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Structure selection based on interval predictor model for recovering static non-linearities from chaotic data

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Abstract: This study introduces a method of structure selection based on interval predictor model (IPM) and sum of squares formulation. The main contribution is to provide polynomial identified models that can recover static non-linearities from chaotic data. Moreover, the dynamical behaviour of the identified models is also examined in the structure selection by considering convex combinations of the polynomial functions that describe the IPM. Numerical experiments contemplating non-linear maps borrowed from the literature are presented to illustrate the potential and efficacy of the proposed approach.

1 Introduction

It has been widely recognised the importance of identification for control as a research area that deals with modelling, the design of experiments, identification of dynamic models appropriate for control design and evaluation of the quality of estimated models [1–4]. Although the quality of models, also known as model validation, has been evaluated in many different ways and numerous works have been published [5–7], it is also well known that non-linear systems pose challenging difficulties to find a suitable model. For instance, it has been known for a number of years that most conventional approaches for model validation are not attractive when the models are chaotic [8].

The search of models for identifying chaotic models is a common place for practitioners of control theory and investigators of non-linear science. In both areas, an important topic has been the investigation of how to predict chaotic time series [9–11]. To achieve such a goal, the detection of the dynamical structure is of great importance and it has received great attention over the past few decades [12–16]. Although a great effort has been made in this direction, the choice of the model structure remains an open question [17, 18]. A simple and suitable structure is usually a hard task to overcome. The determination of the terms or the structure of the model has been considered critically important [18]. This issue may be even more difficult when system identification is devoted to finding a structure from chaotic data.

We easily find many works in which a priori knowledge, or some sort of auxiliary information, has been used to build parsimonious models [19–25]. Among those types of auxiliary information, the static non-linearities, which appear as non-linear curves in a bidimensional embedding space, have been used to choose the model structure [17, 26–28]. This has provided substantial help and many chaotic systems have been effectively modelled. However, it has been pointed out that non-smooth static non-linearity can barely be approximated by a polynomial with a finite degree of non-linearity [17]. For instance, it has been reported that no polynomial model of a lower degree had been found to reproduce the tent attractor. In such studies, the solution came up with the use of rational structures [17, 26, 29, 30]. Despite these structures exhibiting good agreement with many chaotic systems, they are not very useful for acquiring global differential models because they involve poles that may eject the trajectory to infinity [29]. Apart from that, it is well accepted that the polynomial structure is very appealing because of the simplicity and the intuition it offers about the system properties [18, 30].

Therefore, it would be very desirable to develop a framework, wherein static non-linearity could be used together with polynomial structures. In this study, we introduce the interval predictor model (IPM) to select a structure for a polynomial non-linear auto regressive moving average model with exogenous input (NARMAX) [18], which allows the use of static non-linearity and polynomial structures. The IPM is a recent technique and can be viewed as a new field in system identification [31–33]. Here, we use this approach to identify polynomials that allow one to make a prediction that embraces the data range, from the point of view of its amplitude. The sum of squares (SOS) [34] method was employed to estimate two polynomials that give us the lower and upper limits containing all the data set. Moreover, we have shown that better approximation of static non-linearity, as well the largest Lyapunov exponent (LLE) [35, 36] can be achieved by means of a convex combination between the polynomial of lower and upper limit. Additionally, we have found that although some models suggested in the literature offer a good fitting for static non-linearity, they do not present chaotic behaviour, as their LLE is negative or zero. With the suggested approach, the convex combination also allows us to find among pre-selected models, those ones which are indeed chaotic.

The proposed method does not require the knowledge of the system mathematical model since the IPM makes use of empirical data. To the best of author's knowledge, this is the first time that an IPM-based approach has been used to select a structure of NARMAX models in order to recover static non-linearities from chaotic data.

The remainder of the paper is organised as follows. Section 2 presents some background on the main issues of the paper, namely static non-linearity, polynomial NARMAX, IPM, and SOS. The description of the proposed method is presented in Section 3, while the numerical experiments applied in two case studies are presented in Section 4. Section 5 concludes the paper.

2 Background

This section presents some background material used in this study.

