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On linear co-positive Lyapunov functions for sets of linear positive systems ⋆

Florian Knorn, Oliver Mason, Robert Shorten

Hamilton Institute, National University of Ireland Maynooth, Maynooth, Co. Kildare, Ireland

Abstract

In this paper we derive necessary and sufficient conditions for the existence of a common linear co-positive Lyapunov function for a finite set of linear positive systems. Both the state dependent and arbitrary switching cases are considered. Our results reveal an interesting characterisation of "linear" stability for the arbitrary switching case; namely, the existence of such a linear Lyapunov function can be related to the requirement that a number of extreme systems are Metzler and Hurwitz stable. Examples are given to illustrate the implications of our results.

Key words: Positive systems; switched systems; linear Lyapunov functions; stability theory; time-invariant

1 Introduction

Dynamic systems in which the state is constrained to be positive as trajectories evolve, have been the subject of many recent studies in the control engineering and mathematics literature [4,2,23,15]. The interest in such systems is hardly surprising since they are ubiquitous and can be found in diverse areas such as economics [14,18], biology [12,6,1], communication networks [20,5,25], and in decentralised control [24] or synchronisation / consensus problems [13]. While both nonlinear and linear positive systems have been studied, much recent attention has focused on both time-varying and time-invariant linear positive systems, and on the Metzler matrices that characterise the properties of such systems. Our focus in this paper is on this latter class of systems, and in particular on the existence of linear co-positive Lyapunov functions (LCLF). It is well known that the existence of an LCLF is both necessary and sufficient for the stability of a positive linear time-invariant (LTI) system, [2,4,11]. While studying the existence of such Lyapunov functions for switched systems is certainly conservative, given the previous comment (re. LTI systems) establishing conditions under which such functions exist is a natural place to begin the study of the stability of switched linear positive systems. In fact, many of the interesting properties of positive systems can be attributed to the existence of an LCLF. Of particular interest is the recent paper by Haddad *et al.*, [8], in which the existence of such a function was related to delay independent stability properties that are possessed by many positive systems. Inspired by this and related work, and by our interest in switched systems, we intend in this brief paper to determine tractable conditions for the existence of a *common* LCLF for a finite number of LTI systems that are associated with polyhedral regions in the positive orthant, which can be interpreted as state dependent switching. As we shall see, compact and easily verifiable conditions can be obtained for the existence of such a function, and these results complement and complete initial results reported in [17].

Our brief paper is structured as follows. In Section 2 we will present conditions for the existence of a common LCLF for switched positive systems whose constituent systems are associated with cones that partition the state space; Section 3 then focuses on arbitrarily switching systems (i. e. any constituent system can be switched to anywhere in the positive orthant). Finally, in Section 4 we discuss the significance of our results and give examples of applications that motivate their use.

Notation and mathematical preliminaries

Throughout, $\mathbb{R}$ denotes the field of real numbers, $\mathbb{R}^n$ is the n-dimensional Euclidean space and $\mathbb{R}^{n\times n}$ the space of $n \times n$ matrices with real entries. A subset $\mathcal{C}$ of $\mathbb{R}^n$ is a closed, pointed convex cone if and only if $\alpha \boldsymbol{x} + \beta \boldsymbol{y} \in \mathcal{C}$ for any $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{C}$ and non-negative scalars α, β.

⋆ Corresponding author: F. Knorn.

Email addresses: `florian@knorn.org` (Florian Knorn), `oliver.mason@nuim.ie` (Oliver Mason), `robert.shorten@nuim.ie` (Robert Shorten).

Matrices or vectors are said to be positive (non-negative) if all their entries are positive (non-negative); this is written as $\boldsymbol{A} \succ \mathbf{0}$ resp. $\boldsymbol{A} \succeq \mathbf{0}$, where $\mathbf{0}$ is the zero-matrix of appropriate dimension. The ith column of $\boldsymbol{A}$ is denoted $\boldsymbol{A}^{(i)}$. A matrix $\boldsymbol{A}$ is said to be *Hurwitz* if all its eigenvalues lie in the open left half of the complex plane. A matrix is said to be *Metzler* if all its off-diagonal entries are non-negative.

We use $\Sigma_{\boldsymbol{A}}$ to denote the LTI system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$. Such a system is called *positive* if, for a positive initial condition, all its states remain in the positive orthant throughout time. A classic result shows that this will be the case if and only if $\boldsymbol{A}$ is a Metzler matrix, [4]. Similarly, a switched positive linear system is a dynamical system of the form $\dot{\boldsymbol{x}} = \boldsymbol{A}_{s(\boldsymbol{x},t)}\boldsymbol{x}$ for $\boldsymbol{x}(0) = \boldsymbol{x}_0$ where $s : \mathbb{R}^n \times \mathbb{R} \to \{1, \ldots, N\}$ is the so-called *switching signal* and $\{\boldsymbol{A}_1, \ldots, \boldsymbol{A}_N\}$ are the system matrices of the *constituent systems*, which are Metzler matrices. See [21,7] for more details on systems of this type.

