SUBELLIPTIC POINCARÉ INEQUALITIES: THE CASE $p < 1$

BY
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Abstract. We obtain (weighted) Poincaré type inequalities for vector fields satisfying the Hörmander condition for $p < 1$ under some assumptions on the subelliptic gradient of the function. Such inequalities hold on Boman domains associated with the underlying Carnot-Carathéodory metric. In particular, they remain true for solutions to certain classes of subelliptic equations. Our results complement the earlier results in these directions for $p \geq 1$.

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§1. Introduction

One of the main purposes of this paper is to derive a Poincaré-type inequality of the form

$$\left(\frac{1}{|B|}\int_B |f(x)-f_B|^q dx\right)^{\frac{1}{q}} \le cr\left(\frac{1}{|B|}\int_B \left(\sum_j |\langle X_j, \nabla f(x)\rangle|^2\right)^{\frac{p}{2}} dx\right)^{\frac{1}{p}} \tag{1.1}$$

in Euclidean space $\mathbb{R}^N$ for $0<p<1$ and certain values $q>p$, where $\{X_j\}_{j=1}^m$ is a collection of smooth vector fields which satisfy the Hörmander condition (see [H]) provided that f is a suitable function whose subelliptic gradient satisfies a weak reverse Hölder condition (defined below). Here, B denotes any suitably restricted ball of radius r relative to a metric ϱ which is naturally associated with $\{X_j\}$ (as in, for example, [FP] and [NSW]), f_B is some constant (we may assume $f_B = |B|^{-1}\int_B f(x)\,dx$ if $f\in L^1(B)$), and c is a constant which depends on the reverse Hölder constant of the subelliptic gradient of f but is otherwise independent of f, and is also independent of B. More generally, we shall prove one-weight and two-weight versions of this result, and also results for more general domains.

Inequality (1.1) was derived in [J] for $q=p$ and $1\le p<\infty$, and this result was improved by the third author in the case $p>1$ in [L2] where (1.1) is proved for $1<p<Q$, $q=pQ/(Q-p)$, and $Q(\ge N)$ denotes the homogeneous dimension of $\mathbb{R}^N$ associated with $\{X_j\}$ (see below for the definition). Recently the limiting case $p=1$ and $q=\frac{Q}{Q-1}$ was proved in [FLW] by establishing a new representation formula that improves the previous one proved in [L1]; this inequality was then applied to the relative isoperimetric inequality. In the cases $p=Q$ and $p>Q$, f was shown to be in BMO and Hölder classes, respectively, when $|Xf|$ is assumed to be in L^p_{loc} (see [L3], [L4]); embedding theorems on the Campanato-Morrey spaces and from the Morrey spaces to BMO and Lipschitz spaces were also shown. Some related inequalities have been studied in [BM], [HK], and [MS].

There have been very few Poincaré-type results for $p<1$, mainly because there are easy counterexamples even in the case when $\{X_i\}_{i=1}^N$ are the constant vector fields in the coordinate directions, i.e., $X_i=\frac{\partial}{\partial x_i}$, $1\le i\le N$ (see [BK]). However, in this particular case, it has recently been proved that f satisfies a Poincaré inequality if the gradient of f satisfies a weak reverse Hölder condition [BK]. We show in this paper that a similar result is true in the setting of vector fields of the above type and also in a weighted context. In the unweighted case, we will show that (1.1) holds for such functions f if $0<p<1$, $p<q<\infty$, and p and q are related by a natural balance condition involving the local doubling order of Lebesgue measure for metric balls (see [FLW] for $p\ge 1$). This balance condition (introduced earlier in [CW]) can actually be shown to be necessary and sufficient for the validity of the Sobolev-Poincaré inequality.

To state our theorems, we need to introduce some notation and definitions. Let Ω be a domain in $\mathbb{R}^N$, and let $\{X_j\}_{j=1}^m$ be a collection of C^∞ real vector fields defined in a neighbourhood of the closure $\overline{\Omega}$ of Ω. For a multi-index $\alpha=(i_1,\dots,i_k)$, the commutator $[X_{i_1},[X_{i_2},\dots,[X_{i_{k-1}},X_{i_k}]]\dots]$ of length $k=|\alpha|$ will be denoted by X_α. Throughout this paper we assume that the vector fields satisfy Hörmander's condition: there exists some positive integer s such that $\{X_\alpha\}_{|\alpha|\le s}$ span $\mathbb{R}^N$ at each point of Ω. For the sake of brevity, we shall refer to such a family of vector fields as a Hörmander family.

With any Hörmander family, we can associate a metric as follows. First let us say that $\gamma:[a,b]\to\Omega$ is an *admissible curve* if it is Lipschitz and there exist functions $c_i(t)$, $a\le t\le b$, satisfying

$\sum_{i=1}^m c_i(t)^2 \leq 1$ and $\gamma'(t) = \sum_{i=1}^m c_i(t)\, X_i(\gamma(t))$ for almost every $t \in [a,b]$. A natural metric on Ω associated with $X_1, \ldots, X_m$ is

$$\begin{aligned}\varrho(\xi,\eta) = \inf\{b \geq 0 : \exists \text{ an admissible curve } \gamma : [0,b] \to \Omega \\ \text{such that } \gamma(0) = \xi\,,\ \text{and } \gamma(b) = \eta\}\ .\end{aligned}$$

Such a metric is often called a Carnot-Carathéodory metric. It follows from the work of Busemann [Bu, p. 25] that any two points in Ω can be joined by a geodesic (a rectifiable path whose ϱ-length equals the ϱ-distance between its end-points). We assume that any geodesic is canonically parametrised by the ϱ-arclength of its initial segment. The (open) metric ball with centre x and radius r will be denoted $B(x,r) = \{y : \varrho(x,y) < r\}$. ϱ is locally equivalent to the various other metrics defined in [NSW], and generates the same topology as the Euclidean metric. It is shown there that Lebesgue measure is locally doubling on small balls: if $K \subset\subset \Omega$ and $\delta > 0$ is sufficiently small, then

$$|B(x,2r)| \leq C|B(x,r)|, \qquad x \in K,\ 0 < r < \delta. \tag{1.2}$$

Thus (Ω, ϱ) is (locally) a homogeneous space in the sense of Coifman-Weiss.

If B is a metric ball and $t > 0$, $r(B)$ and r_B both denote the radius of B, tB denotes the "t-dilate" of B (the concentric ball with radius $t \cdot r_B$), and z_B denotes the centre of B. In proofs, C denotes any constant whose exact value is unimportant for the purposes of the proof.