Definition 1 (map static non-linearities [17]): Let us consider an n_yth-order map

$$y_k = f(y_{k-1}, \ldots, y_{k-n_y}), \tag{1}$$

where f is a non-linear function. The obtained iterating past values according to the non-linear law $f(\cdot)$ is the static non-linearity of the dynamical map.

Example 2 (static non-linearity of a logistic map): Let us consider the logistic map, defined as

$$y_k = ry_{k-1} - ry_{k-1}^2 . \tag{2}$$

In this example, the dynamic of the map can be obtained by iterating the sequence according to

$$f(x) = rx - rx^2 . \tag{3}$$

The function $f(x)$ is the static non-linearity of the logistic map. This representation is useful to understand the model dynamics since it is well known that the sequence y_k can exhibit complicated dynamics even when $f(\cdot)$ is simple [37].

Definition 3 (polynomial NARMAX [38]): A polynomial NARMAX can be defined as

$$\begin{aligned} y_k = F^{\ell}[&y_{k-1}, \ldots, y_{k-n_y}, u_{k-1}, \ldots, u_{k-n_u}, \\ &e_{k-1}, \ldots, e_{k-n_e}], \end{aligned} \tag{4}$$

in which y_k, u_k, and e_k are, respectively, the model output, input, and noise terms at time $k \in \mathbb{N}$. The parameters n_y, n_u, and n_e are the maximum lag for output, input, and noise terms. Terms such as e_k are constantly used during the parameter estimation process to avoid bias. In this work, F^{ℓ} is assumed to be a polynomial with non-linearity degree $\ell \in \mathbb{Z}^+$.

A considerable number of non-linear systems have been usually modelled using particular cases of the polynomial NARMAX, such as the polynomial nonlinear autoregressive model with exogenous input (NARX) or nonlinear autoregressive model (NAR). In the former case, noise terms are not used, whereas the polynomial NAR is used to represent autonomous systems in which the input is not used. In this study, we use the acronym NARMAX to name all of them.

Definition 4 (IPM [31]): An IPM is a technique developed to compute the range of an output variable given a set of input–output data. An IPM can be fully described by its upper and lower boundaries. In this study, such boundaries are set to be polynomial functions, so one can write

$$I(x) = [f_l(x), f_u(x)],$$

where $f_l(x) < f_u(x)$ for all x in the domain of interest.

The polynomial functions to be computed can be described generically as

$$f_p(x) = \beta_0 + \beta_1 x + \beta_2 x^2 + \cdots + \beta_c x^{d_p}, \tag{5}$$

where d_p is the maximum degree that is used for the upper or for the lower polynomial function. The decision variables are the coefficients β_i, $i = 0, \ldots, c$, used to construct the upper polynomial function and for the lower polynomial function. Note that the IPM boundaries might not have the same polynomial degree. Even though IPMs are used to set a range for the observed data, in this study, they are used as a tool to provide a polynomial structure for recovering static non-linearities from chaotic data.

2.1 SOS programming

If a polynomial $P(x)$ of degree $2d_p$ can be written in the form $P(x) = z(x)^T Q z(x)$ where $z(x)$ is a vector of monomials with degree less than or equal to d_p and Q is a symmetric positive definite matrix, then one can say that $P(x)$ is a SOS. The SOS problem can be casted as a semi-definite programming as reported in [34]. The SOS decomposition is obtained from the factorisation of the matrix Q. The example in the sequel illustrates this fact.

Example 5: Consider the following polynomial function

$$P(x) = x^4 - 4x^3 + 13x^2 . \tag{6}$$

$P(x)$ can be rewritten in the form $z(x)^T Q z(x)$ as

$$P(x) = \begin{bmatrix} x^2 & x \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -2 & 13 \end{bmatrix} \begin{bmatrix} x^2 \\ x \end{bmatrix}, \tag{7}$$

where

$$Q = \begin{bmatrix} 1 & -2 \\ -2 & 13 \end{bmatrix}$$

is a positive definite matrix. In this way, $P(x)$ admits a SOS decomposition. By applying a Cholesky factorisation one can write

$$P(x) = \begin{bmatrix} x^2 & x \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x^2 \\ x \end{bmatrix}$$

or

$$P(x) = (x^2 - 2x)^2 + 9x^2$$

that is a SOS.