Finally, the function $V(\boldsymbol{x}) = \boldsymbol{v}^\mathsf{T}\boldsymbol{x}$ is said to be a *linear co-positive Lyapunov function* (LCLF) for the positive LTI system $\Sigma_{\boldsymbol{A}}$ if and only if $V(\boldsymbol{x}) > 0$ and $\dot{V}(\boldsymbol{x}) < 0$ for all $\boldsymbol{x} \succ \mathbf{0}$, or, equivalently, $\boldsymbol{v} \succ \mathbf{0}$ and $\boldsymbol{v}^\mathsf{T}\boldsymbol{A} \prec \mathbf{0}$.

2 State dependent switching of positive systems

We first consider the existence of a common LCLF for sets of positive LTI systems, each of which is associated with a closed convex region of the positive orthant. Such problems arise in the study of state dependent switching problems, see for example [21]. To be more precise, assume that there exist N closed pointed convex cones $\mathcal{C}_j$, such that the state space, $\mathbb{R}^n_+$, the closed positive orthant of $\mathbb{R}^n$, can be written as $\mathbb{R}^n_+ = \cup_{j=1}^N \mathcal{C}_j$. Moreover, assume that we are given stable positive LTI systems $\Sigma_{\boldsymbol{A}_j}$ for $j = 1, \ldots, N$ such that the jth system can only be active for states within $\mathcal{C}_j$.

Our first main result gives a necessary and sufficient condition for the existence of a common LCLF for this type of positive system. Formally, we provide a condition for the existence of a vector $\boldsymbol{v} \succ \mathbf{0}$ such that $\boldsymbol{v}^\mathsf{T}\boldsymbol{A}_j\boldsymbol{x}_j < 0$ for all non-zero $\boldsymbol{x}_j \in \mathcal{C}_j$ for $j = 1, \ldots, N$.

Theorem 1 *Given N Metzler and Hurwitz matrices $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_N \in \mathbb{R}^{n\times n}$ and N closed, convex pointed cones $\mathcal{C}_1, \ldots, \mathcal{C}_N$ such that $\mathbb{R}^n_+ = \cup_{j=1}^N \mathcal{C}_j$, precisely one of the following statements is true:*

- *(i) There is a positive vector $\boldsymbol{v} \in \mathbb{R}^n$ such that $\boldsymbol{v}^\mathsf{T}\boldsymbol{A}_j\boldsymbol{x}_j < 0$ for all non-zero $\boldsymbol{x}_j \in \mathcal{C}_j$ and $j = 1, \ldots, N$.*
- *(ii) There are vectors $\boldsymbol{x}_j \in \mathcal{C}_j$, with $j = 1, \ldots, N$, not all zero such that $\sum_{j=1}^N \boldsymbol{A}_j\boldsymbol{x}_j \succeq \mathbf{0}$.*

PROOF. (ii) $\Rightarrow \neg$(i): [1] Assume that (ii) holds. Then, for any $\boldsymbol{v} \succ \mathbf{0}$ we have $\boldsymbol{v}^\mathsf{T}\boldsymbol{A}_1\boldsymbol{x}_1 + \ldots + \boldsymbol{v}^\mathsf{T}\boldsymbol{A}_N\boldsymbol{x}_N \geq 0$ which implies that (i) cannot hold.

$\neg$(ii) $\Rightarrow$ (i): Assume that (ii) does not hold, i.e. there are no vectors $\boldsymbol{x}_j \in \mathcal{C}_j$ not all zero such that $\sum_{j=1}^N \boldsymbol{A}_j\boldsymbol{x}_j \succeq \mathbf{0}$. This means that the following intersection of convex cones is empty:

$$\underbrace{\Big\{ \textstyle\sum_{j=1}^N \boldsymbol{A}_j\boldsymbol{x}_j : \boldsymbol{x}_j \in \mathcal{C}_j, \text{ not all zero} \Big\}}_{\mathcal{O}_1} \cap \underbrace{\Big\{ \boldsymbol{x} \succeq \mathbf{0} \Big\}}_{\mathcal{O}_2} = \emptyset.$$

By scaling appropriately we can see that this is equivalent to:

$$\underbrace{\Big\{ \textstyle\sum_{j=1}^N \boldsymbol{A}_j\boldsymbol{x}_j : \boldsymbol{x}_j \in \mathcal{C}_j, \sum_{j=1}^N \|\boldsymbol{x}_j\|_1 = 1 \Big\}}_{\bar{\mathcal{O}}_1} \cap \underbrace{\Big\{ \boldsymbol{x} \succeq \mathbf{0} \Big\}}_{\mathcal{O}_2} = \emptyset \tag{1}$$

where $\|\cdot\|_1$ denotes the usual spatial 1-norm. Now, $\bar{\mathcal{O}}_1$ and $\mathcal{O}_2$ are disjoint non-empty closed convex sets and additionally $\bar{\mathcal{O}}_1$ is bounded. Thus, we can apply Corollary 4.1.3 from [9] which guarantees the existence of a vector $\boldsymbol{v} \in \mathbb{R}^n$ such that

$$\max_{\boldsymbol{y}\in\bar{\mathcal{O}}_1} \boldsymbol{v}^\mathsf{T}\boldsymbol{y} < \inf_{\boldsymbol{y}\in\mathcal{O}_2} \boldsymbol{v}^\mathsf{T}\boldsymbol{y} \tag{2}$$

