\delta_G(x_i)$. For any such configuration in a uniform domain G, $\operatorname{cap}(E, F; G) \ge \epsilon = \epsilon(C, n, K) > 0$ (see, for example, [GM2]). This holds because of the extension property for Sobolev functions due to Jones [J] and the corresponding estimate in $\mathbb{R}^n$.

On the other hand, F' lies outside B'. Writing w_{n-1} for the surface measure of the boundary of the unit ball, a well-known capacity estimate (see e.g. [V1, Example 7.5]) gives

$$\operatorname{cap}(E', F'; \Omega) \le \operatorname{cap}(B_0', \partial B'; \mathbb{R}^n) \lesssim \frac{w_{n-1}}{(\log(C_1/c))^{n-1}}.$$

By the quasiconformality of f, we conclude (simply by performing a change of variables) that

$$\operatorname{cap}(E, F; G) \le \frac{K w_{n-1}}{(\log(C_1/c))^{n-1}}$$

where K is the dilatation of f. Comparing the last pair of inequalities with the lower bound for $\operatorname{cap}(E, F; G)$ obtained previously, we get a bound for C_1 in terms of C, n, and K. Thus if we choose C_1 larger than this value, the claim follows.

Finally we must prove (b). Suppose that $F' \subset \Omega$ is a closed curve containing y and y_0 (or more generally, any pair of points in $\Omega \setminus \overline{S_i'}$ which are the images of points superseparated by S_i); also let $F = f^{-1}F'$. To establish (b), it suffices to show that $\operatorname{len}(F' \cap S_i) \gtrsim \delta_\Omega(y_i)$. We may assume that C_1 has been chosen so large that $S_i' \cap (1/2)B'$ separates images of points superseparated by S_i.

Let $g : [0,1] \to G$ be any path whose image is F. Suppose that there exists $\epsilon > 0$, $0 < t_1 < t_2 < 1$, such that $g(t)$ is in one supercomponent of $G \setminus \overline{S_i}$ for $t_1 - \epsilon \le t < t_1$, in a different supercomponent of $G \setminus \overline{S_i}$ for $t_2 < \epsilon \le t_2 + \epsilon$, and in $\overline{S_i}$ for $t_1 \le t \le t_2$ (there certainly exists at least one such pair t_1, t_2). At either end of the interval $[t_1, t_2]$ there may be subintervals where $g(t)$ is in ∂S_i but, because of the assumed separation property, $g(t)$ cannot go from the closure of one supercomponent to that of another without going through S_i. Therefore we let t_3 be the largest value of $t \in [t_1, t_2)$ such that $g(t)$ is in the closure of the same supercomponent of $G \setminus \overline{S_i}$ as contains $g(t_1)$, and let t_4 be the smallest value of $t \in (t_4, t_2]$ such that $g(t)$ is in the closure of the same supercomponent of $G \setminus \overline{S_i}$ as contains $g(t_2)$. Let us call the subcurve $g((t_3, t_4))$ a *passage through* S_i. The reader is advised to examine the simple geometry of the slices in Lemma 3.3 to convince him/herself that F has at least one passage through S_i, and that the diameter of all such passages is approximately $\delta(x_i)$.