If a polynomial $P(x)$ is a SOS, then one can say that $P(x)$ is positive. The opposite is not always true [39]. However, for polynomial functions in one variable that is the case considered in this study, the SOS constraint is equivalent to the non-negativity. The SOS constraint can be imposed with the help of numerical tools available in the literature [40, 41].

It is important to emphasise that the proposed method does not require the dynamical system (that generates the input–output pairs) to be described as an SOS polynomial. The SOS polynomial constraint will be imposed to obtain the IPM that contains all data generated by the dynamical system.

3 Structure selection via IPM

In what follows we describe the method that has been used to select the model structure to represent a chaotic system. Instead of requiring the knowledge of the chaotic system mathematical model, the proposed approach uses data to estimate them. This is achieved by means of solving a SOS optimisation problem as detailed below.

The data used to compute the IPM are obtained from the first-order non-linear map

$$y_k = f(y_{k-1}) . \tag{8}$$

Denote (y_{k-1}, y_k) the input–output pairs for $k = 1, \ldots, N$. For the sake of simplicity, hereafter the following change of notation will be considered $x_k = y_{k-1}$. It should be clear that systems under consideration are autonomous. However, a delayed version of the output is used to emulate an input, which takes part in the structure selection procedure.

The conditions for designing an IPM based on N input–output data points (x_k, y_k) from the non-linear map (8) follow the same lines as in [42] and are presented in the following optimisation problem.

Optimisation problem 6: If there exist $\gamma > 0$, polynomials $f_u(x)$ and $f_l(x)$ as in (5), a SOS multiplier $\phi(x)$ and a polynomial $p(x) \le 0$, describing the interval of interest for the design of the IPM, such that the following optimisation problem is satisfied

$$\begin{aligned} &\min \gamma \quad \text{s.t.} \\ &\mathbb{E}_x[f_u(x) - f_l(x)] < \gamma\,. \end{aligned} \tag{9}$$

$$y_k - f_l(x_k) > 0, \quad k = 1, \ldots, N, \tag{10}$$

$$y_k - f_u(x_k) < 0, \quad k = 1, \ldots, N, \tag{11}$$

$$f_u(x) - f_l(x) + \phi(x)p(x) \quad \text{is SOS}, \tag{12}$$

where $\mathbb{E}_x[\,\cdot\,]$ is the expected value with respect to x, then

$$I = [f_l(x), f_u(x)]$$

is an IPM that contains all the data.

When x is a standard joint random vector with joint probability density $f_x(x)$, the cost function in (9) can be calculated analytically for each of the monomials as

$$\mathbb{E}[x^n] = \int_{\underline{x}}^{\overline{x}} x^n f_x(x)\,\mathrm{d}x,$$

where $\{x : \underline{x} \le x \le \overline{x}\}$.

The measurements $f(x_k)$, $k = 1, \ldots, N$ must be located inside the IPM, i.e. between the upper function $f_u(x)$ and the lower function $f_l(x)$. Inequalities (9) and (10) have been used to ensure this condition. Moreover, the SOS constraint (12) is employed to compute the coefficients of the polynomial functions that describe the IPM assuring that the upper function is always above the lower function in the interval of interest. For polynomials in one variable, the SOS constraint is equivalent to the non-negativity. The polynomial $p(x)$ can be constructed as

$$p(x) = (x - \underline{x})(x - \overline{x})\,. \tag{13}$$

In this way, $p(x)$ is guaranteed to be negative or equal to zero in the interval $\underline{x} \le x \le \overline{x}$. It is important to remember that optimisation problem 6 is a convex problem that can be solved using standard semi-definite software such as SeDuMi [43].

Our main goal is to recover a model that, when compared with the original system, provides a trade-off between static non-linearity and dynamical behaviour. Once this goal is achieved in optimisation problem 6, one has an upper polynomial function $f_u(x)$ and a lower polynomial function $f_l(x)$ that provide a region of minimum spread for the data and consequently a good static behaviour for the model. To search for a model which also provides the desired dynamical behaviour, convex combinations of the upper and lower polynomial functions have been considered in the form

$$\hat{f}(x) = \alpha f_u(x) + (1 - \alpha) f_l(x), \quad 0 \le \alpha \le 1\,. \tag{14}$$

Remark 7: The α parameter can be used as a degree of freedom to provide a model with an appropriate trade-off between static and dynamic behaviour. In the numerical experiments, a line search on α has been employed. More sophisticated optimisation methods could be employed to search for α.