As the zero vector is in $\mathcal{O}_2$, it follows $\inf_{\boldsymbol{y}\in\mathcal{O}_2} \boldsymbol{v}^\mathsf{T}\boldsymbol{y} \leq 0$. However, as $\mathcal{O}_2$ is the cone $\{\boldsymbol{x} \succeq \mathbf{0}\}$ it also follows that $\inf_{\boldsymbol{y}\in\mathcal{O}_2} \boldsymbol{v}^\mathsf{T}\boldsymbol{y} \geq 0$. Thus, $\inf_{\boldsymbol{y}\in\mathcal{O}_2} \boldsymbol{v}^\mathsf{T}\boldsymbol{y} = 0$. Hence, $\boldsymbol{v}^\mathsf{T}\boldsymbol{y} \geq 0$ for all $\boldsymbol{y} \in \mathcal{O}_2$ and it follows that $\boldsymbol{v} \succeq \mathbf{0}$. Moreover, from (2), we can conclude that for any $j = 1, \ldots, N$ and any $\boldsymbol{x}_j \in \mathcal{C}_j$ with $\|\boldsymbol{x}_j\|_1 = 1$, $\boldsymbol{v}^\mathsf{T}\boldsymbol{A}_j\boldsymbol{x}_j < 0$. As $\mathcal{C}_j \cap \{\boldsymbol{x} \succeq \mathbf{0} : \|\boldsymbol{x}\|_1 = 1\}$ is compact, it follows from continuity that by choosing $\epsilon > 0$ sufficiently small, we can guarantee that $\boldsymbol{v}_\varepsilon := \boldsymbol{v} + \varepsilon\mathbf{1} \succ \mathbf{0}$ satisfies $\boldsymbol{v}_\varepsilon^\mathsf{T}\boldsymbol{A}_j\boldsymbol{x}_j < 0$ for all $\boldsymbol{x}_j \in \mathcal{C}_j \cap \{\boldsymbol{x} \succeq \mathbf{0} : \|\boldsymbol{x}\|_1 = 1\}$ and all $j = 1, \ldots, N$. Here $\mathbf{1}$ is the vector of all ones.

Finally, it is easy to see that $\boldsymbol{v}_\varepsilon^\mathsf{T}\boldsymbol{A}_j\boldsymbol{x}_j < 0$ is true even without the norm requirement on $\boldsymbol{x}_j$. This completes the proof of the theorem. □

Remark Assume that the application allows to partition the state space using *simplicial* cones. Such cones $\mathcal{C}_j$ are generated by non-negative, non-singular generating matrices $\boldsymbol{Q}_j \in \mathbb{R}^{n\times n}$

$$\mathcal{C}_j := \Big\{ \boldsymbol{x} \,\big|\, \boldsymbol{x} = \textstyle\sum_{i=1}^n \alpha_i \boldsymbol{Q}_j^{(i)}, \alpha_i \geq 0, i = 1, \ldots, n \Big\} \tag{3}$$

where $j = 1, \ldots, N$ and $\boldsymbol{Q}_j^{(i)}$ is the ith column of $\boldsymbol{Q}_j$. In that case, we can include the generating matrices into the theorem and reword it slightly:

[1] That is, we show that if (ii) is true, then (i) cannot hold.

Theorem 2 *Given N Metzler and Hurwitz matrices $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_N \in \mathbb{R}^{n \times n}$ and N closed simplicial cones $\mathcal{C}_j$ of the type (3), such that $\mathbb{R}^n_+ = \cup_{j=1}^N \mathcal{C}_j$, precisely one of the following statements is true:*

(i) There is a positive vector $\boldsymbol{v} \in \mathbb{R}^n$ such that $\boldsymbol{v}^\mathsf{T} \boldsymbol{A}_j \boldsymbol{x}_j < 0$ for all non-zero $\boldsymbol{x}_j \in \mathcal{C}_j$ and $j = 1, \ldots, N$.
(ii) There are vectors $\boldsymbol{w}_j \succeq \boldsymbol{0}$ not all zero such that $\sum_{j=1}^N \boldsymbol{B}_j \boldsymbol{w}_j \succeq \boldsymbol{0}$, where $\boldsymbol{B}_j := \boldsymbol{A}_j \boldsymbol{Q}_j$.