Let A be any passage of F through S_i and let $A' = fA$. We first consider the case where $A' \subset S_i'$ (or equivalently, $A \subset f^{-1}B'$). As we already know, S_i' contains a ball $B(y_i, c_1\delta_\Omega(y_i))$, for some $c_1 = c_1(C, n, K) > 0$, so let us define new sets $E' = B(y_i, c_1\delta_\Omega(y_i)/2)$, and $E = f^{-1}E'$. If A' intersects $B(y_i, 3c_1\delta_\Omega(y_i)/4)$, then $\operatorname{len}(A') \gtrsim \delta_\Omega(y_i)$, so we may suppose that $\delta(E', A') \gtrsim \delta_\Omega(y_i)$ (in fact, since $\operatorname{diam}(S_i) \approx \delta_\Omega(y_i)$, we have $\delta(E', A') \approx \delta_\Omega(y_i)$). Since $\operatorname{diam}(E)$, $\operatorname{diam}(A)$, and $\delta(E, A)$ are all approximately equal to $\delta_G(x_i)$, we see that $\operatorname{cap}(E, A; G) \approx 1$ (for the lower bound, we use the uniformity of G as before). Thus $\operatorname{cap}(E', A'; \Omega) \approx 1$ and so $\operatorname{cap}(E', A'; \mathbb{R}^n) \gtrsim 1$. Since $\delta(E', A') \approx \delta_\Omega(y_i)$ and $\operatorname{diam}(E') \approx \delta_\Omega(y_i)$, we deduce that $\operatorname{len}(A') \ge \operatorname{diam}(A') \gtrsim \delta_\Omega(y_i)$, as required.

Suppose, on the other hand, that A' intersects $fS_i \setminus B'$. But A' also contains points in $(1/2)B'$ because of the assumed separation condition for $(1/2)B'$. By connectedness of A' we deduce that $\operatorname{len}(A') > C_1\delta_\Omega(y_i)/2$, as required. $\square$

§4. <u>Imbeddings</u>

Theorem 1.5 follows from the following more general result. Note that in all three parts of this theorem, sufficiency is valid without the assumption that the domain satisfies a slice property, and that necessity in the first two parts requires only a slice property with respect to the distinguished point $x_0 \in \Omega$. Note also that, by Theorem 3.2, this theorem applies to any domain quasiconformally equivalent to a uniform domain and, in particular to all finitely-connected planar domains, since they are conformally equivalent to a disk with a finite number of disks and points removed, and the domains of the latter type are clearly uniform.

Theorem 4.1. *Let $1 \le p < \infty$ and $C_0 > 1$ be fixed. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain that satisfies the C_0-slice property with respect to all of its points. Then*

- (i) *If $p < n$, Ω supports the Sobolev-Poincaré inequality (1.1) if and only if Ω is a John domain.*
- (ii) *If $p = n$, Ω supports the Trudinger inequality (1.2) if and only if Ω is a weak carrot domain.*
- (iii) *If $p > n$, Ω supports the Hölder imbedding (1.3) if and only if Ω a weak α-cigar domain, where $\alpha = (p - n)/(p - 1)$.*

As with Theorem 1.5, the new parts of this theorem consist of (iii) together with necessity in (ii). Part (i) is a combination of [B] and [BK], and sufficiency of the weak carrot condition (for general domains) in (ii) was shown is shown in [SS]. Also as in the planar case, sufficiency in each case is valid without the assumption that the domain satisfies a slice property.

Before proving Theorem 4.1, we record the following local estimate for future reference (for a proof see e.g. [BI, 1.7]). By $L^{1,p}(\Omega)$ we mean the space of locally integrable functions whose distributional gradients belong to $L^p(\Omega)$.

Lemma 4.2. *Let $p > n$ and suppose $u \in L^{1,p}(B)$, where $B \subset \mathbb{R}^n$ is an open ball. Then there is a constant $C = C(p, n) > 0$, such that for all $x, y \in B$,*

$$|u(x) - u(y)| \leq C|x - y|^{1-n/p} \|\nabla u\|_{L^p(B)}.$$

Proof of Theorem 4.1. We need to prove all of (iii) and necessity in (ii). Suppose that Ω has the C-slice property with respect to the distinguished point x_0, and that it supports the Trudinger inequality. We wish to show that Ω is a weak carrot domain with respect to x_0. Suppose therefore that $x \in \Omega$ and that γ and $\{S_i\}_{i=0}^{j}$ are associated with x by the slice property (we also freely use the other notation introduced in Definition 3.1). If $k_\Omega(x, x_0)$ is bounded, the weak carrot condition is trivially satisfied. Since $j + 1 \geq k_\Omega(x, x_0)/C$, we may therefore assume that j is at least 2.