By the Rothschild-Stein lifting theorem (see [RS]), the vector fields $\{X_i\}_{i=1}^m$ on $\Omega \subset \mathbb{R}^N$ can be lifted to vector fields $\{\tilde{X}_i\}_{i=1}^m$ in $\tilde{\Omega} = \Omega \times T \subset \mathbb{R}^N \times \mathbb{R}^{M-N}$, where T is the unit ball in $\mathbb{R}^{M-N}$, by adding extra variables so that the resulting vector fields are free, *i.e.*, the only linear relation between the commutators of order less than or equal to s at each point of $\tilde{\Omega}$ are the antisymmetric and Jacobi's identity. Let $\mathcal{G}(m,s)$ be the free Lie algebra of steps with m generators, that is the quotient of the free Lie algebra with m generators by the ideal generated by the commutators of order at least $s+1$. Then $\{X_\alpha\}_{|\alpha| \leq s}$ are free if and only if $N = \dim \mathcal{G}(m,s)$. We define the homogeneous dimension of Ω to be $Q \equiv \sum_{j=1}^s j m_j$, where m_j is the number of linearly independent commutators of length j for the lifted vector fields.

A weight function $w(x)$ on an open subset E of Ω is a nonnegative function on E which is locally integrable with respect to Lebesgue measure and not everywhere zero. Given $0 < \alpha < 1$, we say that a positive Borel measure μ on an open subset E of Ω is an *α-strong doubling measure* (or simply a strong doubling measure) on E if there exists $C_\mu > 0$ such that $\mu(2B \cap E) \leq C_\mu \mu(B \cap E)$ for all metric balls B for which $\alpha B \subset E$. If $d\mu(x) = w(x)\,dx$ for some weight w, we say that w is an *α-strong doubling weight*. If $1 \leq p < \infty$, we say that a weight w is in the class (local)-$A_p(E) \equiv A_p(E, \varrho, dx)$ if there is some constant $C_w \geq 1$ such that

$$\left(\frac{1}{|B|}\int_B w\ dx\right)\left(\frac{1}{|B|}\int_B w^{-1/(p-1)} dx\right)^{p-1} \leq C_w, \qquad \text{when } 1 < p < \infty,$$
$$\frac{1}{|B|}\int_B w\, dx \leq C_w \operatorname*{ess\,inf}_B w, \qquad \text{when } p = 1,$$

for all metric balls $2B \subset E$. Since Lebesgue measure is doubling with respect to metric balls, we can develop the usual theory of these Muckenhoupt A_p classes as in [Ca], at least for balls $B = B(x,r)$ with

$0 < r < r_0$ and x belonging to a compact subset of E. In particular, it follows easily from the above definition that A_p weights are (local) doubling weights: $\mu(2B) \le C_\mu \mu(B)$ whenever $4B \subset E$. These weights are not necessarily strongly doubling. We should note that our definition is more "local" than Calderón's (since we are only assuming the defining inequality for balls B such that $2B \subset E$), but Calderón's proofs can be easily adjusted to handle this variant; our more local definition allows us to handle certain important classes of weights that would not belong to a more restricted class (for instance, positive powers of distance to the boundary).

We now introduce the notion of functions satisfying a weak reverse Hölder condition. Given a doubling measure μ on an open subset E of Ω, we say that a non-negative function w on E is in $WRH_p(E, \mu)$ if $w \not\equiv 0$, $w \in L^p_{\mathrm{loc}}(E, \mu)$, and if there is a constant $C_{w,\mu} > 0$ such that

$$\left(\frac{1}{\mu(B)} \int_B w(x)^p \, d\mu\right)^{\frac{1}{p}} \le C_{w,\mu} \left(\frac{1}{\mu(B)} \int_{\sigma B} w(x)^q \, d\mu\right)^{\frac{1}{q}} \qquad \text{for all } B \,:\, \sigma' B \subset E.$$

Here q, σ, and σ' are parameters satisfying $0 < q < p$, $1 < \sigma \le \sigma'$. These definitions are independent of the choice of q, σ, σ', as we shall show in Lemma 1.4 (whose proof is adapted from the proof for Lebesgue measure and Euclidean space in [IN]). When $d\mu(x) = v(x)dx$, we shall abuse notation as before and write $WRH_p(E, v)$ in place of $WRH_p(E, \mu)$; in the particular case $v = 1$, we simply write $WRH_p(E)$. For a convenient choice of q, σ, σ', the smallest constant $C_{w,\mu}$ for which the above defining inequality remains valid is called the $WRH_p(E, \mu)$ constant of w. Since we use this constant only as an upper bound on the variability of the gradient, it follows from Lemma 1.4 below that the exact choices of q, σ, σ', are unimportant for our purposes. Before stating Lemma 1.4, let us first state the following Whitney decomposition result of Coifman and Weiss (see [CoWe, Theorem III.1.3]).

Lemma 1.3. *If E is a proper open subset of a homogeneous space (S, d, μ), then there exists a family $\mathcal{F}$ of disjoint metric balls B and constants $1 < K_1 < K_2 < K_3$, M such that*

(a) $E = \bigcup_{B \in \mathcal{F}} K_1 B$.
(b) $\sum_{B \in \mathcal{F}} \chi_{K_2 B}(x) \le M \, \chi_E(x)$ *for all* $x \in S$.
(c) $K_3 B$ *intersects* E^c *for every* $B \in \mathcal{F}$.

Note that by examining the proof of Lemma 1.3 in [CoWe], it is easily verified that the constants K_1, K_2/K_1, and K_3/K_2 can be chosen arbitrarily and independently, provided that they exceed certain lower bounds. For the first and last of these constants, this is essentially trivial, while increasing K_2/K_1 corresponds to using smaller balls in the proof of this lemma.

Lemma 1.4. *Let G be an open subset of a homogeneous space (S, d, μ) and let $\mathcal{F}(G)$ be the set of metric balls contained in G. Suppose that for some $0 < q < p$ and non-negative $f \in L^p_{\mathrm{loc}}(\mu)$, there are constants $A > 1$ and $1 < \sigma_0 \le \sigma_0'$ such that*

$$\left(\fint_B f^p \, d\mu\right)^{1/p} \le A \left(\fint_{\sigma_0 B} f^q \, d\mu\right)^{1/q} \qquad \forall\, B : \sigma_0' B \in \mathcal{F}(G). \tag{1.5}$$

Then for any $0 < r < q$ and $1 < \sigma \le \sigma' < \sigma_0'$, there exists a constant $A' > 1$ such that

$$\left(\fint_B f^p \, d\mu\right)^{1/p} \le A' \left(\fint_{\sigma B} f^r \, d\mu\right)^{1/r} \qquad \forall\, B : \sigma' B \in \mathcal{F}(G). \tag{1.6}$$

In fact, we can choose $A' = C_0 A^s/(\sigma-1)^{C_0}$, where $s = (r^{-1}-p^{-1})/(q^{-1}-p^{-1})$ and C_0 is a sufficiently large constant independent of f, A, σ, and σ'.