Remark Condition (ii) above can be checked by running a feasibility check on a suitably defined linear program, [3]. For example, it is easy to see that (ii) is fulfilled if and only if the following linear program is feasible:

$$\begin{aligned} &\text{argmax} && \boldsymbol{1}^\mathsf{T} \tilde{\boldsymbol{w}} \\ &\text{subject to} && \tilde{\boldsymbol{B}} \tilde{\boldsymbol{w}} \succeq \boldsymbol{0}, \quad \tilde{\boldsymbol{w}} \succeq \boldsymbol{0}, \quad \tilde{\boldsymbol{w}} \preceq \boldsymbol{1} \end{aligned}$$

where $\tilde{\boldsymbol{B}}$ corresponds to the horizontally concatenated $\boldsymbol{B}_j$, and $\tilde{\boldsymbol{w}}$ to the vertically stacked $\boldsymbol{w}_j$. It is then straightforward to run a feasibility check on this linear program, to provide an answer in polynomial time.

Remark We shall see in the next section that Theorem 2 leads directly to very elegant conditions for the existence of a common LCLF, and give very interesting insights into the stability of positive switched linear systems, completing initial work reported in [17].

3 Arbitrarily switching systems

An important special case of the previous results is when each of the $\boldsymbol{Q}_j$ matrices is the identity matrix, namely when we seek a common linear co-positive Lyapunov function for a finite set of positive linear systems. In this situation it is possible to develop a further condition which allows to check for the existence of a common LCLF for each of the constituent systems, which in turn would guarantee the stability of the overall system. This will be given by Theorem 4 below.

However, before stating that theorem, we need a technical result which will simplify the proof of Theorem 4. The following lemma is in fact very similar to Theorem 2, when each of the generating matrices $\boldsymbol{Q}_j$ is the identity matrix.

Lemma 3 *Given N Metzler and Hurwitz matrices $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_N \in \mathbb{R}^{n \times n}$ the following statements are equivalent:*

(i) There is a non-zero $\boldsymbol{v} \succeq \boldsymbol{0}$ such that $\boldsymbol{v}^\mathsf{T} \boldsymbol{A}_j \preceq \boldsymbol{0}$ for all $j = 1, \ldots, N$. [2]
(ii) There are no $\boldsymbol{w}_j \succ \boldsymbol{0}$ such that $\sum_{j=1}^N \boldsymbol{A}_j \boldsymbol{w}_j = \boldsymbol{0}$.

[2] Note that with the assumptions of the lemma, $\boldsymbol{v}^\mathsf{T} \boldsymbol{A}_j$ will always be non-zero for a non-zero $\boldsymbol{v} \succeq \boldsymbol{0}$.

As the proof of the lemma follows closely the lines of that of Theorem 1, it has been moved to the appendix.

Some additional notation is required to present our second main result. Let the set containing all possible mappings $\sigma : \{1, \ldots, n\} \to \{1, \ldots, N\}$ be called $\mathcal{S}_{n,N}$, for positive integers n and N. Given N matrices $\boldsymbol{A}_j$, these mappings will then be used to construct matrices $\boldsymbol{A}_\sigma(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_N)$ in the following way:

$$\boldsymbol{A}_\sigma(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_N) := \left[\boldsymbol{A}^{(1)}_{\sigma(1)} \; \boldsymbol{A}^{(2)}_{\sigma(2)} \; \ldots \; \boldsymbol{A}^{(n)}_{\sigma(n)} \right] \qquad (4)$$

that is, the ith column $\boldsymbol{A}_\sigma^{(i)}$ of $\boldsymbol{A}_\sigma$ is the ith column of one of the $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_N$ matrices, depending on the mapping $\sigma \in \mathcal{S}_{n,N}$.

Theorem 4 *Given a finite number of Hurwitz and Metzler matrices $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_N \in \mathbb{R}^{n \times n}$, the following statements are equivalent:*

(i) There is a strictly positive vector $\boldsymbol{v} \in \mathbb{R}^n$ such that $\boldsymbol{v}^\mathsf{T} \boldsymbol{A}_j \prec \boldsymbol{0}$ for all $j = 1, \ldots, N$.
(ii) $\boldsymbol{A}_\sigma(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_N)$ is Hurwitz for all $\sigma \in \mathcal{S}_{n,N}$.

PROOF. (i) $\Rightarrow$ (ii): Assuming that there exists a positive vector $\boldsymbol{v} \in \mathbb{R}^n$ such that $\boldsymbol{v}^\mathsf{T} \boldsymbol{A}_j \prec \boldsymbol{0}$ for all $j = 1, \ldots, N$, this implies, when looking at the columns of the matrices $\boldsymbol{A}_j$, that $\boldsymbol{v}^\mathsf{T} \boldsymbol{A}_j^{(i)} < 0$ for any $i = 1, \ldots, n$ and $j = 1, \ldots, N$. Thus, it follows that $\boldsymbol{v}^\mathsf{T} \boldsymbol{A}_\sigma(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_N) \prec \boldsymbol{0}$ for all $\sigma \in \mathcal{S}_{n,N}$. Next, we note that since the $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_N$ are all Metzler matrices, by construction so must be all the $\boldsymbol{A}_\sigma(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_N)$, $\sigma \in \mathcal{S}_{n,N}$. Finally, applying Theorem 2.5.3 from [10], we have that all matrices $\boldsymbol{A}_\sigma(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_N)$, $\sigma \in \mathcal{S}_{n,N}$, must be Hurwitz.