We define functions ρ_i and u_i $(0 < i < j)$ on Ω as follows. Let us fix a point $\gamma(t_i) \in S_i$ for each i and write $\delta_i = \delta(\gamma(t_i))$. Writing C_1 for a constant to be fixed later, we define

$$u_i(y) = \frac{C_1}{\delta_i} \left[\inf_{A \in \mathcal{F}_{y,x_0}} \operatorname{len}(A \cap S_i) \right], \qquad y \in \Omega$$

where $\mathcal{F}_{y,x_0}$ is the set of all rectifiable curves in Ω containing y and x_0. It is clear that u_i is Lipschitz and that, by Definition 3.1(d), we have

$$\int_\Omega |\nabla u_i|^n \lesssim C_1^n.$$

By Definition 3.1(c), we see that $u_i(x) - u_i(x_0) \gtrsim C_1$.

Let $u(y) = \sum_{i=1}^{j-1} u_i(y)$. Then $u = 0$ throughout S_0 and hence on a fixed ball around x_0 with radius comparable with $\delta(x_0)$. We fix $C_1 \approx j^{-1/n}$ in such a way that $\int_\Omega |\nabla u|^n = 1$. Also

$$u(y) - u(0) \gtrsim (j - 1)C_1 \gtrsim j^{(n-1)/n}, \qquad \text{for all } y \in S_j.$$

Writing $\phi(y) = \exp(y^{n/(n-1)}) - 1$, we see that

$$\log \left(\int_\Omega \phi(u(y)/t) \right) \geq \log \left(\int_{S_j} \phi(u(y)/t) \right) \gtrsim \log(\delta(x)^n [e^{jt^{-n/(n-1)}} - 1]) \geq 0$$

whenever $t^{n/(n-1)} \lesssim j/\log(1+1/\delta(x)^n)$. Thus

$$\|u\|_{\phi(L)(\Omega)} \gtrsim [j/\log(1+\delta(x)^{-n})]^{(n-1)/n}$$

and the Trudinger inequality implies that $j/\log(1+\delta(x)^{-n})$ is bounded. It follows that j, and hence $k_\Omega(x,x_0)$, is bounded by some constant times $\log(1+\delta(x)^{-n})$, which is clearly equivalent to the desired inequality.

We next prove sufficiency for part (iii). If x,y lie in a ball $B \subset \Omega$, the required inequality follows immediately from Lemma 4.2, so we assume that this is not so. Let γ be a weak (α,β,C)-cigar joining a fixed but arbitrary pair $x,y \in \Omega$. We cover γ by the balls $B_{\gamma(t)} = B(\gamma(t),\delta(\gamma(t))/2)$, $0 \le t \le 1$. Note that the length of $\gamma \cap B_{\gamma(t)}$ is at least $\delta(\gamma(t))/2$ and that all points in $B_{\gamma(t)}$ are approximately the same distance from $\partial\Omega$. By compactness and the Besicovitch Covering Lemma (see [St2]), we can extract a subcollection $S = \{B_i\}_{i=1}^j$ such that S still covers γ but no point in Ω lies in more than $C = C(n)$ of the balls of S. We arrange the indices so that we can choose points $\{x_i\}_{i=0}^j$ for which $x_0 = x$, $x_j = y$, and $x_i \in B_i \cap B_{i+1}$ for $i = 1,\dots,j-1$. By the triangle inequality, Lemma 4.2, and Hölder's inequality, we get