Remark 8: It is important to remember that the dynamical model can be obtained directly from the static non-linearity $\hat{f}(x)$, computed from the optimisation problem 6. The performance criteria used to evaluate the obtained model make use of the static model $\hat{f}(x)$ and of the dynamical model $\hat{f}(y_{k-1})$ as well.

3.1 Static metrics

To validate the static non-linearity of the identified map $\hat{f}(x_k)$, its static non-linearity should be considered and somehow compared with the original system. The MSE and ME were used to assess the static behaviour of the identified models. Considering the static non-linearity defined by (1), the MSE can be written as

$$\text{MSE} = \frac{\sum_{i=1}^{M}\left(f(x_i) - \hat{f}(x_i)\right)^2}{M}, \tag{15}$$

where M is the number of points used to compute the static non-linearity, $f(x_i)$ is the map static non-linearity computed for x_i and $\hat{f}(x_i)$ its estimate. The ME can be expressed as follows (16):

$$\text{ME} = \max\left|f(x_i) - \hat{f}(x_i)\right|, \quad i = 1, \ldots, M, \tag{16}$$

which is the greatest distance between the original map $f(x)$ and the estimated map $\hat{f}(x)$.

3.2 Dynamic metric

The reckoning of the LLE has been considered as an answer to the issue of distinguishing the presence of chaos in a dynamical system. It consists of computing the exponential divergence of nearby trajectories. There are different methods to compute this exponent, as shown in [35, 36]. To evaluate $\hat{f}(x_k)$, the method proposed in [44] is a very simple way to compute the LLE based on the lower bound error [45, 46] of two pseudo-orbits of a chaotic system. Due to its simplicity and effectiveness, this method was employed in this work to compute the LLE for the identified models.

4 Numerical experiments

Numerical experiments are used to illustrate the potential of the technique presented in this study. The routines were implemented in Matlab, version 8.3.0.532 (R2014a) using YALMIP [40] and SeDuMi [43] in an Intel(R) Core(TM) i5-4210, 2.4 GHz, 8 GB RAM, Windows 10.

The procedure performed in both examples was based on the following steps. Firstly, a set of $N = 100$ input–output pairs (y_{k-1}, y_k) have been generated from each map to be identified. Secondly, optimisation problem 6 was solved to obtain the IPM (upper polynomial function $f_u(x)$ and lower polynomial function $f_l(x)$) that contains all the data points. Finally, (14) was employed to search for α that provides both static and dynamic metrics with good performance. In what follows the IPM-based structure selection, proposed in this study, has been employed for two well-known maps borrowed from the literature.

4.1 Tent map

Consider the tent map described by

$$y_k = 1 - 1.999|y_{k-1} - 0.5|, \tag{17}$$

where $y_0 = 1$ is the initial condition.

Table 1 presents the results obtained by applying the proposed procedure, considering different degrees of polynomial functions in the IPM with convex combinations for a set of α values. Some interesting aspects can be observed from Table 1. For example, it can be seen that the estimated map $\hat{f}(x)$ with an IPM of degree $d = 2$ and for $\alpha = 1$, presents LLE = 0.9239, showing a good agreement with the original LLE of the tent map (LLE = 0.9993 [47]). It is also noteworthy that the dynamical behaviour of the identified polynomial model is even closer to the tent map than the rational model identified in [30] (LLE = 0.7369). However, not all polynomials of degree $d = 2$ can reproduce the dynamical behaviour of the original tent map, in some cases, the obtained model did not present positive LLE. More than that, Table 1 shows that even polynomials of higher degree ($d = 4, 6$) can produce models that do not meet the expected LLE. As can be seen from Table 1, a line search has been used on α in order to get solutions that fit both system static and dynamic behaviour. It can be noted

Table 1 Identification of tent map with a grid in the parameter α. Estimated $\hat{f}(x_k)$ with different degrees *d* and their respective mean squared error (MSE), maximum error (ME) and LLE. (–) means that the identified model has a negative LLE