$\neg$(i) $\Rightarrow$ $\neg$(ii): We show that if there does not exist a vector $\boldsymbol{v}$ as described in (i), then at least one of the matrices $\boldsymbol{A}_\sigma(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_N)$ is not a Hurwitz matrix for some $\sigma \in \mathcal{S}_{n,N}$.

To begin, assume that there is no non-zero $\boldsymbol{v} \succeq \boldsymbol{0}$ such that $\boldsymbol{v}^\mathsf{T} \boldsymbol{A}_j \preceq \boldsymbol{0}$ for all $j = 1, \ldots, N$ (note that this is a stronger assumption than the non-existence of a strictly positive vector $\boldsymbol{v}$, as stated in (i); we will relax this assumption below). From Lemma 3 we then know that there is at least one set of vectors $\boldsymbol{w}_j \succ \boldsymbol{0}$ such that

$$\boldsymbol{A}_1 \boldsymbol{w}_1 + \ldots + \boldsymbol{A}_N \boldsymbol{w}_N = \boldsymbol{0} \qquad (5)$$

Next, we express $\boldsymbol{w}_2, \ldots, \boldsymbol{w}_N$ in terms of $\boldsymbol{w}_1$ using diagonal matrices: $\boldsymbol{w}_j = \boldsymbol{D}_j \boldsymbol{w}_1$ where $\boldsymbol{D}_j = \text{diag}\{d_{j_i}\}$ for all $j = 1, \ldots, N$ and $i = 1, \ldots, n$. We can then rewrite

(5) as

$$\begin{aligned} \boldsymbol{A}_1\boldsymbol{D}_1\boldsymbol{w}_1 + \boldsymbol{A}_2\boldsymbol{D}_2\boldsymbol{w}_1 + \ldots + \boldsymbol{A}_N\boldsymbol{D}_N\boldsymbol{w}_1 &= \boldsymbol{0} \\ \big(\boldsymbol{A}_1\boldsymbol{D}_1 + \ldots + \boldsymbol{A}_N\boldsymbol{D}_N\big)\boldsymbol{w}_1 &= \boldsymbol{0} \end{aligned}$$

and thus, since $\boldsymbol{w}_1 \succ \boldsymbol{0}$, we must have for the determinant

$$\Big|\boldsymbol{A}_1\boldsymbol{D}_1 + \ldots + \boldsymbol{A}_N\boldsymbol{D}_N\Big| = 0 \qquad (6)$$

To simplify notation, define for each mapping $\sigma \in \mathcal{S}_{n,N}$ the following product $p_\sigma := \prod_{i=1}^{n} d_{\sigma(i)_i}$ for which we note that $p_\sigma > 0$ for all $\sigma \in \mathcal{S}_{n,N}$ since $d_{j_i} > 0$ for all i and j. Using the fact that the determinant of a matrix is multilinear in the columns of that matrix, we can now express the left-hand side of (6) as

$$\Big|\sum_{j=1}^{N} \boldsymbol{A}_j\boldsymbol{D}_j\Big| = \sum_{\sigma\in\mathcal{S}_{n,N}} p_\sigma \Big|\boldsymbol{A}_\sigma(\boldsymbol{A}_1,\ldots,\boldsymbol{A}_N)\Big| \qquad (7)$$

Recall that the determinant of a matrix is equal to the product of its eigenvalues. Since an $n\times n$ Hurwitz matrix has strictly negative eigenvalues, its determinant will either be strictly positive (when n is even) or strictly negative (when n is odd), but never zero. Thus, using (7) in (6), we conclude that there must be at least one $\sigma \in \mathcal{S}_{n,N}$ for which $\boldsymbol{A}_\sigma(\boldsymbol{A}_1,\ldots,\boldsymbol{A}_N)$ is not a Hurwitz matrix.

To recapitulate, we have shown so far that if there is no non-zero $\boldsymbol{v} \succeq \boldsymbol{0}$ such that $\boldsymbol{v}^\top\boldsymbol{A}_j \preceq \boldsymbol{0}$ for all j, then at least one of the $\boldsymbol{A}_\sigma(\boldsymbol{A}_1,\ldots,\boldsymbol{A}_N)$ matrices has to be non-Hurwitz. However, in order to finish the proof, we need to extend this result to strictly positive $\boldsymbol{v}$, as stated in the theorem. So let us assume that there is no common $\boldsymbol{v} \succ \boldsymbol{0}$ such that $\boldsymbol{v}^\top\boldsymbol{A}_j \prec \boldsymbol{0}$ for all j. If, additionally, there was no $\boldsymbol{v} \succeq \boldsymbol{0}$ either such that $\boldsymbol{v}^\top\boldsymbol{A}_j \preceq \boldsymbol{0}$ for all j, the result follows from the above discussion. However, if there was such a $\boldsymbol{v} \succeq \boldsymbol{0}$, an additional argument is needed.

