SOBOLEV-POINCARÉ INEQUALITIES FOR $\mathbf{p < 1}$

Stephen M. Buckley and Pekka Koskela

Abstract. If Ω is a John domain (or certain more general domains), and $|\nabla u|$ satisfies a certain mild condition, we show that $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ satisfies a Sobolev-Poincaré inequality $\left(\int_\Omega |u-a|^q\right)^{1/q} \le C\left(\int_\Omega |\nabla u|^p\right)^{1/p}$ for all $0 < p < 1$, and appropriate $q > 0$. Our conclusion is new even when Ω is a ball.

Department of Mathematics
University of Michigan
Ann Arbor, MI 48109
USA

1991 *Mathematics subject classifications:* 46E35, 35J60, 42B25, 26D10, 31B25, 30C65.
The first author was supported in part by NSF Grant No. DMS-9207715 and the second by NSF Grant No. DMS-9305742.

1. Introduction

The (q,p)-Poincaré (or Sobolev-Poincaré) inequality

$$\inf_{a\in\mathbf{R}}\left(\int_\Omega |u-a|^q\right)^{1/q} \le C\left(\int_\Omega |\nabla u|^p\right)^{1/p} \tag{1.1}$$

is known to be true for all $u \in W^{1,p}(\Omega)$ and $1 \le p < n$, $0 < q \le np/(n-p)$, if Ω is a bounded Lipschitz domain, or even a John domain (see [Boj], [M]). There are simple examples which show that this inequality is false for all $p < 1$, even if q is very small, Ω is a ball, and u is smooth (one such example is given near the end of Section 1). Nevertheless, we shall show that, under a rather mild condition on ∇u, one can prove such an inequality in any John domain for all $0 < p < 1$ (see Theorem 1.5).

Presumably because of the simple counterexamples, there has been almost no previous research on Poincaré-type inequalities for $p < 1$. One notable exception is to be found in [K], where the assumptions (quasiconformality and $0 < q < np/(n-p)$) are stronger than in Theorem 1.5.

We shall state the first version of the Poincaré inequality in Theorem 1.5, and prove it in Section 3 after a couple of preparatory lemmas in Section 2. Section 4 contains a weak "converse" to Theorem 1.5 and, finally, Section 5 contains some related results.

Let us first introduce some necessary notation and terminology. A cube Q is always assumed to have faces perpendicular to the coordinate directions, $l(Q)$ is the sidelength of Q, and rQ is the concentric dilate of Q by a factor $r > 0$. Ω will refer to a domain in $\mathbf{R}^n$, assumed to be bounded unless otherwise stated. For any exponent $p > 0$, we write $p' = p/(p-1)$ and $p^* = np/(n-p)$. For any measureable $S \subset \mathbf{R}^n$, $0 < |S| < \infty$,

$$\begin{aligned} \|w\|_{p,S} &= \left(⨍_S |w|^p\right)^{1/p} = \left(\frac{1}{|S|}\int_S |w|^p(x)\,dx\right)^{1/p}, \qquad p \in \mathbf{R}\setminus\{0\} \\ \|w\|_{\infty,S} &= \operatorname*{ess\,sup}_{x\in S} w. \end{aligned}$$

Also, $w_S = \|w\|_{1,S}$. For any open set $G \subset \mathbf{R}^n$, we use M_G to refer to the following local version of the Hardy-Littlewood maximal operator (if $G = \mathbf{R}^n$, we simply write Mf):

$$M_G f(x) = \sup_{\substack{x\in Q\\ 2Q\subset G}} ⨍_Q |f|, x \in G.$$

In proofs, C will be used to refer to any constant which plays no significant rôle in the proof. We also use $A \lesssim B$ as a synonym for $A \le CB$ and $A \approx B$ for $A \lesssim B \lesssim A$.

The following lemma is a version of the Whitney decomposition, as found in [S]. We shall denote by $\mathcal{W}(\Omega)$ the collection of cubes $\{Q_j\}$ in the case $A = 20$.

Lemma 1.2. *Given $A \geq 1$, there is $C = C(A, n)$ such that if $\Omega \subset \mathbf{R}^n$ is open then $\Omega = \bigcup_j Q_j$, where the Q_j are disjoint cubes satisfying*

(i) $5A \leq \operatorname{dist}(Q_j, \partial\Omega)/\operatorname{diam} Q_j \leq 15A$.
(ii) $\sum_j \chi_{AQ_j} \leq C\chi_\Omega$.

The set of functions $\{w \in L^q_{\mathrm{loc}}(\Omega) \mid w \geq 0, w \not\equiv 0\}$ satisfying

$$\|w\|_{q,Q} \leq C\|w\|_{p,\sigma Q}, \qquad \forall Q : \sigma' Q \subset \Omega \tag{1.3}$$

for some $0 < p < q$ will be denoted WRH^Ω_q if $1 < \sigma \leq \sigma'$, and RH^Ω_q if $1 = \sigma < \sigma'$. These definitions are independent of p, σ, σ', as long as they satisfy the defining inequalities (see [Bu], [I-N]). The same is true of the smallest constant C for which (1.3) is true (up to a comparability factor), so we denote (for any convenient choice of σ, σ') this best constant as $WRH^\Omega_{q,p}(w)$ or $RH^\Omega_{q,p}(w)$ (if the q-subscript is omitted, we assume $p = q/2$).

A bounded domain Ω with a distinguished point $x_0 \in \Omega$ is called a *John domain* Ω if there exists a constant $C > 0$ such that, for all $x \in \Omega$, there is a path $\gamma : [0, l] \to \Omega$ parametrized by arclength such that $\gamma(0) = x$, $\gamma(l) = x_0$, and

$$\operatorname{dist}(\gamma(t), \partial\Omega) \geq Ct. \tag{1.4}$$

We call C the *John constant* of Ω and denote it by $\mathrm{John}(\Omega)$. This "twisted cone" condition is satisfied, in particular, by all bounded Lipschitz domains and certain fractal domains (for example, snowflake domains), see [M], [NV], and [V]. Note that if Ω is a John domain, any $y \in \Omega$ can act as the distinguished point (a more "central" point will give smaller constants, though). In Section 5, we shall talk about more general domains which we call John-α domains (where $0 < \alpha \leq 1$). These are domains satisfying the same conditions, except that we replace (1.4) with

$$\operatorname{dist}(\gamma(t), \partial\Omega) \geq Ct^{1/\alpha}.$$

Theorem 1.5. *Suppose Ω is a John domain, $Q_0 \in \mathcal{W}(\Omega)$, $0 < p < n$, and $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$. Suppose also that there exists $v \in WRH^\Omega_1$ such that $v \geq |\nabla u|$. Then*

$$\left(\int_\Omega |u - u_{Q_0}|^q\right)^{1/q} \leq C \left(\int_\Omega v^p\right)^{1/p} \tag{1.6}$$

for $q = p^*$. The constant C has the form $C' \cdot WRH^\Omega_{1,p}(v)$, where C' depends only on n, p, $\mathrm{John}(\Omega)$*, and Q_0.*

We stress that $WRH^\Omega_{1,p}(v)$ could be replaced by an appropriate expression involving $WRH^\Omega_1(v)$ (see [Bu], [I-N]).