Proof. Without loss of generality, we assume that $\sigma = \sigma' < \sigma_0$. Let $E \subset G$ be any metric ball. For simplicity, we normalise the metric and the measure so that $r_E = 1$ and $\mu(E) = 1$. Let W_∞ be the set of all metric balls in the Whitney decomposition of E with constants K_1, K_2, K_3 chosen so that K_2 is larger than $2K_4 \equiv 2\sigma_0' K_1$ (so that we can choose B in (1.5) to be the K_1-dilate of any Whitney ball). For all $k \geq 0$, let W_k be the set of all Whitney balls of radius greater than 2^{-k}, and let $E_k = \bigcup_{B \in W_k} K_1 B$. Since μ is doubling, there exists $0 < b < 1$ such that $\mu(B) \geq b^k$ for all $B \in W_k$, and hence there are at most Mb^{-k} balls in W_k. Letting $t = p(q-r)/q(p-r)$, we see that $0 < t < 1$ and $q^{-1} = tp^{-1} + (1-t)r^{-1}$. Now (1.5) and Hölder's inequality imply that

$$\left(\fint_{K_1 B} f^p \, d\mu \right)^{1/p} \leq A \left(\fint_{K_4 B} f^p \, d\mu \right)^{t/p} \left(\fint_{K_4 B} f^r \, d\mu \right)^{(1-t)/r}$$

and so

$$\int_{K_1 B} f^p \, d\mu \leq A^p \left(\fint_E f^r \, d\mu \right)^{(1-t)p/r} \left(\int_{K_4 B} f^p \, d\mu \right)^t \mu(B)^{(1-t)(1-p/r)}.$$

Now $\mu(B)^{(1-t)(1-p/r)} \leq b_1^{-k}$ for all $B \in W_k$, where $b_1 = b^{(1-t)(p-r)/p}$. Thus

$$\int_{E_k} f^p \, d\mu \leq \sum_{B \in W_k} \int_{K_1 B} f^p \, d\mu \leq A^p b_1^{-k} \left(\fint_E f^r \, d\mu \right)^{(1-t)p/r} \sum_{B \in W_k} \left(\int_{K_4 B} f^p \, d\mu \right)^t.$$

If $x \in K_4 B$ for some $B \in W_k$, then $d(x, E^c) \geq K_4 2^{-k}$ (since $K_2 > 2K_4$). On the other hand, if $x \in K_1 B'$, for some $B' \in W_\infty$, then $d(x, E^c) \leq d(x, z_{B'}) + d(z_{B'}, E^c) \leq (K_1 + K_3) r_{B'}$. It follows that if we fix an integer $m > \log_2[(K_1 + K_3)/K_4]$, then $\bigcup_{Q \in W_k} K_4 B \subset E_{k+m}$, and so writing $b_2 = b \cdot b_1$, we get

$$\int_{E_k} f^p \, d\mu \leq M A^p b_2^{-k} \left(\fint_E f^r \, d\mu \right)^{(1-t)p/r} \left(\int_{E_{k+m}} f^p \, d\mu \right)^t. \tag{1.7}$$

Iterating (1.7) we see that

$$\int_{E_k} f^p \, d\mu \leq \left[M A^p \left(\fint_E f^r \, d\mu \right)^{(1-t)p/r} \right]^{\alpha_l} b_2^{-\gamma_l} \left(\int_{E_{k+ml}} f^p \, d\mu \right)^{t^l}$$

where $\alpha_l = \sum_{j=0}^{l-1} t^j$ and $\gamma_l = \sum_{j=0}^{l-1} (k + mj) t^j$. Letting $l \to \infty$, we see that $\alpha_l \to (1-t)^{-1}$ and $\gamma_l \to (k(1-t) + mt)/(1-t)^2$ and so

$$\left(\fint_{E_k} f^p \, d\mu \right)^{1/p} \leq C A^{1/(1-t)} b_2^{-k/p(1-t)} \left(\fint_E f^r \, d\mu \right)^{1/r}. \tag{1.8}$$

If we choose k to be the least integer larger than $\log_2[(K_1+K_3)\sigma/(\sigma-1)]$, then $\sigma^{-1}E \subset E_k$, and so (1.6) follows for all admissible B by choosing $E=\sigma B$ in (1.8). The last statement of the theorem follows since $1/(1-t)=s$ and $b_2^{-k/p(1-t)} \le C(\sigma-1)^{-C_0}$ for sufficiently large C_0. $\square$

It was shown in [FLW] that, given a compact subset K of Ω and a ball $B=B(x,r)$, $r<r_0$, $x\in K$, there exist positive constants γ and c, depending on K and r_0 (or on B if we wish), so that

$$|J| \le c\left(\frac{r(J)}{r(I)}\right)^{N\gamma}|I| \tag{1.9}$$

for all balls I,J with $I\subset J\subset B$. We shall call γ the doubling order of Lebesgue measure for B. We always have $N\le N\gamma\le Q$, where Q is the homogeneous dimension defined previously. We can of course choose $N\gamma=Q$, but smaller values may arise for particular vector fields, and these values may vary with $B(x,r)$.

If $E\subset\Omega$ is open and $f\in C^1(E)$, we write

$$X_jf(x)=\langle X_j(x),\nabla f(x)\rangle,\ j=1,\dots,m,$$

and

$$|Xf(x)|^2=\sum_{j=1}^m |X_jf(x)|^2,$$

where ∇f is the usual gradient of f and $\langle\ ,\ \rangle$ is the usual inner product on $\mathbb{R}^N$.

We now state the unweighted version of our main Poincaré estimate, which essentially generalises the main result of [BK] and extends the main result of [L2].

Theorem 1.10. *Let K be a compact subset of Ω. There exists $r_0>0$ depending on K, Ω and $\{X_j\}$ such that if $E=B(x,r)$ is a ball with $x\in K$ and $0<r<r_0$, and if $1/q=1/p-1/(N\gamma)$, $0<p<1$, where γ is defined by (1.9) for E, then there exists a constant f_E such that*

$$\left(\frac{1}{|E|}\int_E |f(x)-f_E|^q\,dx\right)^{\frac{1}{q}} \le C_0 r\left(\frac{1}{|E|}\int_E |Xf(x)|^p\,dx\right)^{\frac{1}{p}}$$

for any $f\in C^1(E)$ provided that $|Xf|\in WRH_1(E)$. The constant C_0 depends on p, K, Ω, $\{X_j\}$, the $WRH_1(E)$-constant, and the constants c and γ in (1.9). We may choose $f_E=|E'|^{-1}\int_{E'}f(x)\,dx$ for any compactly contained sub-ball E' of E, (in which case the constant C_0 also depends on the choice of E').