$$\begin{aligned}
|u(x)-u(y)| &\le C\sum_{i=1}^{j} |x_i - x_{i-1}|^{1-n/p}\|\nabla u\|_{L^p(B_i)} \\
&\le C\left(\sum_{i=1}^{j} |x_i - x_{i-1}|^{\frac{p-n}{p-1}}\right)^{(p-1)/p}\left(\sum_{i=1}^{j}\|\nabla u\|^p_{L^p(B_i)}\right)^{1/p} \\
&\le C\left(\sum_{i=1}^{j}\int_{\gamma\cap B_i}\delta(z)^{\frac{1-n}{p-1}}\right)^{(p-1)/p}\|\nabla u\|_{L^p(\Omega)} \\
&\le C\left(\int_{\gamma}\delta(z)^{\frac{1-n}{p-1}}\right)^{(p-1)/p}\|\nabla u\|_{L^p(\Omega)} \\
&\le C|x-y|^{(p-n)/p}\|\nabla u\|_{L^p(\Omega)}
\end{aligned}$$

as required.

Finally suppose that Ω has the C-slice property with respect to all its points and that it supports the Hölder imbedding (1.3). We wish to show that Ω is a weak α-cigar domain (for $\alpha = (p-n)/(p-1)$). Suppose therefore that $x,y \in \Omega$ and that γ and S_i are associated with y by the slicing property with centre x. If $k_\Omega(\gamma)$ is bounded, then so is $\int_\gamma \delta(z)^{-s}|dz|$, and the weak cigar condition is trivially satisfied. We may therefore assume that $j \ge 2$.

For $1 \le i < j$, let

$$u_i(z) = \frac{C_i}{\delta_i}\left[\inf_{A\in\mathcal{F}_{z,x}} \operatorname{len}(A\cap S_i)\right], \qquad z\in\Omega$$

where $\mathcal{F}_{z,x}$ is as above, and C_i is a positive constant to be fixed later. Let r_i be the diameter of the ith slice and let $g_i = \|\nabla u_i\|_{L^p(\Omega)}$. Using Definition 3.1, we readily see that $u_i(y) - u_i(x)$ and $r_i^{1-n/p} g_i$ are comparable.

Defining $u = \sum_{i=1}^{j-1} u_i$, we see that $u(y) - u(x) \gtrsim \sum_{i=1}^{j-1} r_i^{1-n/p} g_i$. Since we have not yet specified C_i, we are free to choose g_i arbitrarily. Let $g_i = c r_i^{(1-n/p)/(p-1)}$ where c is chosen so that $\sum_{i=1}^{j-1} g_i^p = 1$. It follows that $\|\nabla u\|_{L^p(\Omega)} = 1$ and that

$$\begin{aligned} u(y) - u(x) &\gtrsim \left(\sum_{i=1}^{j-1} r_i^{(1-n/p)p/(p-1)} \right)^{(p-1)/p} \\ &= \left(\sum_{i=1}^{j-1} r_i^{(p-n)/(p-1)} \right)^{(p-1)/p} \\ &\approx \left(\int_\gamma \delta(z)^{\frac{1-n}{p-1}} \right)^{(p-1)/p} . \end{aligned}$$

Applying (1.3) to u, we see that γ is a weak (α, β)-cigar for the pair x, y. $\square$

In Theorem 4.1, the geometric conditions characterizing (i) and (ii) involve some sort of carrot condition, while the condition characterizing (iii) involves a type of cigar condition. This is because the Hölder imbedding is a genuinely "two-point" condition, where x and y range over all of Ω, while the Sobolev-Poincaré and Trudinger inequalities essentially control the variation of $u(x) - u(x_0)$ for a single distinguished point x_0. In fact these latter inequalities are equivalent to the corresponding inequalities with u_Ω replaced by the average of u over a compactly contained ball centered at x_0 (as previously noted) or, for solutions to many elliptic equations, to the corresponding inequality with u_Ω replaced by $u(x_0)$ (see, for instance, [Z]). The proof in Theorem 4.1 can easily be modified to prove the following "one-point" version of (iii), a task we leave to the interested reader.