We can view Theorem 1.5 as being about functions u for which $|\nabla u| \in WRH^\Omega_1$. In this case we may take $v = |\nabla u|$, making (1.6) into an ordinary Sobolev-Poincaré inequality. This condition is rather mild — it is much weaker than a RH_1^Ω condition — and is satisfied by several important classes of functions. A wide class of examples is provided by functions $u \in W^{1,1}_{\rm loc}(\Omega)$ such that, for every $c \in \mathbf{R}$, $u - c$ satisfies a Caccioppoli-type inequality

$$\int_\Omega \eta^q |\nabla u|^q \le C \int_\Omega |\nabla \eta|^q |u - c|^q \tag{1.7}$$

for every $\eta \in C_0^\infty(\Omega)$ (and some $q > 1$). To show that $|\nabla u| \in WRH_1^\Omega$, suppose Q is a cube for which $4Q \subset \Omega$. Take η to be a smooth function which equals 1 on Q, 0 off $2Q$, and is such that $\|\nabla \eta\|_{L^\infty} \le Cl(Q)^{-1}$. If $r = \max(1, nq/(n+q))$, then $r < q$, but $r^* \ge q$. Using an (r^*, r)-Poincaré inequality, we see that

$$\begin{aligned} \|\nabla u\|_{q,Q} &\le Cl(Q)^{-1} \|u - u_Q\|_{q,2Q} \le Cl(Q)^{-1} \|u - u_Q\|_{r^*,2Q} \\ &\le Cl(Q)^{-1+n/r-n/r^*} \|\nabla u\|_{r,2Q} = C \|\nabla u\|_{r,2Q}, \end{aligned}$$

and so $|\nabla u| \in WRH_q^\Omega \subset WRH_1^\Omega$. (1.7) is satisfied by weak solutions to many elliptic partial differential equations including all linear self-adjoint elliptic p.d.e.'s with bounded measurable coefficients (in which case $q = 2$). A proof of (1.7) for weak solutions to a more general class of p.d.e.'s can be found in [S] (see also Corollary 5.15 and [H-K-M]). (1.7) is also satisfied by coordinate functions of quasiregular mappings [H-K-M].

Let us give an example at this point to show that some sort of condition on u is necessary in order to get a Poincaré inequality for $0 < p < 1$. Let Ω be the interval $[-1, 1]$ and, for any $\epsilon > 0$, let

$$u_\epsilon = \begin{cases} 0, & x \le -\epsilon \\ \phi(x/\epsilon), & -\epsilon < x < \epsilon \\ 1, & \epsilon \le x \end{cases},$$

where $\phi : [-1, 1] \to [0, 1]$ is any differentiable function satisfying $\phi(-1) = 0$, $\phi(1) = 1$, $\phi'_+(-1) = \phi'_-(1) = 0$. Then, for any $0 < p < 1$,

$$\int_{-1}^1 |\nabla u_\epsilon|^p = \int_{-\epsilon}^\epsilon \epsilon^{-p} |\phi'(x/\epsilon)|^p \, dx = \epsilon^{1-p} \int_{-1}^1 |\phi'(y)|^p \, dy \to 0 \quad (\epsilon \to 0). \tag{1.8}$$

On the other hand, any $a \in \mathbf{R}$ satisfies either $|1 - a| \ge 1/2$ or $|0 - a| \ge 1/2$, and so

$$\inf_{a \in \mathbf{R}} \int_{-1}^1 |u_\epsilon(x) - a|^q \, dx \ge 2^{-q}(1 - \epsilon).$$

Since this infimum is bounded below as $\epsilon \to 0$, (1.8) implies that a one-dimensional Poincaré inequality for $0 < p < 1$ is not possible in general. In higher dimensions, we let $f_\epsilon(x_1, \dots, x_n) = u_\epsilon(x_1)$, where u_ϵ is as above. The functions f_ϵ provide the desired counterexample for any domain containing the origin.

We shall see in Theorem 4.2 that one cannot get a satisfactory "ordinary Poincaré inequality" (i.e. with $v = |\nabla u|$) without assuming $|\nabla u| \in WRH_1^\Omega$. However, if $u \in W^{1,t}_{\rm loc}(\Omega)$ for some $t > 1$, then clearly $v = [M_\Omega(|\nabla u|^t)]^{1/t} \geq |\nabla u|$, and Lemma 2.1 assures us that $v \in WRH_1^\Omega$ (of course, we must also assume $v \in L^p(\Omega)$ to get a non-trivial conclusion). With this choice of v in Theorem 1.5, we get a weaker form of control of the variation of u on Ω than is possible when $|\nabla u| \in WRH_1^\Omega$. Notice, however, that each u with $|\nabla u| \in WRH_1^\Omega$ in fact belongs to $W^{1,t}_{\text{loc}}(\Omega)$ for some $t > 1$ (see [B-I]). We are grateful to Carlos Kenig who suggested looking for a maximal function variant of Theorem 1.5.

2. A Pair of Lemmas

It is well known (see [G-R, II.3.4]) that if $f \in L^1_{\text{loc}}(\mathbf{R}^n)$ and $Mf(x)$ is finite almost everywhere, then $(Mf)^\gamma$ is in Muckenhoupt's A_1 weight class for all $0 < \gamma < 1$. Our first lemma gives a weaker conclusion than this for the more general case of a domain $\Omega \subset \mathbf{R}^n$.

Lemma 2.1. *If $f \in L^1_{\text{loc}}(\Omega)$ and $M_\Omega f(x)$ is finite almost everywhere, then $(M_\Omega f)^\gamma \in WRH_1^\Omega$ for all $0 < \gamma < 1$.*

Proof. Suppose $2Q \subset \Omega$. Let us first consider $M_\Omega f_1 = M f_1$, where $f_1 = f\chi_{2Q}$, and normalize f so that $\|f\|_{1,2Q} = 1$. Writing

$$\begin{aligned} A_k &= \{x \in Q \mid 2^{k-1} < Mf_1(x) \leq 2^k\}, \qquad k > 0 \\ A_0 &= \{x \in Q \mid Mf_1(x) \leq 1\}, \end{aligned}$$

a weak-$(1,1)$ estimate for M gives us that $|A_k| \leq C|Q|/2^k$ and so, for all $0 < r < 1$,

$$\begin{aligned} \|(Mf_1)^\gamma\|_{1,Q} &\leq \sum_{k=0}^{\infty} 2^{k\gamma}|A_k| \leq C\sum_{k=0}^{\infty} 2^{-k(1-\gamma)} \leq C \\ &\leq C \min_{x\in Q} (Mf_1(x))^\gamma \leq C\|(Mf_1)^\gamma\|_{r,Q}. \end{aligned} \tag{2.2}$$

We now turn to $M_\Omega f_2$, where $f_2 \equiv f - f_1$. Suppose $M_\Omega f_2(x) > 0$ for some $x \in Q$. Then there exists some cube Q' containing x such that $M_\Omega f_2(x) \leq 2\|f_2\|_{1,Q'}$. It follows that $Q' \not\subset 2Q$ and that $Q' \cap (2Q \setminus Q)$ contains a cube P of sidelength at least $l(Q)/2$. Therefore, if $0 < r < 1$,

$$M_\Omega f_2(x) \leq 2 \inf_{y\in P} M_\Omega f_2(y) \leq 2\|M_\Omega f_2\|_{\gamma r,P} \leq 2^{1+2n/\gamma r}\|M_\Omega f_2\|_{\gamma r,2Q}$$

and so

$$\|(M_\Omega f_2(x))^\gamma\|_{1,Q} \leq \|(M_\Omega f_2(x))^\gamma\|_{\infty,Q} \leq C_r\|(M_\Omega f_2)^\gamma\|_{r,2Q} \tag{2.3}$$