This result and its proof is a hybrid of the main unweighted results of [FLW] and [L2] with the main result of [BK]. As mentioned earlier, we may always choose $N\gamma=Q$.

Note that if we assume Theorem 1.10 for a particular choice of the sub-ball E', and use also the corresponding Poincaré inequality for $p=1$ (as in [FLW]), a standard argument gives Theorem 1.10 for all valid choices of E'. Accordingly we shall prove this theorem only when E' is the "central" ball in an appropriate Whitney decomposition of E. Similarly, if $f\in L^1(E)$, we readily see that f_E can be chosen to be the average of f over all of E (as it is usually defined when $p \geq 1$). Similar comments apply to the weighted results below which, for simplicity, we state only for a particular choice of f_E.

Given $0 < p < 1, p < q < \infty$, and a metric ball $E \subset \Omega$, we shall be interested in weights w_1, w_2 on E for which the following balance condition holds:

$$\frac{r(I)}{r(J)}\left(\frac{w_2(I)}{w_2(J)}\right)^{\frac{1}{q}} \leq c\left(\frac{w_1(I)}{w_1(J)}\right)^{\frac{1}{p}} \tag{1.11}$$

for all metric balls I, J with $I \subset J \subset E$. Note that in the case of Lebesgue measure ($w_1 = w_2 = 1$), (1.11) reduces to (1.9) if $1/q = 1/p - 1/(N\gamma)$. A balance condition of type (1.11) was introduced previously in [CW] to study weighted Poincaré inequalities (see also [FGuW], [FLW], and [L1]). We shall use notation such as $(1.11)_{p_0,q_0}$ when we wish to refer to (1.11) with parameters $p = p_0$ and $q = q_0$.

We now state a weighted Poincaré inequality for $0 < p < 1$, $p < q$, which complements the case $1 \leq p < q$ considered in [FLW]. It also generalises Theorem 1.10, since Lebesgue measure is doubling on small balls centred in K.

Theorem 1.12. *Let K be a compact subset of Ω. There exists $r_0 > 0$ depending on K, Ω and $\{X_j\}$ such that if $E = B(x,r)$ is a ball with $x \in K$ and $0 < r < r_0$, if $0 < p < 1, p < q < \infty$, and if w_1, w_2 are weights satisfying the balance condition (1.11) for E, with $w_1 \in A_1(E)$ and w_2 α-strongly doubling on E for sufficiently small $\alpha = \alpha(\Omega) > 0$, then*

$$\left(\frac{1}{w_2(E)}\int_E |f(x) - f_E|^q w_2(x)\,dx\right)^{\frac{1}{q}} \leq C_0 r\left(\frac{1}{w_1(E)}\int_E |Xf(x)|^p w_1(x)\,dx\right)^{\frac{1}{p}} \tag{1.13}$$

for any $f \in C^1(E)$, with $f_E = w_2(\frac{1}{2}E)^{-1}\int_{\frac{1}{2}E} f(x)w_2(x)\,dx$, provided that $|Xf| \in WRH_1(E,w_1)$. The constant C_0 depends only on p, K, Ω, $\{X_j\}$, the $WRH_1(E,w_1)$-constant, and the constants in the conditions imposed on w_1 and w_2.

For the necessity of (1.11) see Section 2. In the particular case when $w \equiv w_1 = w_2$ is α-strongly doubling (for $\alpha < 1/11$, say), we claim that (1.11) is always satisfied for some $q^{-1} = p^{-1} - \delta$ where $\delta \in (0, 1/p)$ is dependent only on the strong doubling constant. This is clearly true if $r_I > r_J/10$, so we assume $r_I < r_J/10$. We may also assume that $\varrho(z_J, z_I) > r_J/10$.

We claim that $I \subset J_1 \subset J$ for some metric ball J_1 for which $r_{J_1} = r_J/10$. To construct J_1, let $g : [0,s] \to J$, $s < r_J$, be the geodesic curve for which $g(0) = z_J$ and $g(s) = z_I$. Let B_t be the ball of radius $r_J/10 - r_I$, and centre $g(t)$, and let $t_0 = \inf\{0 < t \leq s : z_I \in B_t\}$. Since g is a geodesic, it readily follows that $I \subset J_1 \subset J$ if $r_{J_1} = r_J/10$ and $z_{J_1} = g(t_0)$.

Clearly, $w(J_1)/w(J) > \beta > 0$ for some $0 < \beta < 1$ dependent only on the strong doubling constant. Continuing this process, we can create a finite nested sequence of metric balls J_i of radius $r_J/10^i$ whose final member J_m contains I but has radius not more than $10r_I$. It follows easily that $w(I)/w(J) > C\beta^m$ if $r_I/r_J < 10^{-m}$, establishing our claim.

The following one-weighted corollary of Theorem 1.12 now follows.

Corollary 1.14. *Let K be a compact subset of Ω. Then there exists r_0 depending on K, Ω and $\{X_j\}$ such that if $E = B(x,r)$ is a ball with $x \in K$ and $0 < r < r_0$, if $w \in A_1(E, \varrho, dx)$ is α-strongly doubling on E for sufficiently small $\alpha = \alpha(\Omega) > 0$, and if $q \le q_0$, $0 < p \le 1$, where $p < q_0 = q_0(p,w)$, then*

$$\left(\frac{1}{w(E)} \int_E |f(x) - f_E|^q w(x)\, dx \right)^{\frac{1}{q}} \le C_0 r \left(\frac{1}{w(E)} \int_E |Xf(x)|^p w(x)\, dx \right)^{\frac{1}{p}}$$

for any $f \in C^1(E)$ provided that $|Xf| \in WRH_1(E,w)$. The constant C_0 depends on p, K, Ω, $\{X_j\}$, w, the $WRH_1(E)$-constant, and the constants c and γ in (1.9). Also, we may take $f_E = w(\frac{1}{2}E)^{-1} \int_{\frac{1}{2}E} f(x)w(x)\, dx$.

As an example, Corollary 1.14 is valid for $w(x) = [\varrho(x, E^c)]^r$, for all $r > 0$. Trivially such weights are in $A_1(E)$. Also, the doubling property $w(2B \cap E) \le Cw(B \cap E)$ follows immediately from (strong) doubling for Lebesgue measure if $4B \subset E$, so it suffices to prove the strong doubling property of w for balls "near" the boundary. Suppose therefore that B is a metric ball with $\alpha B \subset E$ for some fixed $\alpha > 0$, but $4B \not\subset E$. By a geodesic argument similar to that used in the paragraphs before Corollary 1.14, we see that there is some ball $B' \subset \alpha B$ whose radius and distance to the boundary of E are both at least $\alpha r_B/4$. It then follows that $w(B \cap E) \ge (\alpha r_B/4)^r |B'|$, while $w(2B \cap E) \le (6r_B)^r |2B| \le C r_B^r |B'|$, by the strong doubling property of Lebesgue measure.