		$d=2$			$d=4$			$d=6$	
α	MSE	ME	LLE	MSE	ME	LLE	MSE	ME	LLE
0	0.0115	0.2867	—	0.0037	0.2042	—	0.0012	0.1282	—
0.1	0.0083	0.2584	—	0.0027	0.1841	0.2949	0.0009	0.1157	0.2801
0.2	0.0062	0.2300	—	0.0020	0.1640	0.5753	0.0007	0.1033	0.8549
0.3	0.0053	0.2017	—	0.0015	0.1440	0.6354	0.0006	0.0908	0.8597
0.4	0.0056	0.1734	—	0.0014	0.1239	—	0.0006	0.0783	0.7489
0.5	0.0071	0.1451	—	0.0015	0.1038	0.5922	0.0007	0.0659	0.8272
0.6	0.0098	0.1473	—	0.0020	0.0838	0.7390	0.0009	0.0634	0.9651
0.7	0.0137	0.1722	0.5989	0.0027	0.0895	0.8355	0.0013	0.0740	0.6645
0.8	0.0187	0.1972	0.7252	0.0038	0.1042	—	0.0018	0.0847	0.8623
0.9	0.0250	0.2222	0.6444	0.0052	0.1193	—	0.0023	0.0955	0.8775
1	0.0324	0.2474	0.9239	0.0068	0.1347	—	0.0030	0.1064	0.9300

Table 2 Comparison among polynomial and rational models obtained by [30] for the tent map and via IPM structure selection with degree $d=6$

	MSE	ME	LLE
polynomial ($d=3$) [30]	0.0062	0.1971	—
rational [30]	0.0006	0.0925	0.7369
proposed method: polynomial $d=6$			
$\alpha=0.34$	0.0006	0.0858	0.6563
$\alpha=0.56$	0.0008	0.0592	0.9275
$\alpha=0.71$	0.0013	0.0751	0.9901

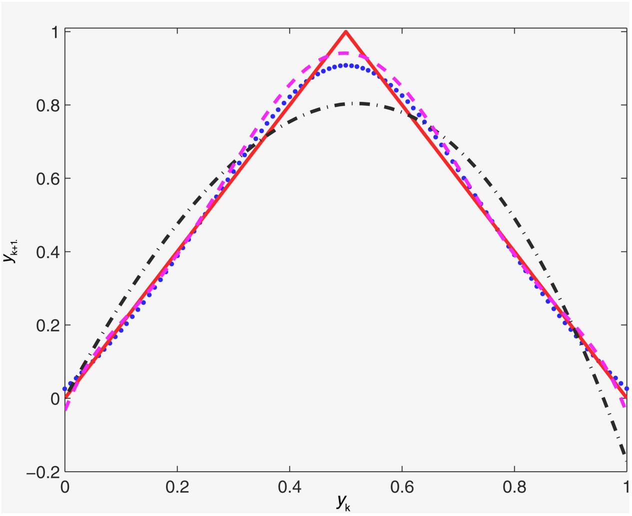

Fig. 1 *First return map. Original tent map (straight red line), polynomial model [30] (dashed dotted black line), rational model [30] (dotted blue line), and SOS model (magenta dashed line)*

also that the estimated LLE is highly dependent on the chosen α while MSE and ME present minor variations. This procedure is particularly interesting because it can be seen as a fine-tuning in order to guarantee the chaotic behaviour of the system. For instance, the polynomial model obtained in [30] (see Table 2) presents some good values of MSE and ME. However, the computation of LLE presented a negative value. In this study, we provide a systematic procedure to find models that exhibit chaotic behaviour. This sort of 'chaotification' has already been investigated in the literature (see [48]). However, in this study, this procedure is made along with the parameter estimation stage, which provides obvious benefits to the system identification field.

As can be seen from Table 1, the parameter $d=6$ produces the largest number of models with a positive LLE. However, it is important to state that there is no evidence that LLE is directly related to the values of *d*, since greater values of *d* can generate overparameterised models that are not capable of reproducing the non-linear behaviour.