It follows from (2.2) and (2.3) that

$$\|(M_\Omega f)^\gamma\|_{1,Q} \leq \|(Mf_1)^\gamma\|_{1,Q} + \|(M_\Omega f_2)^\gamma\|_{1,Q} \leq C_r\|(M_\Omega f)^\gamma\|_{r,Q}$$

for all $0 < r < 1$, as required. $\square$

Lemma 2.4. *If Ω is a bounded Lipschitz domain, $Q_0 \in \mathcal{W}(\Omega)$, $\alpha > (n-1)/n$, and $R > 1$, there exists a constant $C = C(\Omega, \alpha, R)$ such that*

$$\sum_{\substack{Q \in \mathcal{W}(\Omega) \\ Q \subset RQ_0}} |Q|^\alpha \leq C|Q_0|^\alpha. \tag{2.5}$$

Proof. We may assume without loss of generality that Q_0 is centered at the origin, and so $RQ_0 \subset B \equiv \{x \in \mathbf{R}^n \,:\, |x| < r\}$ where $r = R \cdot \mathrm{diam}(Q_0)/2$. We define

$$A_k = \{x \in \Omega \,|\, 2^{-k} \leq \mathrm{dist}(x, \partial\Omega) < 2^{-k+1}\}.$$

Let k_0 be the greatest integer less than $-\log_2[(301 + R/2)\,\mathrm{diam}(Q_0)]$. Since $\mathrm{dist}(Q_0, \partial\Omega) \leq 300\,\mathrm{diam}(Q_0)$, $\Omega \cap B$ is contained in $\bigcup_{k \geq k_0} A_k$. Also,

$$\sum_{\substack{Q \in \mathcal{W}(\Omega) \\ Q \subset RQ_0}} |Q|^\alpha \leq C \int_{\Omega \cap B} \mathrm{dist}(x, \partial\Omega)^{(\alpha-1)n}\, dx \leq C \sum_{k=k_0}^{\infty} 2^{(1-\alpha)nk} |A_k \cap B|.$$

Thus the theorem follows easily if we can show that

$$|A_k \cap B| \leq C(\Omega, \alpha) 2^{-k} r^{n-1}, \qquad k \geq k_0.$$

A bounded Lipschitz domain is the union of finitely many isometric copies of special Lipschitz domains of the form

$$S = \{(x, y) \in \mathbf{R}^{n-1} \times \mathbf{R} \,:\, |x| < r', \phi(x) < y < K\},$$

where $r' > 0$, $K \in \mathbf{R}$, and ϕ is Lipschitz. Under this identification, $\partial\Omega$ is the union of the lower boundaries $\partial S_- = \{(x, \phi(x)) \,:\, |x| < r'\}$. Therefore, it suffices to show that

$$|A'_k \cap B'| \leq C(S, \alpha) 2^{-k} r^{n-1}, \qquad k \geq k_0,$$

where $B' = \{x \in \mathbf{R}^n : |x - x_0| < r\}$ has the same radius as B, and

$$A'_k = \{(x, y) \in S \,:\, 2^{-k} \leq \mathrm{dist}(x, \partial S_-) < 2^{-k+1}\}.$$

We may assume without loss of generality that $r \leq r'$ and that $x_0 = 0$. The Lipschitz condition on S implies that

$$C^{-1}(y - \phi(x)) \leq \mathrm{dist}((x, y), \partial S_-) \leq y - \phi(x), \qquad (x, y) \in S,$$

and so it suffices to show that $|A''_k| \leq C_\alpha 2^{-k} r^{n-1}$ for all $k \geq k_0$, where

$$A''_k = \{(x, y) \in \mathbf{R}^n \,:\, |x| < r, \phi(x) + C^{-1} 2^{-k} \leq y \leq \phi(x) + 2^{-k+1}\}.$$

Let P be the $(n-1)$-dimensional cube which circumscribes $\{x \in \mathbf{R}^{n-1} : |x| < r\}$. Dividing P into subcubes $\{P_j\}_{j=1}^N$ of sidelength less than 2^{-k}, we see that $|\phi(x_1) - \phi(x_2)| \le Cl(P_j)$ for all $x_1, x_2 \in P_j$. Thus $A_k'' \cap (P_j \times \mathbf{R}) \subset P_j \times [\phi(x) - C2^{-k}, \phi(x) + C2^{-k}]$ for any $x \in P_j$, and so

$$|A_k''| = \sum_{j=1}^N |A_k \cap (P_j \times \mathbf{R})| \le C \cdot 2^{-k} \sum_{j=1}^N |P_j| \le C \cdot 2^{-k} r^{n-1},$$

as required. $\square$

3. Proof of Theorem 1.5

Let us first give a proof for the case of a bounded Lipschitz domain Ω. Since the case $p \ge 1$ is known, we assume $p < 1$. We shall prove (1.6) for $p \le q \le p^*$. By Hölder's inequality, the case $q = p^*$ is equivalent to this formally more general case, but the calculations involved will be useful in dealing later with more general domains. Chaining arguments similar to those used here have been used before (see, for example, [Boj], [Bom], [C], and [I-N]). We include all details for the benefit of the reader.

If $4Q \subset \Omega$ then by the $(1^*, 1)$-Poincaré inequality on Q, and Hölder's inequality, we get

$$\begin{aligned} \|u - u_Q\|_{q,Q} &\le \|u - u_Q\|_{n/(n-1),Q} \le Cl(Q)^{-n+1} \int_Q |\nabla u| \\ &\le Cl(Q)\|v\|_{1,Q} \le C\left(WRH_{1,p}^{\Omega}(v)\right) l(Q)\|v\|_{p,2Q} \end{aligned} \tag{3.1}$$

and so, for all $q < 1^*$,

$$\int_Q |u - u_Q|^q \le Cl(Q)^{n+q-nq/p} \left(\int_{2Q} v^p\right)^{q/p}.$$

Here C contains the term $\left(WRH_{1,p}^{\Omega}(v)\right)^q$. Let us fix $Q_0 \in \mathcal{W}(\Omega)$, with center z_0. If $Q \in \mathcal{W}(\Omega)$, with center z, then

$$\int_Q |u - u_{Q_0}|^q \le C\left(|Q| \cdot |u_Q - u_{Q_0}|^q + \int_Q |u - u_Q|^q\right).$$

Since Ω is a bounded Lipschitz domain, it is a John domain. Therefore, there exists a path γ from z to z_0 which satisfies the John conditions given in Section 1. The image of γ is covered by a chain of Whitney cubes $\{Q_j\}_{j=0}^k$, where $Q_k = Q$, and $k < \infty$ depends on $l(Q)$. If Q_j is the first cube in the chain of sidelength 2^s and $\gamma(t_j) \in Q_j$, then the John and Whitney conditions ensure that $t_j \le C2^s$. If $t \le t_j$, then $|\gamma(t) - \gamma(t_j)| \le t_j \le C2^s$, and we can only fit a bounded number of disjoint cubes of sidelength 2^s into a ball of sidelength $C2^s$. This implies that there exist constants $r<1$ and C_0 such that for all j, $l(Q_j)\le C_0r^j$. In addition, there exists $R>0$ such that $RQ_j\supset Q_i$ for every $0\le j\le i\le k$. The constants r, C_0, and R are all independent of i, j, and Q. Also, just by the Whitney condition alone, we have $l(Q_j)>l(Q_{j-1})/4$, and so $Q_{j-1}\subset 9Q_j$ for all $j>0$.