Similarly for any $r_1, r_2 > 0$, we can take $w_i(x) = [\varrho(x, E^c)]^{r_i}$ in Theorem 1.12, as long as these weights satisfy the balance condition.

We mention in passing that it is possible to use the Poincaré estimates above to derive analogous estimates for domains other than balls. In particular, our proof of the Poincaré inequalities can be naturally generalised to handle domains which satisfy the Boman chain condition. Such a generalisation is given after the end of the proof of Theorem 1.12 in §2.

§2. Proof of the Poincaré estimates: Theorems 1.10 and 1.12

As noted in the introduction, Theorem 1.10 is a special case of Theorem 1.12. To prove Theorem 1.12, we first derive a weaker version in which the domain of integration on the left-hand side of the Poincaré inequality is a ball B and, on the right-hand side, it is cB for some $c > 1$. This weaker version for a given B with $2cB \subset E$ will follow readily from the weak reverse Hölder inequality applied to the results of [FLW]. Standard arguments then easily give the case $rB \subset E$ for any $r > 1$, but the case $r = 1$ we want requires more work. The rest of the proof of Theorem 1.12 consists of covering E by a collection of metric sub-balls and chaining together the associated weak inequalities to recover this stronger one. This chaining argument is similar to the proof of the sharp Poincaré inequality for $p > 1$ and $w_1 = w_2 = 1$ in the Carnot-Carathéodory metric case given in [L2], but with the balance condition (1.11) replacing Lemma 4.2 of [L2]. Similar arguments have also been used for the Euclidean case in [Bo], [IN] (see also [BK], [Ch], and [FGuW]).

To carry out the chaining argument, we need to define the so-called Boman chain condition in the context of a homogeneous space. The definition given below may seem slightly different from the corresponding version in Euclidean space, but it suffices for our purpose.

Definition 2.1. Let (S, d, μ) be a homogeneous space in the sense of Coifman-Weiss. A domain (i.e. a connected open set) E in S is said to satisfy the *Boman chain condition* if there exist constants M, $\lambda > 1$, $C_2 > C_1 > 1$, and a family $\mathcal{F}$ of disjoint metric balls B such that

(i) $E = \bigcup_{B \in \mathcal{F}} C_1 B$.

(ii) $\sum_{B \in \mathcal{F}} \chi_{C_2 B}(x) \leq M\, \chi_E(x)$ for all $x \in S$.

(iii) There is a so-called "central ball" $B_* \in \mathcal{F}$ such that for each ball $B \in \mathcal{F}$, there is a positive integer $k = k(B)$ and a chain of balls $\{B_j\}_{j=0}^k$ such that $B_0 = B$, $B_k = B_*$, and $C_1 B_{j-1} \bigcap C_1 B_j$ contains a metric ball D_j whose volume is comparable to those of both B_{j-1} and B_j for all $1 \leq j \leq k$.

(iv) $B \subset \lambda B_j$, for all $j = 0, \ldots, k(B)$.

We shall call such a set E a *(Boman) chain domain.* We shall refer to individual chains as (λ, C_1, C_2)-chains if we wish to specify the parameters. Clearly all chain domains are bounded. M is a "dimensional constant" which is of no great concern to us. If λ is much larger than C_1 and C_2, it indicates the domain is "bad" (for instance, it may be very elongated or it may have narrow bottlenecks). C_1 and C_2 are not important, as there is a lot of freedom in their choice. Trivially for instance, λ, C_1, and C_2 can all be multiplied by the same factor larger than 1 (while holding M constant) if we shrink the balls accordingly. Also, whenever we shall need to assume a domain is a chain domain, the proof can be altered easily to work no matter how much we weaken the chain condition by increasing C_1 or decreasing C_2, as long as $C_1 < C_2$. Let us therefore assume $C_1 = 2$ and $C_2 = 10$, unless otherwise specified.

If $S = \mathbb{R}^N$ and d is the Euclidean metric, this is the standard Boman chain condition, and it is known to be satisfied by bounded Lipschitz domains, (ϵ, ∞) domains, and John domains. In the general homogeneous space, it is difficult to determine whether or not a domain is a chain domain. However, in the Carnot-Carathéodory case, we have the following result which we formally state for future reference.

Lemma 2.2. [L2] *Let $E \subset \Omega$ be a metric ball. Then E is a chain domain.*

We note that in [BKL], metric John domains are defined in general homogeneous spaces and are shown to be chain domains. This allows one to state a version of Lemma 2.2 for metric balls satisfying a weak geodesic condition in an arbitrary homogeneous space which satisfy a certain quasigeodesic condition.

The following lemma is a generalisation of a lemma of Boman [Bom] (where it is stated for the unweighted Euclidean case with $E = \mathbb{R}^n$). A version for doubling weights in $\mathbb{R}^n$ is stated in [ST, p. 1055].

Lemma 2.3. *Suppose $1 < \lambda$, $1 \leq p < \infty$ and w_2 is an α-strong doubling weight for sufficiently small $\alpha = \alpha(S, \lambda) > 0$ on a subset E of a homogeneous space S. Let $\{B_\beta\}_{\beta \in I}$ be an arbitrary family of open metric balls contained in E, and let $\{a_\beta\}_{\beta \in I}$ be non-negative numbers. Then*

$$\Big\| \sum_{\beta \in I} a_\beta \chi_{\lambda B_\beta \cap E} \Big\|_{L^p_{w_2}(E)} \leq C \Big\| \sum_{\beta \in I} a_\beta \chi_{B_\beta} \Big\|_{L^p_{w_2}(E)},$$

where C is independent of $\{a_\beta\}$ and $\{B_\beta\}$.

As the above lemma is proved in a similar fashion to all previous versions, we omit a formal proof. As with the other versions, the proof reduces to the boundedness of the relevant maximal operator M on the conjugate space $L^{p'}_{w_2}(E)$, which in turn follows from the boundedness of M from $L^1_{w_2}(E)$

to weak-$L^1_{w_2}(E)$. For this last result, we simply use the correct covering lemma, specifically Theorem III.1.2 of [CoWe]. The maximal operator needed here is

$$Mg(x) = \sup_{B(x,r)} \frac{1}{w_2(B(x,r))} \int_{B(x,r)} |g(y)| w_2(y)\, dy.$$

where the supremum is taken over all balls B for which $\lambda^{-1}B \subset E$.

