# Boman equals John

*S. Buckley, P. Koskela and G. Lu**

**Abstract.** In the abstract setting of homogeneous spaces, we prove the equivalence of two geometric conditions, namely the defining conditions for John domains and Boman domains.

1991 Mathematics Subject Classification: 43A85, 22E30, 31C45, 30C65

## 1. Introduction

In 1961 John [Jo] introduced the notion of a twisted interior cone condition in connection with his work on elasticity. This condition was later employed by Reshetnyak [R] in 1976 to study quasiconformal mappings of small dilatation. In 1979 Martio and Sarvas [MS] renamed the class of domains satisfying a twisted interior cone condition as the class of John domains; see 2.2 below for a precise definition. They used this condition and certain variants of it to study global injectivity properties of locally injective mappings.

As examples of John domains let us point out that smooth domains, Lipschitz domains and certain fractal domains (for example the snowflake domain) are John domains. Moreover, John domains arise naturally in the iteration of complex polynomials. Carleson, Jones and Yoccoz [CJY] have recently been able to characterise for polynomial mappings the situations when the basin of attraction at infinity and the bounded Fatou components are John domains; in particular they show that the basin at infinity is John if and only if the polynomial satisfies a weak version of hyperbolicity.

By now the class of John domains has been extensively studied in connection with quasiconformal analysis. For example, a quasiconformal mapping of a ball onto a John domain has interesting properties as pointed out by Pommerenke in 1982 [P], Väisälä in 1989 [V2], and Heinonen in 1989 [He]. In particular, the Jacobian of a quasiconformal mapping of a ball onto a domain $G$ is the restriction of an $A_\infty$-weight if and only if $G$ is a John domain; for this see the paper by Heinonen and Koskela [HeK] from 1994. The geometry of simply connected plane John domains is well understood by the work of Näkki and Väisälä from 1991 [NV]; also see the paper [M] by Martio from 1988 for a nice characterisation without restrictions on connectivity.

In 1982 Boman [Bom] introduced a chain condition that nowadays is referred to as the Boman chain condition. We say that a domain is a Boman domain if it satisfies a Boman chain condition;

---

* The first author was partially supported by NSF Grant DMS–9207715 and by Forbairt. The second author was partially supported by NSF Grant DMS–9305742 and by the Academy of Finland. The third author was partially supported by NSF Grant DMS–9315963.

see 2.1 below for the precise definition. Boman used his condition for $L^p$-estimates for elliptic systems and his condition has since found a number of other applications. In 1985 Iwaniec and Nolder [IN] proved a Hardy-Littlewood type inequality for quasiregular mappings defined in a Boman domain. Bojarski [Boj] showed in 1989 that Boman domains admit a Sobolev-Poincaré inequality with the best possible exponents (same as for a ball). Chua [C] extended Bojarski's result in 1993 to a weighted setting, Lu [L] established a sharp Sobolev-Poincaré inequality for Hörmander vector fields when $p > 1$, and Franchi, Lu, and Wheeden [FLW] proved the sharp inequalities for all $p \geq 1$ in general. One should also note the paper of Franchi, Gutierrez, and Wheeden [FGW] from 1994 where they prove Sobolev-Poincaré inequalities for metric balls associated with Grushin type operators, and the paper by Buckley and Koskela [BK] from 1994 where the authors study Sobolev-Poincaré inequalities for the "unnatural" exponents $p < 1$ for Sobolev functions and for solutions to certain elliptic equations; also see [BKL] for the vector field situation. The important observation in connection with vector fields is that metric balls in a Carnot-Caratheodory metric are Boman domains which allows one to patch up global estimates from local ones. The first to notice that metric balls are some kind of chain domain was apparently Jerison in 1986 [Je] who used arguments similar to those used earlier by R.Kohn in the Euclidean setting.