Therefore, using (3.1) for $q=1$,

$$\begin{aligned}
|u_Q-u_{Q_0}|^q &\le \left(\sum_{j=1}^k |u_{Q_j}-u_{Q_{j-1}}|\right)^q \le \left(\sum_{j=1}^k \fint_{Q_{j-1}} |u-u_{Q_j}|\right)^q \\
&\lesssim \left(\sum_{j=1}^k \fint_{9Q_j} |u-u_{Q_j}|\right)^q \lesssim \left(\sum_{j=1}^k \fint_{9Q_j} |u-u_{9Q_j}|\right)^q \\
&\lesssim \left(\sum_{j=1}^k \left(\fint_{18Q_j} v^p\right)^{1/p} l(Q_j)\right)^q \le \left(\sum_{j=1}^k \left(\int_{18Q_j} v^p\right)^{1/p} l(Q_j)^{(p-n)/p}\right)^q
\end{aligned}$$

We now split up $\int_\Omega |u-u_{Q_0}|^q$ into a sum of integrals over the Whitney cubes of Ω and attach a chain of Whitney cubes as above to each, giving us

$$\begin{aligned}
\int_\Omega |u-u_{Q_0}|^q &\lesssim \sum_{Q\in\mathcal{W}(\Omega)} \left(\int_{2Q} v^p\right)^{q/p} |Q|^t + \sum_{Q\in\mathcal{W}(\Omega)} |Q| \left(\sum_{j=1}^k l(Q_j)^{\frac{p-n}{p}} \left(\int_{18Q_j} v^p\right)^{1/p}\right)^q \\
&\equiv I+II.
\end{aligned}$$

where $t\equiv 1+q/n-q/p\ge 0$, since $p<n$ and $q\le p^*$. Thus,

$$I\lesssim \sum_{Q\in\mathcal{W}(\Omega)} \left(\int_{2Q} v^p\right)^{q/p} \lesssim \left(\int_\Omega v^p\right)^{q/p}.$$

The latter inequality is true since $q/p\ge 1$, and the cubes $\{2Q\}$ have finite overlap at every point. To handle II, we need to consider two cases separately.

Case 1: $q\le 1$.

In this case,

$$\begin{aligned}
II &\lesssim \sum_{Q\in\mathcal{W}(\Omega)} |Q| \sum_{j=1}^k l(Q_j)^{(p-n)q/p} \left(\int_{18Q_j} v^p\right)^{q/p} \\
&= \sum_{Q'\in\mathcal{W}(\Omega)} |Q'|^{-q/p^*} \left(\int_{18Q'} v^p\right)^{q/p} \left(\sum_{Q:Q'=Q_j} |Q|\right),
\end{aligned}$$

where the sum in parentheses is over all cubes Q whose chain goes through Q'. By the John condition, it follows that $Q \subset RQ'$ for some $R < \infty$, and so the parenthesized sum is dominated by $R^n|Q'|$. Thus, by the finite overlap of the cubes $\{18Q'\}$, we get

$$II \lesssim \sum_{Q' \in \mathcal{W}(\Omega)} \left(\int_{18Q'} v^p \right)^{q/p} \lesssim \left(\int_\Omega v^p \right)^{q/p}.$$

Case 2: $1 < q \le p^* < \dfrac{n}{n-1}$.

In this case,

$$\begin{aligned} II &\lesssim \left(\sum_{Q \in \mathcal{W}(\Omega)} |Q|^{1/q} \sum_{j=1}^{k} |Q_j|^{-1/p^*} \left(\int_{18Q_j} v^p \right)^{1/p} \right)^q \\ &= \left(\sum_{Q' \in \mathcal{W}(\Omega)} |Q'|^{-1/p^*} \left(\int_{18Q'} v^p \right)^{1/p} \sum_{Q : Q' = Q_j} |Q|^{1/q} \right)^q. \end{aligned}$$

Since $1/q \ge 1/p^* > 1/1^* = (n-1)/n$, we can use Lemma 2.4 to finish the proof as in Case 1.

Let us now consider an arbitrary John domain Ω. Except for the estimation of II, the above proof carries over unchanged for any bounded domain. For a general domain, however, the chain of Whitney cubes employed satisfies no useful condition and we cannot hope to estimate II; the John condition is precisely what we need to complete the proof. We can no longer use Lemma 2.4, which is false for general John domains (since they can have "too many" small Whitney cubes). Of course, simple geometry ensures that (2.5) is valid for all domains if $\alpha = 1$, so the estimation of II is as before if $q \le 1$. Therefore, we may assume that $q > 1$.

First of all, suppose that $q < p^*$. The l^q Hölder's inequality

$$\sum_{j=1}^{\infty} a_j \le \left(\sum_{j=1}^{\infty} a_j^q s^j \right)^{1/q} \left(\sum_{j=1}^{\infty} s^{-j/(q-1)} \right)^{(q-1)/q} \le C_{s,q} \left(\sum_{j=1}^{\infty} a_j^q s^j \right)^{1/q}, \quad s > 1 \tag{3.2}$$

is valid for all non-negative sequences of numbers $\{a_j\}$. We apply (5.2) to the inner sum of II to get that

$$II \le C_{s,q} \sum_{Q \in \mathcal{W}(\Omega)} |Q| \sum_{j=1}^{k} |Q_j|^{-q/p^*} \left(\int_{18Q_j} v^p \right)^{q/p} s^j,$$

where $s > 1$ is arbitrary. Since $q/p^* < 1$, we can fix s so close to 1 that $|Q_j|^{-q/p^*} s^j \le C|Q_j|^{-1}$, for some constant C independent of j. This allows us to finish the proof as before, since we now need (2.5) only for the case $\alpha = 1$, after a change in the order of summation.

Finally, let us handle the case $q = p^* > 1$. Here we make essential use of the ideas of Boman [Bom] and Bojarski [Boj], who consider similar problems for the case $p > 1$. The following lemma will be useful to us; its proof, a simple application of the Hardy-Littlewood maximal operator and L^p-duality, can be found in both [Bom] and [Boj].

Lemma 3.3. *Let $F = \{Q_\alpha\}_{\alpha \in I}$ be an arbitrary family of cubes in $\mathbf{R}^n$. Assume that for each Q_α, we are given a non-negative number a_α. Then, for $1 \le q < \infty$ and $R \ge 1$, we have*

$$\int_{\mathbf{R}^n} \left(\sum_\alpha a_\alpha \chi_{RQ_\alpha} \right)^q \le D \int_{\mathbf{R}^n} \left(\sum_\alpha a_\alpha \chi_{Q_\alpha} \right)^q,$$

where the constant D depends only on n, q, and R.