Theorem 4.3. *Let $n < p < \infty$ and $C_0 > 1$ be fixed. Suppose that $\Omega \subset \mathbb{R}^n$ is bounded domain that satisfies the C_0-slice property with respect to $x_0 \in \Omega$. Then Ω supports the one-point Hölder imbedding*

$$|u(x) - u(x_0)| \le C|x - x_0|^{1-n/p} \left(\int_\Omega |\nabla u|^p \, dx \right)^{1/p}$$

if and only if Ω is a weak α-carrot domain for $\alpha = (p-n)/(p-1)$.

§5. Further results

Let us begin by stating a more general version of Theorem 4.1(iii). We leave the proof to the reader, as it requires only a few easy modifications to the original proof.

Theorem 5.1. *Let $n < p < \infty$, $0 < t \le 1 - n/p$, and $C_0 > 1$ be fixed. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain that satisfies the C_0-slice property with respect to all of its points. Then Ω supports the Hölder imbedding*

$$|u(x) - u(y)| \le C|x - y|^t \left(\int_\Omega |\nabla u|^p \, dx \right)^{1/p}$$

if and only if Ω *is a weak* (α,β)*-cigar domain, where* $\alpha=(p-n)/(p-1)$ *and* $\beta=pt/(p-1)$.

For Hölder imbeddings, the case of unbounded domains is not much different than the case of bounded domains and follows fairly easily from the bounded case, as we shall now see. We say a domain Ω is a *local weak* α*-cigar domain* if there exist $C,\delta>0$ such that whenever $|x-y|<\delta$, the pair x,y satisfies a weak (α,α,C)-cigar condition (in the sense of Section 2). Clearly the value of δ is not significant and this concept coincides with that of a weak α-cigar domain in the class of bounded domains.

Theorem 5.2. *If* $\Omega\subset\mathbb{R}^n$ *is a local weak* α*-cigar domain, where* $\alpha=(p-n)/(p-1)$*, then*

$$W^{1,p}(\Omega)\subset C^{0,1-n/p}(\overline{\Omega}). \tag{5.3}$$

Conversely, if (5.3) is valid for a domain $\Omega\subset\mathbb{R}^n$ *that satisfies the* C_0*-slice property with respect to all of its points (for some fixed* C_0*), then* Ω *is a local weak* α*-cigar domain.*

Proof. It is shown in [KR] that $W^{1,p}(\Omega)\subset C^{0,1-n/p}(\overline{\Omega})$ if and only if (1.3) is true for all $x,y\in\Omega$, $|x-y|<\delta$ for some $\delta>0$ (and hence all such δ with a constant depending on δ). Using this fact, the proof is essentially the same as that given in Theorem 4.1, so we omit the details. □

Note that, in the statement of this last result, it suffices to assume that the C_0-slice property is valid only for those points x,x_0 which are within a distance δ of each other.

Each part of Theorem 4.1 says that if a domain satisfies a certain geometric condition—which, to cover all cases, we refer to as condition X_p below—then it supports the Sobolev imbeddings of order p and, conversely, that a domain supporting a Sobolev imbedding of order p which also satisfies certain extra assumptions, must satisfy condition X_p. As pointed out in Section 1, some extra assumptions are necessary to prove this converse direction in each part of the theorem.

It would be more natural to replace our previous extra assumptions with the assumption that the domain is quasiconformally equivalent to a domain satisfying condition X_p. The earlier proof of Theorem 4.1 (above and in [BK]) does not work in such generality for any value of p, but we shall see below that such an improvement is possible when $p=n$. In general such a general statement is actually false (at least for the Hölder imbedding on unbounded domains). For instance, if A is the the complement of the closed unit disk in the plane, let us remove from A closed disks of radius $1/10$ centered at each integer lattice point $x\in A$ and call the new domain G. By inverting G with respect to the origin, we get a bounded domain Ω for which $W^{1,3}(\Omega)\subset C^{0,1-2/3}(\overline{\Omega})$. To see that this imbedding is true, we need only note that Ω differs from the unit disk D only by a collection of disks whose double dilates do not intersect, allowing us to extend any function $u\in W^{1,3}(\Omega)$ to a function $u\in W^{1,3}(D)$ with comparable norm. A straightforward computation shows that Ω does not satisfy the weak 1/2-cigar condition, even though G is a local 1/2-cigar domain (in fact it is an (ϵ,δ) domain in the terminology of P. Jones).