Assume that no $\boldsymbol{v} \succ \boldsymbol{0}$ satisfies $\boldsymbol{v}^\top\boldsymbol{A}_j \prec \boldsymbol{0}$ for all j. It then follows that for $\boldsymbol{A}_j(\varepsilon) := \boldsymbol{A}_j + \varepsilon\boldsymbol{1}_{n\times n}$ where $\varepsilon > 0$ and $\boldsymbol{1}_{n\times n}$ is the $n\times n$ matrix of all ones, there cannot be a non-zero $\boldsymbol{v} \succeq \boldsymbol{0}$ achieving $\boldsymbol{v}^\top\boldsymbol{A}_j(\varepsilon) \preceq \boldsymbol{0}$ for all j. This can be shown again by argument of contradiction: Consider there was a vector $\boldsymbol{v} \succeq \boldsymbol{0}$ such that $\boldsymbol{v}^\top\boldsymbol{A}_j(\varepsilon) \preceq \boldsymbol{0}$ for all j and $\varepsilon > 0$. Then

$$\begin{aligned} \boldsymbol{v}^\top\big(\boldsymbol{A}_j + \varepsilon\boldsymbol{1}_{n\times n}\big) &\preceq \boldsymbol{0} \\ \boldsymbol{v}^\top\boldsymbol{A}_j &\preceq \boldsymbol{0} - \varepsilon\boldsymbol{v}^\top\boldsymbol{1}_{n\times n} \\ \boldsymbol{v}^\top\boldsymbol{A}_j &\prec \boldsymbol{0} \end{aligned}$$

for $\varepsilon > 0$ and $j = 1,\ldots,N$, which contradicts the first assumption; thus, there is no non-zero $\boldsymbol{v} \succeq \boldsymbol{0}$ so that $\boldsymbol{v}^\top\boldsymbol{A}_j(\varepsilon) \preceq \boldsymbol{0}$ for all j.

Now, choosing $\varepsilon > 0$ small enough to ensure all $\boldsymbol{A}_j(\varepsilon)$ are still Hurwitz and Metzler matrices, it follows from our earlier argument that there is at least one $\sigma \in \mathcal{S}_{n,N}$ so that $\boldsymbol{A}_\sigma\big(\boldsymbol{A}_1(\varepsilon),\ldots,\boldsymbol{A}_N(\varepsilon)\big)$ is non-Hurwitz.

Finally consider a sequence of (ε_k) such that $\varepsilon_k \to 0$ as $k \to \infty$ and with ε_k small enough so that all $\boldsymbol{A}_j(\varepsilon_k)$ are still Hurwitz and Metzler matrices. Since these matrices and thus all $\boldsymbol{A}_\sigma\big(\boldsymbol{A}_1(\varepsilon_k),\ldots,\boldsymbol{A}_N(\varepsilon_k)\big)$ depend continuously on ε_k, it follows for all $\sigma \in \mathcal{S}_{n,N}$ that $\boldsymbol{A}_\sigma\big(\boldsymbol{A}_1(\varepsilon_k),\ldots,\boldsymbol{A}_N(\varepsilon_k)\big) \to \boldsymbol{A}_\sigma(\boldsymbol{A}_1,\ldots,\boldsymbol{A}_N)$ as $\varepsilon_k \to 0$. And since there is at least one $\sigma \in \mathcal{S}_{n,N}$ for which $\boldsymbol{A}_\sigma\big(\boldsymbol{A}_1(\varepsilon_k),\ldots,\boldsymbol{A}_N(\varepsilon_k)\big)$ is non-Hurwitz this will also be the case for $\boldsymbol{A}_\sigma(\boldsymbol{A}_1,\ldots,\boldsymbol{A}_N)$. □

Theorem 4 states that N positive LTI systems have a common linear co-positive Lyapunov function $V(\boldsymbol{x}) = \boldsymbol{v}^\top\boldsymbol{x}$ if and only if all the $\boldsymbol{A}_\sigma(\boldsymbol{A}_1,\ldots,\boldsymbol{A}_N)$ matrices are Hurwitz matrices, for all $\sigma \in \mathcal{S}_{n,N}$. In that case, switched system formed by these subsystems is uniformly asymptotically stable under arbitrary switching.

Clearly, when $\boldsymbol{A}_j\boldsymbol{Q}_j$ in Section 2 are Metzler and Hurwitz, then this Hurwitz condition can also be used to give a solution to the state dependent switching problem.

Remark Note that the above result may also be deduced from the more general results on **P**-matrix sets given in [22].

4 Final remarks

To conclude our brief paper we note some situations where our results may be of use.