The main purpose of this paper is to point out that a domain $\Omega$ is a Boman domain if and only if $\Omega$ is a John domain. This is Theorem 3.1 below. We establish this result in the abstract setting of a homogeneous space so as to cover various situations at once (for example, the Carnot-Carathéodory metrics associated with vector fields: see [BKL], [NSW]). In the general homogeneous space it is difficult to determine whether or not a domain is a Boman domain. The point we want to make is that it is often easier to establish that a domain is a metric John domain, and, since these classes coincide, one should aim for this. For example, as a simple corollary of the easier half of our main result we shall show that, in a rather general setting, metric balls satisfy the Boman chain condition; the setting is more general than that of a similar result in [FGW] and the proof involves considerably less work.

In the Euclidean setting it appears to be folklore to people working on quasiconformal analysis that Boman domains are John domains even though we have no references to give. It is also relatively easy to show that a John domain is a Boman domain even in a homogeneous space. The converse statement is more complicated in the abstract setting of a homogeneous space (and false without some sort of geodesic assumption, as we shall see in Section 3).

## 2. Definitions

Let $(S, d, \mu)$ be a homogeneous space in the sense of Coifman-Weiss. Thus $d$ is a pseudometric and $\mu$ is a measure that is doubling with respect to metric balls.

**Definition 2.1.** A domain (i.e. connected open set) $E$ in $S$ is said to satisfy the *Boman chain condition* if there exist positive constants $M$, $\lambda > 1$, $C_2 > C_1 > 1$, and a family $\mathcal{F}$ of disjoint metric balls $B$ such that

(i) $E = \bigcup_{B \in \mathcal{F}} C_1 B$ .
(ii) $\sum_{B \in \mathcal{F}} \chi_{C_2 B}(x) \leq M \chi_E(x)$ for all $x \in S$ .

(iii) There is a so-called "central ball" $B_* \in \mathcal{F}$ such that for each ball $B \in \mathcal{F}$, there is a positive integer $k = k(B)$ and a chain of balls $\{B_j\}_{j=0}^k$ such that $B_0 = B$, $B_k = B_*$, and $C_1 B_j \bigcap C_1 B_{j+1}$ contains a metric ball $D_j$ whose measure is comparable to those of both $B_j$ and $B_{j+1}$.
(iv) $B \subset \lambda B_j$, for all $j = 0, \ldots, k(B)$.

We shall call such a set $E$ a *(Boman) chain domain*. We shall refer to individual chains as $(\lambda, C_1, C_2)$-chains if we wish to specify the parameters. Clearly all chain domains are bounded. $M$ is a "dimensional constant" which is of no great concern to us. If $\lambda$ is much larger than $C_1$ and $C_2$, it indicates the domain is "bad" (for instance, it may be very elongated or it may have narrow bottlenecks). The other parameters are not important, as there is a lot of freedom in their choice. Trivially for instance, $\lambda$, $C_1$, and $C_2$ can all be multiplied by the same factor larger than 1 (while holding $M$ constant) if we shrink the balls accordingly.

If $S = \mathbb{R}^N$ equipped with the Lebesgue measure and $d$ is the Euclidean metric, this is the standard Boman chain condition. Notice that in the above definition we require that the intersection of the expanded balls contains a ball of large volume instead of a ball of large radius. The reason for this is that in applications of this definition one wishes to compare averages over balls in the chains associated with the domain, see [BKL], [FGW], [FLW].

Let us briefly discuss rectifiability of curves in a general metric space $(X, d)$. We define the arclength of a curve $\gamma : [a, b] \to X$ as in the Euclidean case. More precisely, we first define its arclength with respect to a partition $P = \{a = t_0 < t_1 < \cdots < t_m = b\}$ of $[a, b]$ to be $\sum_{i=1}^m d(\gamma(t_i), \gamma(t_{i-1}))$, and then take the supremum over all partitions of the associated lengths. A rectifiable curve $\gamma$ can always be reparametrized to give a curve $\hat{\gamma} : [0, l] \to X$ such that the arclength of $\hat{\gamma}|_{[0,t]}$ is $t$ for all $0 \le t \le l$ (we say that $\hat{\gamma}$ is *parametrized by arclength*). To see this it suffices to show that the arclength of $\hat{\gamma}|_{[0,t]}$ is a continuous function of $t$; this is proved as in the Euclidean case (which can be found in [V1, Chapter 1], for instance).