We now state and prove the improved imbedding theorem of Trudinger type. Although this result is more general than the previous one, and has a very short proof, we have chosen to give precedence to the other proof because the one below seems somewhat unnatural and applies only to the case $p=n$. The class $\operatorname{loc}\operatorname{Lip}_\alpha(\Omega)$ referred to below is the class of functions which satisfy the Lipschitz (or Hölder, as we have called it) inequality of order α, $|f(x)-f(y)|\le m|x-y|^\alpha$ whenever x, y lie in any ball contained in Ω (and the constant m is independent of the ball).

Theorem 5.4. *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain that is quasiconformally equivalent to a weak carrot domain. Then Ω supports the Trudinger inequality (1.2) if and only if Ω is a weak carrot domain.*

Proof. Suppose $f : D \to \Omega$ is a K-quasiconformal mapping from a weak carrot domain D to Ω. Since Ω supports a Trudinger inequality, we claim that there is a constant $M > 0$ such that

$$\operatorname{cap}(E, F; \mathbb{R}^n) \leq M \operatorname{cap}(E, F; \Omega) \tag{5.5}$$

for a fixed closed ball $E \subset \Omega$ and all closed balls $F \subset \Omega \backslash E$. Assuming this claim to be true for the moment, and combining it with Theorem 5.2 of [HeK], we deduce that f belongs to $\operatorname{loc}\operatorname{Lip}_\alpha(\Omega)$ for some α dependent only on the given data. But this condition on f, together with Lemma 3.20 of [GM1] and the fact that D is a weak carrot domain, implies that Ω is a weak carrot domain, as required.

To justify our claim, we fix any ball E for which $4E \subset \Omega$. If F intersects $2E$, it is easy to see that $\operatorname{cap}(E, F; \mathbb{R}^n) \leq M_1 \operatorname{cap}(E, F; \Omega)$ (for a proof, see Lemma 3.3 of [HeK]).

We may therefore assume that $2E$ and F are disjoint. Let d be the distance between E and F and let r_E, r_F be the radii of the balls E, F. By a *good constant*, we shall mean any constant depending only on n, r_E, and the diameter of Ω. Note that, since Ω is bounded, r_F/r_E and r_F/d are bounded above by good constants. If $u \in W^{1,n}(\Omega)$ is a function which equals 0 on E and 1 on F, and ϕ is the function in (1.2), it is clear that

$$\int_\Omega \phi(|u - u_\Omega|/s) \geq c_1 |F| \phi(1/2s),$$

where $c_1 > 0$ is a good constant (we can take $c_1 = 1$ if $|F| \leq |E|$). It follows readily that

$$\|u - u_\Omega\|_{\phi(L)(\Omega)} \geq c(\log(2 + r_F^{-1}))^{-(n-1)/n}$$

for some good constant $c > 0$, and so (1.2) implies that

$$\operatorname{cap}(E, F; \Omega) \geq (c/C)^n (\log(2 + r_F^{-1}))^{1-n}. \tag{5.6}$$

But by elementary estimates,

$$\operatorname{cap}(E, F; \mathbb{R}^n) \leq C_1 (\log d/r_F)^{1-n} \leq C_2 (\log(2 + r_F^{-1}))^{1-n},$$

where C_1 and C_2 are good constants. Condition (5.5) now follows by combining this last inequality with (5.6). $\square$

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