4.1 Numerical example

As a short example for Theorem 4, consider three Metzler and Hurwitz matrices

$$\boldsymbol{A}_1 = \begin{bmatrix} -12 & 6 & 6 \\ 1 & -10 & 2 \\ 5 & 3 & -10 \end{bmatrix}, \boldsymbol{A}_2 = \begin{bmatrix} -12 & 4 & 0 \\ 6 & -10 & 9 \\ 4 & 3 & -13 \end{bmatrix}, \boldsymbol{A}_3 = \begin{bmatrix} -9 & 2 & 8 \\ 6 & -10 & 4 \\ 3 & 0 & -11 \end{bmatrix}$$

It turns out that the $\boldsymbol{A}_\sigma(\boldsymbol{A}_1,\boldsymbol{A}_2,\boldsymbol{A}_3)$ are all Hurwitz matrices, for any $\sigma \in \mathcal{S}_{3,3}$; hence a switched linear positive system with these matrices will be uniformly asymptotically stable under arbitrary switching. If, however, the (3,1)-element of $\boldsymbol{A}_3$ is changed from 3 to 5 — note that after change $\boldsymbol{A}_3$ is still a Metzler and Hurwitz matrix — then the matrix $\boldsymbol{A}_{(3,1,3)} = \big[\boldsymbol{A}_3^{(1)}\ \boldsymbol{A}_1^{(2)}\ \boldsymbol{A}_3^{(3)}\big]$ will have an eigenvalue $\lambda \approx 0.06$ which violates the Hurwitz condition.

4.2 Classes of switched time-delay systems

Consider the class of linear positive systems with time-delay considered by Haddad *et al.*, [8]:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{A}_{\mathrm{d}}\boldsymbol{x}(t-\tau)$$

where $\boldsymbol{A}_{\mathrm{d}}$ is a non-negative matrix and where $\big(\boldsymbol{A} + \boldsymbol{A}_{\mathrm{d}}\big)$ is a Hurwitz and Metzler matrix. As shown in [16], our results can be used to prove the stability of time-delay systems where $\boldsymbol{A}$ switches between a finite set of Metzler, Hurwitz matrices, which would be a slight improvement of the results from [8].

4.3 Switched positive systems with multiplicative noise

Consider the class of switched positive systems

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(t)\boldsymbol{x}, \quad \boldsymbol{A}(t) \in \big\{\boldsymbol{A}_1, \ldots, \boldsymbol{A}_N\big\}$$

If all N constituent systems share a co-positive linear Lyapunov function, then it follows that the system

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(t)\boldsymbol{D}(t)\boldsymbol{x}, \quad \boldsymbol{A}(t) \in \big\{\boldsymbol{A}_1, \ldots, \boldsymbol{A}_N\big\}$$

where $\boldsymbol{D}(t) = \mathrm{diag}\{d_j(t)\}$ for $j = 1, \ldots, N$ is a diagonal matrix, is also exponentially stable, provided that the $d_j(t)$ are strictly positive and bounded for all t and j. Systems of this type arise in situations where the state is reset (for example, by quantisation).

4.4 Robustness of switched positive systems with channel dependent multiplicative noise

An important class of positive systems is the class that arises in certain networked control problems. Here, the system of interest has the form:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(t,\boldsymbol{x})\boldsymbol{x} + \big[\boldsymbol{C}_1(t,\boldsymbol{x}) + \ldots + \boldsymbol{C}_n(t,\boldsymbol{x})\big]\boldsymbol{x}$$

where we assume $\big(\boldsymbol{A}(t,\boldsymbol{x}) + \boldsymbol{C}_1(t,\boldsymbol{x}) + \ldots + \boldsymbol{C}_n(t,\boldsymbol{x})\big)$ to be always Metzler and Hurwitz (for all t and $\boldsymbol{x} \in \mathbb{R}^n_+$), where $\boldsymbol{A}(t,\boldsymbol{x}) \in \mathbb{R}^{n\times n}$ is Metzler, and where $\boldsymbol{C}_i(t,x) \succeq \boldsymbol{0}$ is an $n \times n$ matrix that describes the communication path from the network states to the ith state; namely it is a matrix of unit rank with only one non-zero row. Further, we allow the network interconnection structure to vary with time between N different configurations, so that $\boldsymbol{A}(t,\boldsymbol{x}) \in \big\{\boldsymbol{A}_1, \ldots, \boldsymbol{A}_N\big\}$ and $\boldsymbol{C}_i(t,\boldsymbol{x}) \in \big\{\boldsymbol{C}_{i1}, \ldots, \boldsymbol{C}_{iN}\big\}$ for $i = 1, \ldots, n$. Our principal result can then be used to give conditions such that this system is exponentially stable. Further, by exploiting simple properties of Metzler matrices (all off-diagonal entries are non-negative), we get the robust stability of the related system:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(t,\boldsymbol{x})\boldsymbol{x} + \big[\boldsymbol{C}_1(t,\boldsymbol{x})\boldsymbol{D}_1(t) + \ldots + \boldsymbol{C}_n(t,\boldsymbol{x})\boldsymbol{D}_n(t)\big]\boldsymbol{x}$$

where $\boldsymbol{D}_i(t)$ is a non-negative diagonal matrix whose diagonal entries are strictly positive, but with entries bounded less than one, $i = 1, \ldots, n$.