**Definition 2.2.** A bounded open proper subset $E$ of a metric space $(X, d)$ with a distinguished point $x_* \in E$ is called a *(metric) John domain* if it satisfies the following "twisted cone" condition: there exists a constant $c > 0$ such that for all $x \in E$, there is a curve $\gamma = \gamma : [0, l] \to E$ parametrized by arclength such that $\gamma(0) = x$, $\gamma(l) = x_*$, and $d(\gamma(t), E^c) \ge ct$. We call such a curve a *John curve for $x$*.

In the above definition, we shall always assume without loss of generality that $c < 1$. Note also that $l$ is bounded above by $\operatorname{diam}(E)/c$, and that John domains are automatically connected. If $E$ is a John domain, any $y \in E$ can act as the distinguished point (but a more "central" point will give a smaller constant). For examples of John domains in a Euclidean space see the introduction. In a more general setting, any metric ball satisfying a weak geodesic condition (which is always valid in the case of a Carnot-Carathéodory metric) is a metric John domain; a precise statement and its easy proof are given in Corollary 3.5 below.

In Definition 2.2, one could also use the diameter of the curve segment from $t = 0$ to $t = l$ in place of the arclength $l$ (dropping our standing assumption that curves are regular). Obviously such "arclength-John" domains are "diameter-John" domains. In $R^n$ these two definitions were shown to be equivalent by Martio and Sarvas [MS]. More generally, if we simply assume $X$ has the very weak geometric property that any two distinct points in it can be joined by a "quasigeodesic" curve of length at most a fixed multiple of the distance between them, it is not hard to show that the two notions are still equivalent. For completeness, let us outline the main idea here. We replace the diameter curve by a finite sequence of such quasigeodesics sewn together. More exactly, we rectify the curve one segment at a time starting from $x$, replacing each curve segment by a quasigeodesic with the same endpoints, and choosing the segments so small that the distance from the boundary to any point on the curve segment is approximately constant in a relative sense. As we move away from $x$ (and hence from $E^c$), it is clear that we can in this manner rectify pieces whose diameters grow geometrically.

## 3. John domains and Boman domains

From now on, we assume that $(S, d, \mu)$ is a homogeneous space, but we assume the doubling condition only for metric balls in $S$ of radius less than $\delta$ for some fixed $0 < \delta \leq \infty$ (so the results are applicable to the Carnot-Carathéodory metrics associated with vector fields). We shall also assume that $d$ is a genuine metric, rather than just a pseudometric. We shall denote by $z(B)$ and $r(B)$ the centre and radius respectively of any metric ball.

If $B = B(x_0, r)$ is an open metric ball in $S$, we say that $B$ satisfies a *weak geodesic condition* if for every $x \in B$ there exists a curve $\gamma : [0, l] \to B$ of length less than $r$ for which $\gamma(0) = x$ and $\gamma(l) = x_0$. We shall say that a connected open set $E \subset S$ satisfies a *strong geodesic condition* if every sub-ball of $E$ satisfies a weak geodesic condition. The geodesic condition used in [FGW] and [FLW] to prove that metric balls are chain domains is similar to, but logically stronger than, our strong geodesic condition. If $d$ is a Carnot-Carathéodory metric, i.e. $d(x, y)$ is equal to the infimum of the lengths of curves joining $x$ and $y$, it follows immediately that connected open sets satisfy a strong geodesic condition.