Since $RQ_j \supset Q_k = Q$ for all $1 \le j \le k$, it is clear that

$$II \le \sum_{Q \in \mathcal{W}(\Omega)} |Q| \left(\sum_{\substack{Q' \in \mathcal{W}(\Omega) \\ RQ' \supset Q}} a_{Q'} \right)^{p^*}$$

where $a_{Q'} = l(Q')^{(p-n)/p} \left(\int_{18Q'} v^p \right)^{1/p}$. Thus, using Lemma 3.3, we get

$$\begin{aligned} II &\le \sum_{Q \in \mathcal{W}(\Omega)} \int_Q \left(\sum_{Q' \in \mathcal{W}(\Omega)} a_{Q'} \chi_{RQ'}(x) \right)^{p^*} dx \le \int_{\mathbf{R}^n} \left(\sum_{Q' \in \mathcal{W}(\Omega)} a_{Q'} \chi_{RQ'} \right)^{p^*} \\ &\le C \int_{\mathbf{R}^n} \left(\sum_{Q' \in \mathcal{W}(\Omega)} a_{Q'} \chi_{Q'} \right)^{p^*} \le C \int_{\mathbf{R}^n} \sum_{Q' \in \mathcal{W}(\Omega)} a_{Q'}^{p^*} \chi_{Q'} \\ &= C \sum_{Q' \in \mathcal{W}(\Omega)} a_{Q'}^{p^*} |Q'| = C \sum_{Q' \in \mathcal{W}(\Omega)} \left(\int_{18Q'} v^p \right)^{p^*/p} \le C \left(\int_\Omega v^p \right)^{p^*/p}, \end{aligned}$$

as required. This finishes the proof of Theorem 1.5. □

The proof for $q = p^* > 1$ of course implies the previously considered $1 < q < p^*$ case, but a variant of the method used in that previous case will also be used for the more general John-α domains considered in Section 5, while the method for $q = p^*$ cannot be used for more general domains.

4. A weak "converse" to Theorem 1.5

Suppose $0 < p < n$, $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$, and $2Q \subset \Omega$, where Q is a cube. If $|\nabla u| \in WRH_1^\Omega$, it follows easily from Theorem 1.5 that

$$\left(\int_Q |u - u_Q|^{p^*} \right)^{1/p^*} \le C \left(\int_Q |\nabla u|^p \right)^{1/p}, \tag{4.1}$$

where C is independent of Q since all cubes have the same John constant. We now show that if the partial derivatives of u do not change sign, then even a weak form of (4.1) implies that $|\nabla u| \in WRH_1^\Omega$.

Theorem 4.2. *Assume $0 < p < 1$, $0 < q$, and $u \in W^{1,1}_{\rm loc}(\Omega)$, where Ω is an arbitrary domain in $\mathbf{R}^n$. Assume also that the partial derivatives of u are each of constant sign on Ω. If there exists a constant C such that, for all cubes Q for which $5Q \subset \Omega$,*

$$\inf_{a\in\mathbf{R}} \left(\int_Q |u-a|^q \right)^{1/q} \le C|Q|^{1-q/p^*} \left(\int_{3Q} |\nabla u|^p \right)^{1/p},$$

it follows that $|\nabla u| \in WRH_1^\Omega$.

Proof. Without loss of generality, we may assume that $\partial_i u > 0$ on Ω, for all i. Suppose for the purposes of contradiction that $|\nabla u| \notin WRH_1^\Omega$ and so, for every $m > 0$, there exists a cube Q_m such that $5Q_m \subset \Omega$ and $\|\nabla u\|_{1,Q_m} \ge m\sqrt{n}\|\nabla u\|_{p,3Q_m}$. Therefore,

$$\max_{1\le i\le n} \|\partial_i u\|_{1,Q_m} \ge m\|\nabla u\|_{p,3Q_m}.$$

Without loss of generality, we assume that this maximum occurs for $i = 1$. Let us choose $\phi \in C_0^\infty(\mathbf{R}^n)$ such that $\phi \ge 0$ and $\int_{\mathbf{R}^n} \phi = 1$. If $\phi_\epsilon(x) = \epsilon^{-1}\phi(x/\epsilon)$, then $u_\epsilon = u * \phi_\epsilon$ is a smooth approximation to u as $\epsilon \to 0^+$. Since $\partial_i u_\epsilon = \partial_i u * \phi_\epsilon \ge 0$, every u_ϵ is monotonically increasing in coordinate directions (MICD, for short). Also, as $\epsilon \to 0^+$, $u_\epsilon(x) \to u(x)$ a.e. $x \in \Omega$. It follows that $u_-(x) = \liminf_{\epsilon\to 0^+} u_\epsilon(x)$ is MICD and equal to $u(x)$ almost everywhere. We may therefore assume that $u = u_-$, and so u is MICD. Next, let α be the corner of Q_m whose components are all less than those of the centre point of Q_m, and let β be the opposite corner (so that each component of $\beta - \alpha$ equals $l(Q_m)$). Let $Q_m^+ = Q_m + (\beta - \alpha)$ and let $Q_m^- = Q_m - (\beta - \alpha)$, so that Q_m touches each of Q_m^+ and Q_m^- at one corner. We claim that for $v = u$,

$$\inf_{\substack{x\in Q_m^- \\ y\in Q_m^+}} |v(x) - v(y)| \ge l(Q_m) \left(\fint_{Q_m} \partial_1 v \right). \tag{4.3}$$

First of all, note that $\sup_{x\in Q_m^-} u(x) = u(\alpha) \le u(\beta) = \inf_{y\in Q_m^+} u(y)$, since u is MICD.

If we can prove (4.3) with $v = u_\epsilon$ (for all sufficiently small $\epsilon > 0$), it follows for $v = u$ without difficulty since

$$\int_{Q_m} \partial_1 u_\epsilon \to \int_{Q_m} \partial_1 u \quad (\epsilon \to 0^+)$$

and there is some sequence $\{\epsilon_n\}$, decreasing to 0, for which

$$u_{\epsilon_n}(\alpha) \to u(\alpha),\ u_{\epsilon_n}(\beta) \to u(\beta) \quad (n \to \infty).$$

Suppose therefore that $v = u_\epsilon$ for some $\epsilon > 0$ small enough that v is defined on $3Q_m$. Let us view $\mathbf{R}^n$ as $\mathbf{R} \times \mathbf{R}^{n-1}$, writing $x = (x_1, x')$ for any $x \in \mathbf{R}^n$. If $x \in Q_m$, then we define $\gamma_{x'}$ to be the path in Q_m consisting of three straight line segments from α to (α_1, x') to (β_1, x') to

β. Let us call the straight line pieces of this path (in the same order) $\gamma^i_{x'}$ for $i = 1, 2, 3$. Since v is smooth,

$$v(\beta) - v(\alpha) = \int_{\gamma_{x'}} \nabla v(x) \cdot dx \geq \int_{\gamma^2_{x'}} \nabla v(x) \cdot dx = \int_{\alpha_1}^{\beta_1} \partial_1 v(x_1, x')\, dx_1.$$

Averaging over all x' for which $(\alpha_1, x') \in \partial Q_m$, we get (4.3).

Using (4.3) we see that, for any $a \in \mathbf{R}$,

$$\begin{aligned}\int_{3Q_m} |u-a|^q &\geq \int_{Q_m} (|u(x+\beta-\alpha)-a|^q + |u(x-\beta+\alpha)-a|^q) \\ &\geq Cl(Q_m)^{n+q} \|\partial_1 u\|^q_{1,Q_m} \geq Cm^q |Q_m|^{1+q/n} \|\nabla u\|^q_{p,3Q_m} \\ &\geq Cm^q |Q_m|^{1-q/p^*} \left(\int_{3Q_m} |\nabla u|^p \right)^{q/p}.\end{aligned}$$

Letting $m \to \infty$, this contradicts the assumed uniform Poincaré inequality, and so we are done. $\square$

5. Other Results

We begin by stating an abstract Poincaré inequality due to Boas and Straube [B-S] for exponents $p \geq 1$, to which we can then apply our methods to get a similar result for certain exponents $p < 1$. In these theorems, $W^{1,p}(\Omega, \alpha)$ denotes the space of functions with norm $\|u\|_{p,\Omega} + \|\delta^\alpha \nabla u\|_{p,\Omega}$, where $\delta(x) = \mathrm{dist}(x, \partial\Omega)$ from here on.