5 Conclusion

In this paper we have presented necessary and sufficient conditions for the existence of a certain type of Lyapunov function for switched linear positive systems. Examples are given to illustrate some of the implications of our results. Future work will consider switched positive systems with time-delay. We suspect that the results presented here will be of great value in this future study.

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A Proof of Lemma 3

(i) $\Rightarrow$ (ii): Assume there is a non-zero vector $\boldsymbol{v} \succeq \boldsymbol{0}$ such that $\boldsymbol{v}^\mathsf{T}\boldsymbol{A}_j \preceq \boldsymbol{0}$ for all $j = 1, \ldots, N$. Thus,

$$\boldsymbol{v}^\mathsf{T}\boldsymbol{A}_1 + \ldots + \boldsymbol{v}^\mathsf{T}\boldsymbol{A}_N \preceq \boldsymbol{0}$$

and for any set of strictly positive vectors $\boldsymbol{w}_j \succ \boldsymbol{0}$,

$$\boldsymbol{v}^\mathsf{T}\boldsymbol{A}_1\boldsymbol{w}_1 + \ldots + \boldsymbol{v}^\mathsf{T}\boldsymbol{A}_N\boldsymbol{w}_N < 0$$
$$\boldsymbol{v}^\mathsf{T}\big(\boldsymbol{A}_1\boldsymbol{w}_1 + \ldots + \boldsymbol{A}_N\boldsymbol{w}_N\big) < 0$$

so that

$$\boldsymbol{A}_1\boldsymbol{w}_1 + \ldots + \boldsymbol{A}_N\boldsymbol{w}_N \neq \boldsymbol{0}$$

In other words, there are no vectors $\boldsymbol{w}_j \succ \boldsymbol{0}$ such that $\sum_{j=1}^N \boldsymbol{A}_j\boldsymbol{w}_j = \boldsymbol{0}$.

(ii) $\Rightarrow$ (i): Assuming that there are no vectors $\boldsymbol{w}_j \succ \boldsymbol{0}$ such that $\sum_{j=1}^N \boldsymbol{A}_j\boldsymbol{w}_j = \boldsymbol{0}$, we can write

$$\big\{\boldsymbol{A}_1\boldsymbol{w}_1 + \ldots + \boldsymbol{A}_N\boldsymbol{w}_N : \boldsymbol{w}_j \succ \boldsymbol{0}\big\} \cap \big\{\boldsymbol{0}\big\} = \emptyset$$

Since the $\boldsymbol{A}_j$ are all Metzler and Hurwitz matrices, it is easy to show that this implies

$$\underbrace{\big\{\boldsymbol{A}_1\boldsymbol{w}_1 + \ldots + \boldsymbol{A}_N\boldsymbol{w}_N : \boldsymbol{w}_j \succ \boldsymbol{0}\big\}}_{\mathcal{O}_1} \cap \underbrace{\big\{\boldsymbol{x} \succ \boldsymbol{0}\big\}}_{\mathcal{O}_2} = \emptyset$$

This corresponds to the intersection of two open convex cones, $\mathcal{O}_1$ and $\mathcal{O}_2$. As this intersection is empty, the two cones are disjoint and there must exist a separating hyperplane between them, see for instance [19]. In other words, there is a vector $\boldsymbol{v} \in \mathbb{R}^n$ such that

$$\boldsymbol{v}^\mathsf{T}\boldsymbol{y} < 0 \text{ for all } \boldsymbol{y} \in \mathcal{O}_1 \quad \text{and} \quad \boldsymbol{v}^\mathsf{T}\boldsymbol{y} > 0 \text{ for all } \boldsymbol{y} \in \mathcal{O}_2$$

From the second inequality we get that $\boldsymbol{v}$ has to be non-negative (and non-zero). The first inequality, in turn, can be written as

$$\boldsymbol{v}^\mathsf{T}\boldsymbol{A}_1\boldsymbol{w}_1 + \ldots + \boldsymbol{v}^\mathsf{T}\boldsymbol{A}_N\boldsymbol{w}_N < 0 \quad \text{for all } \boldsymbol{w}_j \succ \boldsymbol{0}$$

Furthermore, since $\boldsymbol{v} \succeq \boldsymbol{0}$, and since the inequality has to hold for any choice of (strictly positive) vectors $\boldsymbol{w}_j$, each individual summand must be less than or equal to zero. However, this can only be the case if $\boldsymbol{v}^\mathsf{T}\boldsymbol{A}_j \preceq \boldsymbol{0}$ for $j = 1, \ldots, N$, which completes the proof. □