**Theorem 3.1.** *Suppose $E$ is a proper open subset of $S$. Then*

*(a) If $E$ is a John domain, it is a Boman chain domain.*
*(b) If $E$ is a Boman chain domain, has diameter less than $\delta/2$, and satisfies a strong geodesic condition, then $E$ is a John domain.*

Before giving the proof of Theorem 3.1 let us briefly comment on the necessity of the strong geodesic condition. First of all notice that an open set $E$ satisfying the conditions (i)–(iv) of Definition 2.1 can fail to be connected. More precisely, let $F$ be a closed subset of a ball $B(x, 2r)$ in the Euclidean space $\mathbb{R}^n$ and equip $S = B(x, 2r) \setminus F$ with the restrictions of the Lebesgue measure and the Euclidean metric. Then it follows that $(S, d, \mu)$ is a homogeneous space, and that $\Omega = B(x, r) \setminus F$ satisfies (i)–(iv), provided that $F$ is of measure zero. Hence we may select $F$ so that $\Omega$ is a Boman chain domain but fails to be John.

Thus some local connectivity condition is necessary for $(S, d, \mu)$ in (b) above. The condition we employ appears rather optimal and we would like to remark that the strong geodesic condition holds in all the situations we know of where the Boman condition has been applied.

For the proof of Theorem 3.1 we need the following Whitney decomposition lemma of Coifman and Weiss [CW,III.1.3].

**Lemma 3.2.** *If $E$ is a proper open subset of a homogeneous space $(S, d, \mu)$, then there exists a family $\mathcal{F}$ of disjoint metric balls and constants $M, 1 < K_1 < K_2 < K_3$ such that*

*(a)* $E = \bigcup_{B \in \mathcal{F}} K_1 B$ .
*(b)* $\sum_{B \in \mathcal{F}} \chi_{\mathrm{K_2 B}}(x) \leq M \, \chi_{\mathrm{E}}(x)$ *for all* $x \in S$ .
*(c)* $K_3 B$ *intersects* $E^c$ *for every* $B \in \mathcal{F}$.

Note that by examining the proof of this lemma in [CW], it is easily verified that the constants $K_1, K_2/K_1$, and $K_3/K_2$ can be chosen arbitrarily and independently, provided that they exceed certain lower bounds. For the first and last of these constants, this is essentially trivial, while increasing $K_2/K_1$ corresponds to using smaller balls in the proof of this lemma.

*Proof of Theorem 3.1.* Suppose $E$ is a John domain. We define auxiliary constants $K_3' = K_3 + K_1$ and $K_4 = 8K_1K_3/K_2$, where the constants $K_1, K_2, K_3$ are from Lemma 3.2 above. Since $E$ is bounded, the comments above allow us to assume that the ratio $K_2/K_1$ exceeds 100, and is so large that $r(K_4B) < \delta$ for all $B \in \mathcal{F}$.

We show that $E$ satisfies conditions (i)–(iv) of Definition 2.1. Using Lemma 3.2, (i) and (ii) are automatically satisfied — we can choose $C_2 = K_2$ and $C_1 = 4K_1$ (the "4" factor will be used later for verifying (iii)). We now use the John condition to verify (iii) and (iv).

Let $x_*$ be the distinguished point of $E$, and choose as the centre ball any $B_* \in \mathcal{F}$ such that $x_* \in K_1B_*$. Fixing $B \in \mathcal{F}$, we write $x_0 = z(B)$, $B_0 = B$, and $t_0 = 0$. We choose a curve $\gamma : [0, l] \to E$, parametrized by arclength, for which $\gamma(0) = x_0$, $\gamma(l) = x_*$, and $d(\gamma(t), E^c) > ct$. We shall inductively define $t_j$ for $j > 0$. Once $t_j$ is chosen, we choose $B_j \in \mathcal{F}$ such that $x_j \equiv \gamma(t_j) \in K_1B_j$ (any such $B_j$ will be suffice, except that we insist that $B_j = B_*$ if $t = l$). For $j > 0$, we define $t_j = \min\{l, t_{j-1} + K_1r(B_{j-1})\}$. Now the Whitney decomposition implies that for all $i$,

$$r(B_i) \geq d(x_i, E^c)/K_3' \geq ct_i/K_3' \tag{3.3}$$

and so for all $j \geq 0$,

$$r(B_j) \geq cr(B_0)/K_3' \tag{3.4}$$

Thus the radius of $B_j$ is bounded below and we eventually reach an integer $k = k(B)$ for which $x_k = x_*$. Note that (3.4) immediately implies that $k \leq l/(cr(B)/K_3)$; a little more thought shows that the sharper (3.3) gives geometrically growing lower bounds for the radii and hence $k$ is bounded by $C \log(1/r(B))$, where $C$ depends only on $E$ and the metric.