Theorem 5.1. *Let Ω be a bounded domain in $\mathbf{R}^n$ whose boundary is locally the graph of a Hölder continuous function of exponent α, where $0 \leq \alpha \leq 1$, and suppose $1 \leq q < \infty$. Let H be a cone in $W^{1,q}_{\mathrm{loc}}(\Omega)$ such that the closure of $H \cap W^{1,q}(\Omega, \alpha)$ in $W^{1,q}(\Omega, \alpha)$ contains no non-zero constant function. Then there is a constant C such that*

$$\int_\Omega u^q \leq C \int_\Omega |\delta^\alpha \nabla u|^q \tag{5.2}$$

for every function u in H.

Suppose now that $v \geq |\nabla u|$ and $v \in WRH^\Omega_q$. By the defining properties of Whitney cubes we have

$$\int_\Omega |\delta^\alpha \nabla u|^q \leq C \sum_{Q \in \mathcal{W}(\Omega)} |Q|^{\alpha q/n} \int_\Omega |\nabla u|^q.$$

Also, for all $Q \in \mathcal{W}(\Omega)$, $0 < p < q$, we have

$$\int_Q |\nabla u|^q \le C \left(\int_{2Q} v^p \right)^{q/p} |Q|^{1-q/p}.$$

Therefore, choosing $p = nq/(n+\alpha q)$, we get

$$\int_\Omega |\delta^\alpha \nabla u|^q \le C \sum_{Q \in \mathcal{W}(\Omega)} \left(\int_{2Q} v^p \right)^{q/p} \le C \left(\int_\Omega v^p \right)^{q/p}. \tag{5.3}$$

Taking these calculations together with Theorem 5.1, we have proved the following theorem.

Theorem 5.4. *Suppose Ω, α, and H are as in Theorem 5.1. Suppose also that $0 < p < n/\alpha$ is such that $q \equiv np/(n-\alpha p) > 1$, and that $v \in WRH_q^\Omega$ satisfies $v \ge |\nabla u|$ on Ω for some $u \in H$. Then*

$$\left(\int_\Omega |u|^q \right)^{1/q} \le C \left(\int_\Omega v^p \right)^{1/p}, \tag{5.5}$$

where $C = C' \dot{WRH}{}^\Omega_{q,p}(v)$ and C' depends only on Ω, p, α, and H.

As with Theorem 1.5, we can take $v = |\nabla u|$, if $|\nabla u| \in WRH_q^\Omega$ or, if $u \in W^{1,t}_{\mathrm{loc}}(\Omega)$ for some $t > q$, we can take $v = [M(|\nabla u|^t)]^{1/t}$. The concept of a cone H is a further abstraction in Theorems 5.1 and 5.4. It implies the more usual Poincaré inequalities with left-hand sides of the form $\left(\int_Q |u - u_{Q_0}|^q \right)^{1/q}$, as previously considered, since we can take $H = \{u \in W^{1,p}_{\mathrm{loc}}(\Omega) \mid u_{Q_0} = 0\}$. If $v \in L^p(\Omega)$, Theorem 5.4 guarantees that $u \in L^q(\Omega) \subset L^1(\Omega)$, so we can in fact replace u_{Q_0} with u_Ω on the left-hand side.

The proof of Theorem 5.4 shows that powers of δ on the right-hand side of the Poincaré inequality can be exchanged for a lower exponent of integrability on the left-hand side. The following variation of Theorem 1.5 makes such an exchange in the opposite direction.

Theorem 5.6. *Suppose Ω is a John domain, $Q_0 \in \mathcal{W}(\Omega)$, $0 < p \le 1$, and $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$. Suppose also that there exists $v \in WRH_1^\Omega$ such that $v \ge |\nabla u|$. Then*

$$\int_\Omega |u - u_{Q_0}|^p \le C \int_\Omega (v\delta)^p, \tag{5.7}$$

where $C = [C' \dot{WRH}{}^\Omega_{1,p}(v)]^p$, and C' depends only on n, p, John(Ω), and Q_0.

Proof. We estimate $\int_\Omega |u - u_{Q_0}|^p$ as in Theorem 1.5 (for the case $q = p$), until we get to the sums I and II. Now,

$$I = \sum_{Q \in \mathcal{W}(\Omega)} |Q|^{p/n} \int_{2Q} v^p \le \sum_{Q \in \mathcal{W}(\Omega)} \int_{2Q} (v\delta)^p \le C \int_\Omega (v\delta)^p,$$

as required. As for the second sum,

$$II \leq \sum_{Q \in \mathcal{W}(\Omega)} |Q| \sum_{j=1}^{k} l(Q_j)^{p-n} \int_{18Q_j} v^p \leq \sum_{Q \in \mathcal{W}(\Omega)} |Q| \sum_{j=1}^{k} |Q_j|^{-1} \int_{18Q_j} (v\delta)^p.$$

II can now be estimated as before by changing the order of summation. We only need (2.5) for the case $\alpha = 1$, so the result follows immediately for all John domains. □

One can also state a version of Theorem 5.6 for $p > 1$. In this case, (5.7) is true with $v = |\nabla u|$ and we do not need to assume $v \in WRH_1^{\Omega}$. A proof using the techniques of Section 3 is not difficult, so we omit the details. We believe this result is already known, but do not have a reference.

Let us now look at examples relevant to Theorems 5.1, 5.4, and 5.6 to explore the sharpness of the exponents in their statements. Consider the cusp

$$\Omega \;=\; \{x = (x_1, x') \in \mathbf{R} \times \mathbf{R}^{n-1} \;:\; 0 < x_1 < 1, |x'| < x_1^{1/\alpha}\},$$

whose boundary is the graph of a Hölder continuous function of exponent α, and suppose $u(x) = x_1^{-\gamma}$, $0 < \alpha \leq 1$, and $\beta > \alpha$. Then $\int_\Omega u^p = \infty$ if

$$\gamma p \geq 1 + (n-1)\alpha^{-1}, \tag{5.8}$$

and $\int_\Omega |\delta^\beta \nabla u|^p \approx \gamma^p \int_0^1 x_1^{\beta p/\alpha - (\gamma+1)p + (n-1)\alpha^{-1}}\, dx_1 < \infty$ if

$$\beta p/\alpha - (\gamma+1)p + (n-1)\alpha^{-1} > -1 \tag{5.9}$$

Since (5.9) can be written as $\gamma p < 1 + (n-1)\alpha^{-1} + (\beta - \alpha)p/\alpha$ and $\beta > \alpha$, we can choose $\gamma > 0$ satisfying (5.8) and (5.9). This is easily seen to imply the sharpness of the exponent α on the right-hand side of (5.2).

We now turn to Theorem 5.4. If Ω and u are as above, then $\int_\Omega u^q = \infty$ if we fix γ so that $\gamma q = 1 + (n-1)\alpha^{-1}$. However, $\int_\Omega |\nabla u|^p < \infty$ if $(\gamma + 1)p < 1 + (n-1)\alpha^{-1}$. If we combine this inequality with the previous equation, we get

$$q > q_0 \equiv \frac{p}{1 - \alpha p/(\alpha + n - 1)}$$

as long as $p < 1 + (n-1)/\alpha$. Note also that $|\nabla u| \in WRH_1^{\Omega}$ since $|\nabla u|$ is essentially constant on any cube Q for which $5Q \subset \Omega$. It follows that if $p < 1 + (n-1)/\alpha$ and if we replace the value of q in (5.5) by any value larger than q_0, the inequality is false. For any $\alpha < 1$,

$$q_0 > \frac{p}{1 - \alpha p/n} = \frac{np}{n - \alpha p}$$

which indicates a probable lack of sharpness in the relationship between q and p in Theorem 5.4 (intuitively, one would expect the above example to indicate the sharp relationship between the exponents). Note that the second inequality in (5.3) is the only non-reversible step we added to Theorem 5.1 to get Theorem 5.4, so any loss of sharpness must occur at this step.