Proof of Theorem 1.12. For any (compactly contained) metric sub-ball B of E used in this proof, f_B will denote the average $w_2(B)^{-1} \int_B f(x)w_2(x)\,dx$ (rather than an average over $(1/2)B$). By assumption, w_1, w_2 satisfy the condition $(1.11)_{p,q}$ with $p < 1$, and so they also satisfy $(1.11)_{1,q}$. Thus, if B is any metric ball for which $c\sigma B \subset E$, it follows from [FLW] that,

$$\left(\frac{1}{w_2(B)} \int_B |f(x) - f_B|^q w_2(x) dx\right)^{1/q} \leq C\, r_B \left(\frac{1}{w_1(B)} \int_{cB} \Big(\sum_{i=1}^m |X_i f|\Big) w_1(x) dx\right). \tag{2.4}$$

Since $|Xf| \in WRH_1(E, w_1)$, we deduce that

$$\left(\frac{1}{w_2(B)} \int_B |f(x) - f_B|^q w_2(x) dx\right)^{1/q} \leq C\, r_B \left(\frac{1}{w_1(B)} \int_{c\sigma B} \Big(\sum_{i=1}^m |X_i f|\Big)^p w_1(x) dx\right)^{1/p}. \tag{2.5}$$

By an easy covering argument, we can (for the sake of simplicity) reduce (2.5) to

$$\left(\frac{1}{w_2(B)} \int_B |f(x) - f_B|^q w_2(x) dx\right)^{1/q} \leq C\, r_B \left(\frac{1}{w_1(B)} \int_{2B} \Big(\sum_{i=1}^m |X_i f|\Big)^p w_1(x) dx\right)^{1/p}. \tag{2.6}$$

We note that (2.6) is equivalent to

$$\int_B |f - f_B|^q w_2 \leq C\, r_B^q\, w_2(B) w_1(B)^{-q/p} \left(\int_{2B} \Big(\sum_{i=1}^m |X_i f|\Big)^p w_1\right)^{q/p}, \tag{2.7}$$

for the p, q, w_1, w_2 satisfying the balance condition (1.11).

Next fix the central ball B_* as in the definition of the chain domain selected by Lemma 2.2. We have

$$\begin{aligned} \|f - f_{2B_*}\|^q_{L^q_{w_2}(E)} &\leq \max(1, 2^{q-1}) \sum_{B \in \mathcal{F}} \|f - f_{2B}\|^q_{L^q_{w_2}(2B)} \\ &\quad + \max(1, 2^{q-1}) \sum_{B \in \mathcal{F}} \|f_{2B} - f_{2B_*}\|^q_{L^q_{w_2}(2B)} \\ &= \mathrm{I} + \mathrm{II}. \end{aligned} \tag{2.8}$$

Before we proceed, let us note that, in a completely abstract setting, $(f, g) \mapsto \|f - g\|_{L^q}^p$ is a metric on L^q whenever $0 < p < \min(1, q)$. This follows easily from the fact that $(f, g) \mapsto \|f - g\|_{L^q}^t$ is a metric for $t = \min(1, q)$, and the fact that a metric raised to a power strictly between 0 and 1 is also a metric (and $p, p/q < 1$).

Now we temporarily fix $B \in \mathcal{F}$ and consider the chain $\mathcal{F}(B) = \{B_0, \ldots, B_{k(B)}\}$ constructed according to Lemma 2.2. Thus

$$\begin{aligned}
\|f_{2B} - f_{2B_*}\|^p_{L^q_{w_2}(2B)} &\le C \sum_{j=0}^{k(B)-1} \|f_{2B_j} - f_{2B_{j+1}}\|^p_{L^q_{w_2}(2B)} \\
&\le C \sum_{j=0}^{k(B)-1} \Big(\frac{w_2(B)}{w_2(4B_j \bigcap 4B_{j+1})}\Big)^{p/q} \\
&\qquad \cdot \|f_{2B_j} - f_{2B_{j+1}}\|^p_{L^q_{w_2}(4B_j \cap 4B_{j+1})} \\
&\le C \sum_{j=0}^{k(B)-1} \Big(\frac{w_2(B)}{w_2(B_j)}\Big)^{p/q} \|f - f_{2B_j}\|^p_{L^q_{w_2}(4B_j)} \\
&\qquad + C \sum_{j=0}^{k(B)-1} \Big(\frac{w_2(B)}{w_2(B_{j+1})}\Big)^{p/q} \|f - f_{2B_{j+1}}\|^p_{L^q_{w_2}(4B_{j+1})} \\
&\le 2\, C \sum_{j=0}^{k(B)} \Big(\frac{w_2(B)}{w_2(B_j)}\Big)^{p/q} \|f - f_{2B_j}\|^p_{L^q_{w_2}(4B_j)} .
\end{aligned}$$

We observe that

$$\|f - f_{2B_j}\|^p_{L^q_{w_2}(4B_j)} \le \|f - f_{4B_j}\|^p_{L^q_{w_2}(4B_j)} + \|f_{4B_j} - f_{2B_j}\|^p_{L^q_{w_2}(4B_j)} ,$$

and

$$\begin{aligned}
\|f_{4B_j} - f_{2B_j}\|_{L^q_{w_2}(4B_j)} &\le w_2(4B_j)^{1/q} \left(\frac{1}{w_2(2B_j)} \int_{2B_j} |f - f_{4B_j}| w_2 \right) \\
&\le C \left(\int_{4B_j} |f - f_{4B_j}|^q w_2 \right)^{1/q} .
\end{aligned}$$

Therefore, we get

$$\|f_{2B} - f_{2B_*}\|^p_{L^q_{w_2}(2B)} \le C \sum_{j=0}^{k(B)-1} \Big(\frac{w_2(B)}{w_2(B_j)}\Big)^{p/q} \|f - f_{4B_j}\|^p_{L^q_{w_2}(4B_j)} .$$

Since, by the chain condition, $B \subset \lambda B_j$ for each $B_j \in \mathcal{F}(B)$, we then have

$$\begin{aligned}\|f_{2B} - f_{2B_*}\|^p_{L^q_{w_2}(2B)} \frac{\chi_B(\xi)}{w_2(B)^{p/q}} &\le C \sum_{A\in\mathcal{F}} \Big(\frac{1}{w_2(A)}\Big)^{p/q} \|f - f_{4A}\|^p_{L^q_{w_2}(4A)} \chi_{\lambda A}(\xi) \\ &\le C \sum_{A\in\mathcal{F}} \Big(r_A w_1(A)^{-1/p} ||Xf||_{L^p_{w_1}(8A)}\Big)^p \chi_{\lambda A}(\xi) \\ &= C \sum_{A\in\mathcal{F}} a_A\, \chi_{\lambda A}(\xi)\,.\end{aligned}$$