It is clear that for all $0 \leq j < k$, $4K_1B_j \cap 4K_1B_{j+1}$ includes whichever of $B_j$, $B_{j+1}$ has the smaller radius. Since $K_2 > 100K_1$, neighbouring balls are of comparable radius (in fact, conditions (b) and (c) of Lemma 3.2 imply that the larger radius is at most $2K_3/K_2$ times the smaller one). It follows that $K_4B_j \supset B_{j+1}$ and $K_4B_{j+1} \supset B_j$, and the doubling property ensures that $B_j$ and $B_{j+1}$ are of comparable measure. We have therefore proved (iii).

Finally, we prove (iv). Since $d(x_j, x_0) \leq d(x_j, E^c)/c \leq K_3r(B_j)/c$, we get from (3.4) that

$$\begin{aligned} d(x, z(B_j)) &\leq d(x, x_0) + d(x_0, x_j) + d(x_j, z(B_j)) \\ &\leq r(B_0) + K_3r(B_j)/c + K_1r(B_j) \\ &\leq (K_3' + K_3 + cK_1)r(B_j)/c \end{aligned}$$

for all $x \in B_0$.

We next prove (b). Suppose $E$ satisfies the hypotheses of (b). If for all chains $\{B_j\}_{j=0}^k$, we have $r(B_j)/r(B_0) \geq Ct^j$ for some $C > 0$, $t > 1$, it is easy to construct the required John curves by a connect-the-dots process, as we shall see below. Unfortunately, the chain condition is not strong enough to imply such a geometric growth rate. Nevertheless we shall show that $E$ is a John domain by first constructing new Boman chains (taken from the same family $\mathcal{F}$) with such a growth rate.

Consider one such $B \in \mathcal{F}$ and its associated chain of balls $\{B_j\}_{j=0}^k$, where $B_0 = B$, $B_k = B_*$. We shall define new chains $\mathcal{F}_\rangle(\mathcal{B})$ for all $1 \leq i \leq \infty$. First let us show that there exists a constant $M > 0$, dependent only on the Lebesgue doubling constant and the parameters of the chain domain, for which either $k \leq M$, or $r(B_j) > 2r(B)$ for some $0 < j \leq M$. Let $t = \min\{2\lambda + 4, \delta/2r(B)\}$. By 2.1 (iv), $r(B_i) \geq r(B)/\lambda$ for all $0 \leq i \leq k$. Since $r(tB) \leq \delta/2$ and $r(tB) \leq (2\lambda + 4)r(B)$, the doubling condition ensures the existence of some constant $M_0$ such that $\mu(B_i) \geq \mu(tB)/M_0$ for all $B_i \subset tB$, and so the number of chain balls in $tB$ is at most $M_0$. Now, $M = M_0 + 1$ is the number we require. To see this, suppose $k > M$ and so, for some $0 < j \leq M$, $B_j$ is not contained in $tB$. We cannot have $r(tB) = \delta/2$, since then $tB \supset E \supset B_j$, a contradiction. Therefore $B_j$ is not contained in $(2\lambda + 4)B$ and so either $r(B_j) > 2r(B)$, in which case we are done, or $B_j$ does not intersect $2\lambda B$. In the latter case, condition (iv) ensures $r(B_j) > 2r(B)$ anyway, so $M$ has the required properties.