If a thinner grid is performed in the parameter α one might find better performance in terms of both MSE and LLE, as can be seen in Table 2. Moreover, by using the proposed technique, one can find polynomial models that outperform the rational structure proposed in [30] with smaller values for MSE, ME and with LLE closer to its real value.

Fig. 1 depicts the static recovered by using the polynomial and rational models identified in [30], the original tent map and the model estimated using the technique presented in this study.

For the sake of completeness, the rational and polynomial dynamical models from [30] are repeated here. A polynomial model for the tent map from [30] is given by

$$y_k = 2.7627y_{k-1} - 1.6938y_{k-1}^2 - 1.2405y_{k-1}^3 \tag{18}$$

and a rational model for the tent map from [30] is given by

$$\begin{aligned} D &= 1 - 2.416y_{k-1} + 2.407y_{k-1}^2, \\ y_k &= \frac{1}{D}\left(0.02608 - 1.325y_{k-1}^2 + 1.325y_{k-1}\right). \end{aligned} \tag{19}$$

The estimated polynomial for the tent map from this study with $d=6$ and $\alpha=0.56$ has been compared with the polynomial and rational models identified by Correa *et al.* [30]. The performance metrics of this model are presented in Table 2 and the model can be written as

$$\begin{aligned} y_k = &-0.03304 + 3.91075y_{k-1} - 25.73878 \\ &y_{k-1}^2 + 131.54011y_{k-1}^3 - 286.49149 \\ &y_{k-1}^4 + 265.92687y_{k-1}^5 - 89.15479y_{k-1}^6. \end{aligned} \tag{20}$$

From Fig. 1 one can see that the polynomial model (20) provides smaller ME than the polynomial and rational models from [30]. Besides, the structure selected by SOS programming still retains a very small MSE, showing a static non-linearity in good agreement with the one presented by the tent map. It is worthwhile to mention that the difference between the LLE of this polynomial and the LLE of the tent map is smaller than the one presented by the models identified by Correa *et al.* [30], allowing to conclude that there is a good commitment between static and dynamic behaviour of the identified model.

4.2 Sine map

Let us now consider the sine map

$$y_k = 3.142\sin(y_{k-1}), \tag{21}$$

where $y_0 = 1.5$ is the initial condition.

Table 3 Comparison among polynomial and rational models obtained by Correa *et al.* [30] for the sine map and via IPM structure selection with degree $d = 6$

	MSE	ME	LLE
polynomial ($d = 3$) [30]	1.1443	1.8780	1.4820
rational [30]	0.1426	0.5473	1.0983
Proposed method: polynomial $d = 6$			
$\alpha = 0.49$	0.0002	0.0853	0.8383
$\alpha = 0.45$	0.0002	0.0782	0.9722
$\alpha = 0.01$	0.0007	0.1420	1.0303

Following the procedure introduced in this study, the polynomial models of degree $d = 6$ were identified for different values of α. Table 3 shows the results obtained for the metrics MSE, ME and LLE considered in this work. The results are compared with polynomial and rational models presented in [30]. The polynomial model and the rational NAR model for the sine map identified by Correa *et al.* [30] are presented in the sequel. The polynomial model for the sine map from [30] is given by (22)

$$y_k = 2.9893y_{k-1} - 0.2479y_{k-1}^3 \tag{22}$$

and the rational model for the tent map from [30] is given by (23)

$$\begin{aligned} D = 1 &- 0.5117 \times 10^{-3}y_{k-2}^2 - 0.3722 \times 10^{-2} \\ &y_{k-3}^2 + 0.8053 \times 10^{-2}y_{k-1}y_{k-2} - 0.2728 \times 10^{-2}y_{k-1} \\ &y_{k-3} + 0.1174y_{k-1}^2, \\ y_k &= \frac{1}{D}(3.9570y_{k-1} - 0.3951y_{k-1}^3). \end{aligned} \tag{23}$$