In the above expression, a_A is notationally defined in an obvious way. For the term II in (2.8), we have

$$\begin{aligned}\text{II} &\le C \sum_{B\in\mathcal{F}} \int_E \big(\|f_{2B} - f_{2B_*}\|^p\big)^{q/p}_{L^q_{w_2}(2B)} \frac{\chi_B(\xi)}{w_2(B)} w_2(\xi) d\xi \\ &\le C \int_E \Big| \sum_{A\in\mathcal{F}} a_A\, \chi_{\lambda A}\Big|^{q/p} w_2(\xi) d\xi.\end{aligned}$$

Since $q/p \ge 1$, we can use Lemma 2.3 to get

$$\text{II} \le C \int_E \Big| \sum_{A\in\mathcal{F}} a_A\, \chi_A\Big|^{q/p} w_2(\xi) d\xi.$$

Since the balls in $\mathcal{F}$ are disjoint, we have

$$\text{II} \le C \sum_{A\in\mathcal{F}} a_A^{q/p} \int_E \chi_A(\xi) w_2(\xi) d\xi \le C \sum_{A\in\mathcal{F}} \Big(r_A^p w_1(A)^{-1} ||Xf||^p_{L^p_{w_1}(8A)}\Big)^{q/p} w_2(A)\,.$$

Therefore,

$$\begin{aligned}\text{II} &\le C \sum_{A\in\mathcal{F}} w_2(A) w_1(A)^{-q/p} r_A^q \left(\int_{8A} \Big(\sum_{i=1}^m |X_i f|\Big)^p w_1\right)^{q/p} \\ &\le C\, w_2(E) w_1(E)^{-q/p} r_E^q \sum_{A\in\mathcal{F}} \left(\int_{8A} \Big(\sum_{i=1}^m |X_i f|\Big)^p w_1\right)^{q/p} \\ &\le C\, w_2(E) w_1(E)^{-q/p} r_E^q \left(\int_{E} \Big(\sum_{i=1}^m |X_i f|\Big)^p w_1\right)^{q/p}.\end{aligned} \tag{2.9}$$

In the last inequality we used the fact that $q \ge p$, and that $\sum_{A\in\mathcal{F}} \chi_{8A}(\xi) \le C\chi_E(\xi)$, while in the middle inequality, we used the balance condition (1.11).

For the term I in (2.8), the estimate is the same by replacing $4A$ by $2A$ in the estimate of II. □

Remarks.

(1) The argument used above in order to obtain Theorem 1.12 from its weaker version can be adapted to derive an analogue of Theorem 1.12 for suitable chain domains. In fact, let D be a chain domain as defined above. Define

$$A = \sup_{\substack{I,J \\ C_2 I, C_2 J \subset D \\ I \subset \lambda J}} \left\{ \frac{r(I)}{r(J)} \left(\frac{w_2(I)}{w_2(J)} \right)^{\frac{1}{q}} \left(\frac{w_1(I)}{w_1(J)} \right)^{-\frac{1}{p}} \right\}.$$

If $\bar{D}$ is a compact subset of Ω whose diameter is small compared with its distance to the boundary, then we only have to change all parenthesised instances of "E" to "B_*" in (2.9) above to get a proof of the following Poincaré inequality:

$$\|f - f_D\|_{L^q_{w_2}(D)} \le C_0 \|Xf\|_{L^p_{w_1}(D)},$$

where f_D is the w_2-average of f over B_*, and C_0 now depends also on D.

(2) As for the case $p \ge 1$, some sort of balance condition is needed to prove a two-weighted inequality like (1.13) (assuming w_1, w_2 are as in Theorem 1.12). Let us show that (1.11) must hold for all $I \subset J = E$. It suffices by strong doubling to prove this for $r_I < r_J/100$, $8I \subset J$. There is a bump function f which equals 1 on I, is supported on $2I$, and such that $|Xf| \le C/r_I$ (see [L1, Lemma 7.12]). It is not clear that $|Xf| \in WRH_1(E)$, but Lemma 2.1 of [BK] can be adapted to show that $g = M(|Xf|^2)^{1/2}$ belongs to $WRH_1(4I)$, where $M(|Xf|^2)$ is the restricted maximal function

$$M(|Xf|^2)(x) = \sup_{\substack{B=B(x,r) \\ 2B \subset 4I}} \frac{1}{w_2(B)} \int_B |Xf|^2 w_2 dx.$$

Extending g to be zero on $E \setminus 4I$, it is clear that $g \in WRH_1(E)$ (since it has compact support in $3I$) and that $|Xf| \le g$. The proof of Theorem 1.12 and the L^p-boundedness of the maximal function then show that (1.13) holds for f. By the doubling property of w_2, we see that for $t \in \{0,1\}$, $w_2(\{x \in E \,|\, f(x) \ne t\}) \ge C w_2(I)$ and so, no matter what choice of constant f_E we make, the left hand side of (1.13) is at least $C(w_2(I)/w_2(E))^{1/q}$. However, the right-hand side of (1.13) is at most

$$C \frac{r(E)}{r(I)} \left(\frac{w_1(I)}{w_1(E)} \right)^{1/p},$$

and (1.11) readily follows. Thus, if we have a weighted Poincaré inequality on E and on all of its sub-balls (with a uniform constant C_0), (1.11) holds for all $I \subset J \subset E$.

(3) One can also prove some results for more general domains, for example metric versions of the John-α domains defined in [BK]. We shall not state or prove such results here as the statements lack elegance and we suspect this method does not give sharp results for such domains. Suffice it to say that one can prove such results in certain cases where it is possible to modify the above proof as in [BK] to avoid the use of Lemma 2.3. In particular this can be done if $q < 1$.

Finally, let us give an application of the above to solutions of a class of quasilinear subelliptic equations. We show that the conclusion of Theorem 1.12 holds for these solutions with f_E replaced by $f(x)$, where x is an arbitrary point of $\frac{1}{2}E$. In order to deduce this from Theorem 1.12 we have to verify two facts: a reverse Hölder inequality for $|Xu|$ when u is a solution, and an estimate for $|u(x)-u_E|$, where u_E is the average of u in $\frac{1}{2}E$. The first of these is a consequence of a Caccioppoli type inequality and the second essentially follows from a mean-value inequality established in [L4].

Results of this type have appeared in [Z], [BK] (with $X_i = \frac{\partial}{\partial x_i}$), and in [L5] for Hörmander families. In particular we shall show here that Theorem 5.14 of [BK] for $0<p<1$ remains true for Hörmander vector fields, thereby extending the result of [L5] to the case $0<p<1$.

We consider quasilinear second order subelliptic partial differential equations of the form

$$\sum_{j=1}^{m} X_j^* A_j(x,u,X_1u,X_2u,\cdots,X_mu) + B(x,u,X_1u,X_2u,\cdots,X_mu) = 0\,, \tag{2.10}$$

under certain structral assumptions. Harnack inequalities for weak solutions, subsolutions, and supersolutions of (2.10) have been established in [L4].