If $k \leq M$, we define $\mathcal{F}_\rangle(\mathcal{B}) = \mathcal{F}(\mathcal{B})$ for all $1 \leq i \leq \infty$. Otherwise, we write $B^1 = B_j$, where $r(B_j) > 2r(B)$ and $0 < j \leq M$. We get $\mathcal{F}_\infty(\mathcal{B})$ by discarding all balls in $\mathcal{F}(\mathcal{B})$ after $B^1$ and appending $\mathcal{F}(\mathcal{B}^\infty)$ (this hybrid sequence of balls is actually a $(\lambda + \lambda^2, C_1, C_2)$-chain, but this does not concern us right now). Since the new part of the chain is a Boman chain in its own right, we see that the new balls have radius at least $r(B_j)/\lambda$, and so in at most $M$ more steps, we either reach the centre ball or we encounter a ball $B^2$ for which $r(B^2) > 2r(B^1) > 4r(B)$. In the first case, we let $\mathcal{F}_\rangle(\mathcal{B}) = \mathcal{F}_\infty(\mathcal{B})$ for all $2 \leq i \leq \infty$, while in the latter case we replace the balls in $\mathcal{F}_\infty(\mathcal{B})$ after $B^2$ by $\mathcal{F}(\mathcal{B}^\in)$ to create $\mathcal{F}_\in(\mathcal{B})$. Continuing inductively, we must at some stage reach the centre ball because the domain is bounded. Let us refer to the (finite number of) balls $B^k, k \geq 1$ as link balls for obvious reasons. The last new chain $\mathcal{F}_\infty(\mathcal{B})$ obviously has the desired geometric growth rate. Also, we claim that it is a $(\lambda + 2\lambda^2, C_1, C_2)$-chain. To see this, note that if $B'$ occurs in $\mathcal{F}_\infty(\mathcal{B})$ after $B''$ and they are separated by exactly one link ball, $B^k$, then $(\lambda + \lambda^2)B' \supset B''$ since $\lambda B' \supset B^k$ and $\lambda B^k \supset B''$. As one goes back further in the chain, induction and the supergeometric increase of $r(B^k)$ readily imply that $(\lambda + 2\lambda^2)B' \supset B''$ whenever $B''$ occurs before $B'$ (the "2" factor occurs as $\sum_{i=0}^\infty 2^{-i}$).

Now let $\gamma_j$ be a curve joining the centres of $B'_j$ and $B'_{j+1}$ whose length is less than $C_1$ times the sum of the radii (such a curve exists because of (iii) and the assumed geodesic condition) and let $\gamma$ be the curve that glues the $\gamma_i$'s together. The geometric growth of $r(B_i)$ implies that $\sum_{i=0}^j r(B'_i) \leq Cr(B'_j)$, and so the length of the initial part of $\gamma$ joining $z(B)$ to $z(B'_j)$ has length bounded by a constant times $r(B'_j)$. Since all points in $C_1 B'_j$ are at least a distance $(C_2 - C_1)r(B'_j)$ from $E^c$, it follows that $\gamma$ is a John curve joining $z(B)$ to the $z(B_*)$. For a general point $x \in B$, the assumed geodesic property allows us to join $x$ to $z(B)$ by means of a curve of length less than $r(B)$. Sewing this curve onto the John curve for $z(B)$, we get a John curve for $x$, as required. □

In the Carnot-Carathéodory setting, the fact that metric balls are chain domains is implicit in [Je], and explicitly proved in [L]. The same result is given in a more general setting in [FGW], where a certain geodesic condition is assumed. Here we prove it in still greater generality as a simple corollary of Theorem 3.1 (a), assuming only the weak geodesic condition defined before that theorem.

**Corollary 3.2.** *Any metric ball $B = B(x_*, r) \subset S$, $B \neq S$, which satisfies a weak geodesic condition is a John domain, and hence a chain domain.*

*Proof.* Let us fix $x \in B$. There exists a curve $\gamma : [0, b] \to B$ parametrized by arclength, such that $b < r$, $\gamma(0) = x$, $\gamma(b) = x_*$. Since $d(\gamma(t), x_*) = b - t$, we have

$$d(\gamma(t), B^c) \geq d(x_*, B^c) - d(x_*, \gamma(t)) \geq r - (b - t) > t.$$

and so $B$ is a John domain. □

## References

[Boj] Bojarski, B., Remarks on Sobolev imbedding inequalities. In: Springer-Verlag Lecture Notes in Math. 1351, 52–68. Springer-Verlag, New York-Berlin-Heidelberg 1989.