By applying the approach presented in [44], the LLE for the sine map (21) is LLE = 1.1016. In this sense, one can note from Table 3 that the rational model from [30] provides the closest LLE value. However, it can also be noted from Table 3 that the polynomial models can provide structures that present smaller MSE and ME than the rational model from [30]. These polynomial models are still able to produce a chaotic behaviour close to the original sine map. The first-order identified map for $\alpha = 0.45$ and degree $d = 6$ is given by

$$\begin{aligned} y_k = &- 0.00184 + 3.11530y_{k-1} + 0.00276 \\ &y_{k-1}^2 - 0.49364y_{k-1}^3 - 0.00095 \\ &y_{k-1}^4 + 0.01829y_{k-1}^5 + 0.00007y_{k-1}^6. \end{aligned} \tag{24}$$

Fig. 2 illustrates the static recovered by using the polynomial and rational models identified in [30], the original sine map and the model estimated using the technique presented in this study with $d = 6$ and $\alpha = 0.45$. One can see that the polynomial obtained with the proposed technique provides the best static estimation with a LLE that can reproduce the chaotic behaviour. Therefore, it is possible to conclude that the model identified from the proposed approach can reproduce static non-linearity and the chaotic dynamic of the identified sine map. The same discussion about the α parameter can be applied to this example. For some values of α, the LLE is not positive, meaning that the estimated model is not chaotic.

Although, the models obtained so far represent a significant improvement to the previous work, we believe that there is some room to reduce the number of terms of our models. Future work should address this direction, with particular attention, for instance, for chaotic systems which exhibits hysteresis such as those proposed in [49]. This family of chaotic systems proposed by Sprott is usually designed by a jerk equation. Some of the non-linearity could be addressed by Hammerstein models as described in [27]. An additional perspective can be seen in [50], where the authors have proposed a technique to consider separately linear and non-linear behaviour from data. In the current examples, this procedure could lead to simpler models both for tent or sine map.

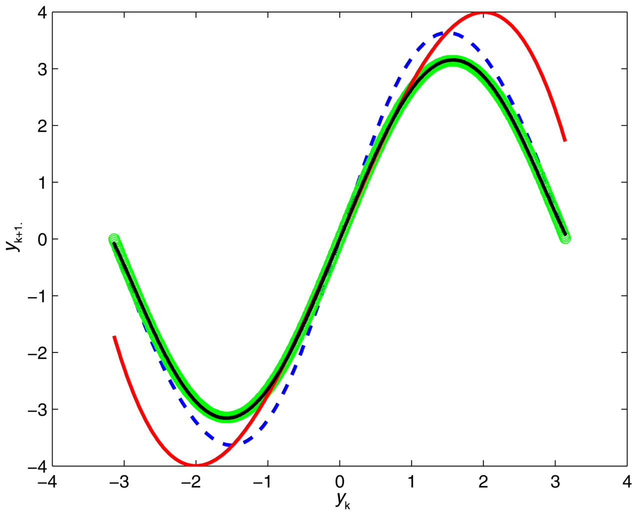

Fig. 2 *First return map. Original sine map (straight black line), polynomial model (straight red line), rational model (dotted blue line), and SOS model (dotted green line)*

However, it should be emphasised that instead of separating linear and non-linear behaviour, our work treats them as a single phenomenon to be identified by a model. This task is well achieved by using the SOS to identify a model structure from the chaotic data. It is also noteworthy that the model structures obtained by the proposed technique are easily handled and suitable to control problems, especially the ones focused on the data-based identified model.

5 Conclusions

This study has presented an IPM-based procedure for the structure selection problem to recover static non-linearities from the chaotic data. The SOS formulation has been employed to provide the upper polynomial function $f_u(x)$ and the lower polynomial function $f_l(x)$ that describe the IPM. The convex combinations of $f_u(x)$ and $f_l(x)$ have been used to search for a specific model that satisfies the user requirement for a given purpose, just varying a single parameter α. The MSE, ME and LLE metrics have been adopted to evaluate the identified models. In this way, it has been possible to estimate low degree representative polynomials for a given chaotic data set. We have used a line search on α to provide solutions that fit both system static and dynamic behaviour. This procedure has been shown to be a good tool to provide models with chaotic behaviour. The tent map and sine map were used to illustrate the proposed approach. In both cases, the identified models, with less complex structures, were capable of outperforming existing results from the literature.

Although the line search has provided better models, there is no guarantee that they represent optimal solutions. Future work should, therefore, include more sophisticated optimisation methods in this stage. It is also desirable to investigate techniques to further simplify the models.

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