We let x, η denote vectors in $\mathbb{R}^N$ and $\mathbb{R}^m$ respectively, and $Xu = (X_1u,\cdots,X_mu)$. $A(x,u,\eta) = (A_1(x,u,\eta),\cdots,A_m(x,u,\eta))$ and $B(x,u,\eta)$ are, respectively, vector and scalar measurable functions defined on $\Omega\times\mathbb{R}\times\mathbb{R}^m$, where Ω is a domain in $\mathbb{R}^N$.

For all $M<\infty$ and for all $(x,u,\eta)\in\Omega\times(-M,M)\times\mathbb{R}^N$, we assume the structure of (2.10) satisfies the following inequalities:

$$\begin{aligned} |A(x,u,\eta)| &\le a_0|\eta|^{s-1} + (a_1(x)|u|)^{s-1}\,, \\ \eta\cdot A(x,u,\eta) &\ge |\eta|^s - (a_2(x)|u|)^s \\ |B(x,u,\eta)| &\le b_0|\eta|^s + b_1(x)|\eta|^{s-1} + (b_2(x))^s\,|u|^{s-1} \end{aligned} \tag{2.11}$$

where $s>1$, a_0, b_0 are constants, $a_i(x), b_i(x)$ are nonnegative measurable functions; $s, a_0, b_0, a_i(x), b_i(x)$ may possibly depend on M. All the coefficients $a_i(x), b_i(x)$ are assumed to be in certain subspaces of $L^t_{\mathrm{loc}}(\Omega)$, where $t=\max(s,Q)$ (see [L4] for the details).

We now define the notion of a solution to the equation (2.10). First, for a domain $D\subset\Omega$, the Sobolev class $W^{1,s}(D)$ ($W^{1,s}_0(D)$), is defined as the closure of C^1-smooth (and compactly supported) functions in the norm $||u|| = ||u||_{s,D} + ||Xu||_{s,D}$. Here $||\cdot||_{s,D}$ is the usual L^s-norm in D. Integration by parts shows that each $u\in W^{1,s}(D)$ has a unique subgradient $Xu\in L^s(D)$. We say u is a weak solution of (2.10) in Ω if u belongs to $W^{1,s}_{\mathrm{loc}}(\Omega)$ and

$$\int_\Omega \{X\phi\cdot A(x,u,Xu) - \phi B(x,u,Xu)\}\,dx = 0$$

for all bounded $\phi(x)\in W^{1,s}_0(\Omega)$.

Given $a_i(x), b_i(x)\in L^Q_{\mathrm{loc}}(\Omega)$, $u(x)\in L^\infty_{\mathrm{loc}}$, a standard approximation argument shows that, if the above equation holds for all $\phi(x)\in C^1_0(\Omega)$, then it still holds for all $\phi(x)$ given in the definition.

As an application of Theorem 1.10 we obtain the following result. We note that the boundedness assumption on u can be dropped if $b_0=0$ (and (2.11) is true for all $M>0$ with parameters independent of M).

Theorem 2.12. *Let K be a compact subset of Ω. There exists r_0 such that whenever $E = B(x, r)$, $x \in K$, and $0 < r < r_0$, the following holds. Let $0 < p < 1$ and $1/q = 1/p - 1/(N\gamma)$, where γ is defined by (1.9) for E (we can always choose $N\gamma = Q$). Let $u \in W^{1,s}_{\mathrm{loc}}(E)$, $|u| \le M$, be a weak solution of (2.10). Then there is a constant C depending on the structure conditions (2.11), p, s, q, $||u||_{s,\frac{1}{2}E}$, Ω, and b_0M such that for any point $x_0 \in \frac{1}{2}E$*

$$\left(\frac{1}{|E|}\int_E |u - u(x_0)|^q\right)^{\frac{1}{q}} \le Cr\left(\frac{1}{|E|}\int_E |Xu|^p\right)^{\frac{1}{p}},$$

where $\rho(E)$ is the radius of E.

In order to deduce Theorem 2.12 from Theorem 1.10 we have to verify that Xu satisfies a weak reverse Hölder inequality and then replace the average of u over $\frac{1}{2}E$ by $u(x_0)$. We first establish a weak reverse Hölder inequality.

Caccioppoli Estimate. If $u \in W^{1,s}(E)$ is a solution of (2.10), then for any metric ball B such that $2B \subset E$, any $\xi \in C_0^\infty(2B)$, and any constant c, we have

$$\int_{2B} \xi^s |Xu|^s \le C \int_{2B} \left(|\xi|^s + |X\xi|^s\right) |u - c|^s.$$

This is a special case of formula (3.26) in [L4] derived by setting $q = 1$ and replacing u by $u - c$ (note that $u - c$ is a solution to an equation also satisfying structural conditions (2.11)). In particular, if we select a cut-off function relative to the Carnot-Carathéodory metric (see Lemma (7.12) in [L1]), we can choose $0 \le \xi \le 1$, $\xi \in C_0^\infty(2B)$, $\xi = 1$ on B and $|X\xi| \le Cr_B^{-1}$. Thus

$$\int_B |Xu|^s \le Cr_B^s \int_{2B} |u - c|^s,$$

and so by using the Sobolev-Poincaré inequality on $2B$ (see [L2]) to control the right-hand side of the above inequality we see that $|Xu| \in WRH_s(E)$.

Therefore, the solutions satisfy the hypothesis of Theorem 1.10. Let $B = \frac{1}{2}E$, and let u_B be the average value of u on B. For simplicity, we normalise so that $r_B = 1$. Now

$$|u(x) - u(x_0)| \le |u(x) - u_B| + |u_B - u(x_0)|,$$

so it suffices to control the q-integrals of the two right-hand side terms. For the first term, we use Theorem 1.10. By Hölder's inequality, and Theorem 2.13 below, the second term can be bounded the $L^s(B)$-norm of $|u - u_B|$. Using the classical Poincaré inequality, this can then be bounded by the L^s-norm of $|Xu|$, which is comparable with the L^p-norm of $|Xu|$ by Lemma 1.4, since we have shown that $|Xu| \in WRH_s(E)$. This reasoning establishes Theorem 2.12.

Theorem 2.13. *Suppose u is a weak solution of (2.10) in a metric ball $2B$, $|u| \le M$ in $2B$. Then*

$$\max_{x \in B} |u(x) - u_B| \le C\left(\frac{1}{|B|}\int_B |u - u_B|^q\right)^{\frac{1}{q}}$$

for any $q > s - 1$, where C depends on s, Q, the structure conditions (2.11), and b_0M.

We omit the proof of this theorem, as it follows readily from Theorem 3.15 of [L4]).

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