[Bom] Boman, J., $L_p$-estimates for very strongly elliptic systems, Report no. 29, Department of Mathematics, University of Stockholm, Sweden 1982.

[BK] Buckley, S., Koskela, P., Sobolev-Poincaré inequalities for $p < 1$. Indiana Univ. Math. J. 43 (1994), 221–240.

[BKL] Buckley, S., Koskela, P., Lu, G., Subelliptic Poincaré inequalities: the case $p < 1$. Publicacions Mathemàtiques (to appear).

[CJY] Carleson, L., Jones, P.W., Yoccoz, J.-C., Julia and John. Bol. Soc. Brasil. Mat. 25 (1994), 1–30.

[C] Chua, S.-K., Weighted Sobolev's inequality on domains satisfying the chain condition. Proc. Amer. Math. Soc. 117 (1993), 449–457.

[CW] Coifman, R., Weiss, G., Analyse harmonique non-commutative sur certains espaces homogène (Springer-Verlag Lecture Notes in Math. 242). Springer-Verlag, New York-Berlin-Heidelberg 1971.

[FGW] Franchi, B., Gutierrez, C., Wheeden, R., Weighted Sobolev-Poincaré inequalities for Grushin type operators. Comm. P.D.E. 19 (1994), 523–604.

[FLW] Franchi, B., Lu, G., Wheeden, R., Representation formulas and weighted Poincaré inequalities for Hörmander vector fields. Ann. Inst. Fourier (Grenoble) 45, 2 (1995), 577–604.

[He] Heinonen, J., Quasiconformal mappings onto John domains. Rev. Mat. Iberoamericana 5 (1989), 97–123.

[HeK] Heinonen, J., Koskela, P., $A_\infty$-condition for the Jacobian of a quasiconformal mapping, Proc. Amer. Math. Soc. 120 (1994), 535–543.

[IN] Iwaniec, T., Nolder, C., Hardy-Littlewood inequality for quasiregular mappings in certain domains in $\mathbb{R}^n$. Ann. Acad. Sci. Fenn. Ser. A I Math. 10 (1985), 267–282.

[Je] Jerison, D., The Poincaré inequality for vector fields satisfying Hörmander's condition. Duke Math. J. 53 (1986), 503–523.

[Jo] John, F., Rotation and strain. Comm. Pure Appl. Math. 14 (1961), 391–413.

[L] Lu, G., The sharp Poincaré inequality for free vector fields: An endpoint result. Rev. Mat. Iberoamericana 10 (1994), 453-466.

[M] Martio, O., John domains, bilipschitz balls and Poincaré inequality, Rev. Rom. Math. Pures Appl. 33 (1988), 107–112.

[MS] Martio, O., Sarvas, J., Injectivity theorems in plane and space. Ann. Acad. Sci. Fenn. Ser. A I Math. 4 (1979), 383–401.

[NSW] Nagel, A., Stein, E.M., Wainger, S., Balls and metrics defined by vector fields I: basic properties. Acta Math. 155 (1985), 103–147.

[NV] Näkki, R., Väisälä, J., John disks. Expo. Math. 9 (1991), 3–43.

[P] Pommerenke, Ch., One-sided smoothness conditions and conformal mapping. J. London Math. Soc. 26 (1982), 77–88.

[R] Reshetnyak, Yu. G., Stability in Liouville's theorem on conformal mappings of a space for domains with nonsmooth boundary. Siberian Math. J. 17 (1976), 281–288.

[V1] Väisälä, J., Lectures on $n$-dimensional quasiconformal mappings (Springer-Verlag Lecture Notes in Math. 229). Springer-Verlag, New York-Berlin-Heidelberg 1971.

[V2] Väisälä, J., Quasiconformal maps of cylindrical domains. Acta Math. 162 (1989), 201–225.