Finally, we note that Theorem 5.6 is sharp. To see this let Ω and u be as before, except that we must choose $\alpha = 1$. If $\gamma = n/p$ then $\int_\Omega u^p = \infty$, while $\int_\Omega \delta^q |\nabla u|^p < \infty$ for any $q > p$.

It is not difficult to see that the Hölder domains of exponent α considered in Theorems 5.1 and 5.4 are special cases of John-α domains. We now consider Poincaré inequalities on domains of this latter type. For the case $p \geq 1$ see [S-S]. As with Theorem 5.4, we do not know if the exponent q in the following theorem is sharp.

Theorem 5.10. *Suppose Ω is a John-α domain, $Q_0 \in \mathcal{W}(\Omega)$, $0 < p < 1$, and $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$. Suppose also that there exists $v \in WRH_1^\Omega$ such that $v \geq |\nabla u|$. Then (1.6) is true for $q = \min(\alpha p^*, 1)$. If $\alpha p^* > 1$, (1.6) is also true for some $q > 1$. The constant C in (1.6) has the form $C' WRH^\Omega_{1,p}(v)$, where C' depends only on p, α, Ω, and Q_0.*

Proof. We just need to estimate the sum II from the proof of Theorem 1.5. Let us first assume that $q = \alpha p^* \leq 1$. In this case,

$$II \leq C \sum_{Q' \in \mathcal{W}(\Omega)} |Q'|^{-\alpha} \left(\int_{18Q'} v^p \right)^{q/p} \left(\sum_{Q : Q' = Q_j} |Q| \right).$$

If Q is a cube whose chain runs through Q' then by the John-α condition, $Q \subset Q''$, where Q'' is a concentric dilate of Q' for which $l(Q'') = C(l(Q'))^\alpha$. Therefore,

$$\sum_{Q : Q' = Q_j} |Q| \leq |Q''| \leq C |Q|^\alpha,$$

and so

$$II \leq C \sum_{Q' \in \mathcal{W}(\Omega)} \left(\int_{18Q'} v^p \right)^{q/p} \leq C \left(\int_\Omega v^p \right)^{q/p}.$$

To finish the proof, we need to consider the case $\alpha p^* > q > 1$. We shall imitate the proof of the corresponding case for John domains. However, we need to estimate more delicately, because the chain of cubes in a John-α domain do not have to decrease geometrically in size (we shall also need to further restrict the allowable values of q before we are done). First of all, a volume argument shows that the number of cubes of sidelength 2^{-s} is less than $C 2^{s(1-\alpha)n}$, for any integer s (C independent of s), but we can do better than this. Suppose M is a mesh of cubes of equal size which partition $\mathbf{R}^n$. If a path has points in more than $m 2^{n-1}$ cubes of M, it has to be of length greater than $(m-1)d$, where d is the sidelength of the cubes in M. Therefore the number of cubes in M which include points on a path of length L is not more than CL/d. Applying this to our chain of cubes, we see that the number of cubes of sidelength 2^{-s} is less than $C 2^{s(1-\alpha)}$. It follows that $|Q_j| < C j^{-n/(1-\alpha)}$.

The l^q Hölder inequality

$$\sum_{i=1}^{\infty} a_j \le \left(\sum_{i=1}^{\infty} |Q_j|^{-s} a_j^q\right)^{1/q} \left(\sum_{i=1}^{\infty} |Q_j|^{s/(q-1)}\right)^{(q-1)/q} \le C_{s,q} \left(\sum_{i=1}^{\infty} |Q_j|^{-s} a_j^q\right)^{1/q}, \tag{5.11}$$

is valid for any $s > (q-1)(1-\alpha)/n$ and all non-negative sequences $\{a_j\}$. Applying this to the inner sum of II, we get that

$$II \le C_{s,q} \sum_{Q\in\mathcal{W}(\Omega)} |Q| \sum_{j=1}^{k} |Q_j|^{-s-q/p^*} \left(\int_{18Q_j} v^p\right)^{q/p}.$$

We can finish as in the case $q \le 1$ as long as $s + q/p^* \le \alpha$. We can fix s satisfying this inequality as long as $(q-1)(1-\alpha)/n + q/p^* < \alpha$, which is equivalent to

$$q < \frac{\alpha p^* n + (1-\alpha)p^*}{n + (1-\alpha)p^*} \equiv q_0,$$

and so the theorem follows for all $q < q_0$. Since $\alpha p^* > 1$, we have $q_0 > 1$ as required. $\square$

Let us now look at variations of the Poincaré inequality in which we replace u_{Q_0} in the left-hand side of (1.6) by some other constant. It is often possible to replace u_{Q_0} by the average of u on Q_0 with respect to some measure $w(x)\,dx$. As an example, we give such a variant of Theorem 1.5, where we write $u_{Q_0,w}$ for the average $\int_{Q_0} uw/\int_{Q_0} w$.

Theorem 5.12. *Suppose Ω, Q_0, p, u, and v are as in Theorem 1.5. Then*

$$\left(\int_\Omega |u - u_{Q_0,w}|^{p^*}\right)^{1/p^*} \le C \left(\int_\Omega v^p\right)^{1/p}$$

in either of the following two cases:

(i) $p^* > 1$ *and* $w \in L^{p^*/(p^*-1)}_{\mathrm{loc}}(\Omega)$.

(ii) $p^* \le 1$, *and there exists* $r > 1$ *such that* $w \in L^{r'}_{\mathrm{loc}}(\Omega)$ *and* $|u-c| \in WRH_r^\Omega$ *for all* $c \in \mathbf{R}$.

Proof. Without loss of generality, we may assume that $w_{Q_0} = 1$. By Theorem 1.5, we need only estimate $|u_{Q_0} - u_{Q_0,w}|$. For any $r > 1$,

$$\begin{aligned} |u_{Q_0} - u_{Q_0,w}| &= \left| \fint_{Q_0} (u - uw) \right| = \left| \fint_{Q_0} (u - u_{Q_0})(1-w) \right| \\ &\le \|u - u_{Q_0}\|_{r,Q_0} \cdot \|1-w\|_{r',Q_0}. \end{aligned}$$

In case (i), we simply take $r = p^*$ and use Poincaré's inequality on the first factor. In case (ii), we first use the inequality $\|u - u_{Q_0}\|_{r,Q_0} \le C\|u - u_{Q_0}\|_{p^*,Q_0}$ and then a (p^*,p)-Poincaré inequality. $\square$

Remarks.

(a) As usual the assumption $v \in WRH_1^\Omega$ is unnecessary if $p \geq 1$.
(b) The condition $|u-c| \in WRH_r^\Omega$ in (ii) is satisfied by weak solutions to elliptic p.d.e.'s if positive subsolutions to those equations satisfy a weak Harnack inequality (for instance, linear self-adjoint elliptic p.d.e.'s with bounded coefficients).

Finally, we shall replace u_{Q_0} by $u(x_0)$, for some $x_0 \in \Omega$. It is not difficult to see that this is impossible without an additional assumption which implies the local boundedness of differences $u(x)-u(y)$ $(x, y \in \Omega)$. We shall content ourselves with extending a result of Ziemer [Z] to the case $p<1$ (and arbitrary John domains). The result concerns weak solutions $u \in W^{1,q}(\Omega)$ of the partial differential equation

$$\operatorname{div} A(x, u, \nabla u) = B(x, u, \nabla u)$$

where A and B are Borel measurable vector-valued and scalar-valued functions, respectively, defined on $\Omega \times \mathbf{R} \times \mathbf{R}^n$. The functions A and B are such that

$$\begin{aligned} |A(x,z,w)| &\leq a_0|w|^{q-1} + a_1|z|^{q-1} + a_2 \\ |B(x,z,w)| &\leq b_0|w|^{q-1} + b_1|z|^{q-1} + b_2 \\ A(x,z,w) \cdot w &\geq |w|^q - c_1|z|^q - c_2, \end{aligned}$$

where a_0 is a positive constant, the other subscripted coefficients are in suitable $L^p(\Omega)$ spaces, and $x \in \Omega$, $z \in \mathbf{R}$, $w \in \mathbf{R}^n$ are arbitrary. For any $q>1$, let us call such a partial differential equation a *type-q equation.* In particular, elliptic equations of the form $\operatorname{div} a_{i,j}(x)\nabla u = 0$, where $a_{i,j} \in L^\infty$, are type-2 equations and, more generally, $\operatorname{div} |\nabla u|^{p-2} a_{i,j}(x)\nabla u = 0$ is a type-p equation.

Theorem 5.13 [Z]**.** *Let Ω be a bounded Lipschitz domain and let $u \in W^{1,q}(\Omega)$ be a weak solution of a type-q equation for some $q>1$. Then, for each $x_0 \in \Omega$, there is a constant C that depends only on a_i, b_i, c_i, q, $\|u\|_{q,\Omega}$, and Ω such that*

$$\|u-u(x_0)\|_{q,\Omega} \leq C\|\nabla u\|_{q,\Omega}.$$

If we assume that $A(x,z,w)$ and $B(x,z,w)$ are homogeneous (of degree $q-1$) in z and w, C can be assumed to be independent of $\|u\|_{q,\Omega}$.

Theorem 5.14. *Let Ω be a John domain and let $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ be a weak solution of a type-q equation for some $q>1$. Suppose also that $|\nabla u| \leq v \in WRH_q^\Omega$. Then, for each $x_0 \in \Omega$ and $0<p<n$, there is a constant C such that*

$$\left(\int_\Omega |u-u(x_0)|^{p^*}\right)^{1/p^*} \leq C\left(\int_\Omega v^p\right)^{1/p}.$$

If we write Q_0 for the Whitney cube which contains x_0, then $C = [C' W\dot{R}H^{\Omega}_{q,p}(v)]^{q/p^*}$, where C' depends only on a_i, b_i, c_i, n, p, q, $\mathrm{John}(\Omega)$, x_0, and $\|u\|_{q,Q_0}$. C' is independent of this last parameter if A and B are homogeneous as in Theorem 5.13.

Proof. Since our assumptions are stronger than those of Theorem 1.5, we only need to estimate $|u_{Q_0} - u(x_0)|$. Theorem 5.13 implies that

$$\begin{aligned}\|u - u(x_0)\|_{1,Q_0} &\le C\|u - u(x_0)\|_{q,Q_0} \le C\|\nabla u\|_{q,Q_0} \\ &\le C\|v\|_{q,Q_0} \le C\|v\|_{p,Q_0},\end{aligned}$$

and so we are done. $\square$

Corollary 5.15. *Let Ω be a John domain and let $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ be a weak solution of a type-q equation for some $q > 1$. Suppose that A and B are homogeneous, as in Theorem 5.13, and that each of a_i, b_i, c_i, $i = 1, 2$, is zero. Then, for all $x_0 \in \Omega$ and $0 < p < n$,*

$$\left(\int_\Omega |u - u(x_0)|^{p^*}\right)^{1/p^*} \le C\left(\int_\Omega |\nabla u|^p\right)^{1/p}.$$

where $C = [C' W\dot{R}H^{\Omega}_{q,p}(|\nabla u|)]^{q/p^*}$, and C' depends only on a_0, b_0, n, p, q, $\mathrm{John}(\Omega)$, and x_0.

Proof. It suffices to note that our assumptions on A, B guarantee that $|\nabla u| \in WRH^{\Omega}_q$ (see [S]). $\square$

References

[B-S] H. Boas and E. Straube, *Integral inequalities of Hardy and Poincaré type*, Proc. Amer. Math. Soc. **103** (1988), 172–176.

[Boj] B. Bojarski, *Remarks on Sobolev imbedding inequalities*, in Proc. of the Conference on Complex Analysis, Joensuu, 1987, Lecture Notes in Math. 1351, Springer-Verlag, 1988.

[B-I] B. Bojarski and T. Iwaniec, *Analytical foundations of the theory of quasiconformal mappings in $\mathbf{R}^n$*, Ann. Acad. Sci. Fenn. Ser. A I Math. **8** (1983), 257–324.

[Bom] J. Boman, *L^p-estimates for very strongly elliptic systems*, unpublished manuscript.

[Bu] S. Buckley, *Pointwise multipliers for reverse Hölder spaces*, to appear in Studia Math.

[C] S.-K. Chua, *Weighted Sobolev inequalities on domains satisfying the chain condition*, Proc. Amer. Math. Soc. **117** (1993), 449–457.

[G-R] J. García-Cuerva and J. Rubio de Francia, *Weighted norm inequalities and related topics*, North Holland, Amsterdam, 1985.

[H-K-M] J. Heinonen, T. Kilpeläinen and O. Martio, *Nonlinear potential theory of degenerate elliptic equations*, Oxford University Press, Oxford, 1993.

[I-N] T. Iwaniec and C. Nolder, *Hardy-Littlewood inequality for quasiregular mappings in certain domains in* $\mathbf{R}^n$, Ann. Acad. Sci. Fenn. Ser. A I Math. **10** (1985), 267–282.

[K] P. Koskela, *An inverse Sobolev lemma*, to appear in Rev. Mat. Iberoamericana.

[M] O. Martio, *John domains, bilipschitz balls and Poincaré inequality*, Rev. Rom. Math. Pures Appl. **33** (1988), 107–112.

[NV] R. Näkki and J. Väisälä, *John disks*, Exposition. Math. **9** (1991), 3–43.

[S] E. Sawyer, *Norm inequalities relating singular integrals and the maximal function*, Studia Math. **75** (1983), 253–263.

[Se] J. Serrin, *Local behavior of solutions of quasilinear equations*, Acta Math. **111** (1964), 247–302.

[S-S] W. Smith and D.A. Stegenga, *Hölder domains and Poincaré domains*, Trans. Amer. Math. Soc. **319** (1990), 67–100.

[V] J. Väisälä, *Uniform domains*, Tohoku Math. J. **40** (1988), 101–118.

[Z] W. P. Ziemer, *A Poincaré-type inequality for solutions of elliptic differential equations*, Proc. Amer. Math. Soc. **97** (1986), 286–290.

Department of Mathematics, St. Patrick's College, Maynooth, Co. Kildare, Ireland (Stephen Buckley).
E-mail: sbuckley@maths.may.ie

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA (Pekka Koskela).
E-mail: koskela@math.lsa.umich.edu