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A model validation scale based on multiple indices

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Abstract Validation of an estimated model is not a trivial task because it depends on the purpose of the model, which usually defines the most important features of the model. Thus, in a validation process, the use of diverse tools that exploit different domains is recommended. Here, with this aim, a scale for model validation is proposed that combines the Normalized Root Mean Square Error (NRMSE) with two new indices: the coherence-based index and the fourth-order cross-cumulant index. The proposed scale was used for the validation of three models: the Logistic Map, the Duffing–Ueda oscillator, and the Buck converter. The results demonstrated that the proposed model validation scale produces a more complete validation process that takes into account both time and frequency information and provides robustness against Gaussian noise.

Keywords Coherence · Higher order statistics · Model validation · Fourth-order cross-cumulant · NRMSE

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1 Introduction

System identification is the part of control engineering that uses statistical methods to build mathematical models of dynamic systems from measured data. Despite the approach being used to solve the identification problem and regardless the structure of the final model, the last step in any identification procedure should be the validation of the estimated model. The main objective of the validation is to seek answers to questions such as the following: is the identified model valid? If so, under what conditions? According to [1], models are validated to see how they meet certain validation criteria.

Most conventional approaches for model validation are not effective when the models are chaotic; as a result, alternative invariant indices should be used to quantify the quality and adequacy of the estimated model [2]. In this respect, bifurcation diagrams and the dynamics of attractors have been shown to be far more sensitive to variations in the model structure than many other methods used in model validation. However, the evaluation of bifurcation diagrams and attractors sometimes become subjective [3], and in some cases, these tools are not suitable for validating a model regarding some specific feature.

In several applications, validation is focused on properties regarding time and frequency domains. For example, for series prediction, an index that is able to determine accuracy regarding time prediction is important. A known index frequently used for model validation is the Root Mean Square Error (RMSE), which takes into account similarities and/or changes between time series from the system and the model. However, this index is not able to detect features on frequency and can present erroneous results for noisy signals.

In the literature, reports can be found involving comparisons among different indices. Penaranda and Saavedra-Montes [4] presented a comparison of eight different indices.

Willmott and Matsuura [5] argued that the mean absolute error (MAE) is superior to the RMSE. Although the RMSE presents some weakness, several applications have been reported [5,6].

In this paper, a different perspective is proposed. Instead of making comparisons or claiming that one index is superior to another, we propose the use of multiple indices. The core idea is to combine multiple indices to build a scale for model validation. From this scale, an objective result of validation is available if desired. In addition, the validation result of each index used in the scale is also available, and from this, the user can set the scale to achieve a model with the desired features.

In this work, we will concentrate our efforts on the indices that cover both time and frequency domains. To build the proposed scale, we propose two new indices to be combined with the Normalized Root Mean Square Error (NRMSE) index: the coherence-based index and the fourth-order cross-cumulant. The first index is based on the coherence function [7], which is defined as the parcel of the power spectrum of a given signal that is accounted for by a second signal. The index is analogous to the square of the correlation coefficient, but in the frequency domain [8]. This new index uses the coherence, in a weighted version, as a measurement of the similarity between the original system and the model at the frequency domain. The other proposed index also measures the similarities between the time series from the model and system, but it uses the fourth-order cross-cumulant [9], which is a measure of independence among random variables. However, the use of the fourth-order cross-cumulant for independence measurement avoids the estimation, given a finite data set, of marginal probability density functions and the true high-dimensional entropy of the original process, which is an extremely difficult task in general [9].

The main motivations for the use of the fourth-order cross-cumulant as an index for model validation are the following: (i) higher order cumulants are blind to any kind of Gaussian process, whereas second-order statistics are not; i.e., fourth-order cumulant-based signal processing methods handle colored Gaussian measurement noise automatically, whereas second-order-based methods do not [10]; (ii) the measure of statistical independency is stronger than correlation; (iii) higher order statistics can characterize nonlinear processes better than second-order statistics.

In this paper, the NRMSE, the coherence and the fourth-order cross-cumulant are combined to provide an objective decision for model validation, mixing aspects of both frequency and time domains with robustness against noise. Most important, one of the aims of this paper is to demonstrate the importance of combining indices for the validation process.

Nonlinear models are known to simulate a broader range of systems more precisely than linear models, and for this reason, they are widely studied. Techniques of control and system analysis, as well as of signal processing and system identification, have been developed specifically for nonlinear models [11–19].

In this paper, the proposed method is tested for three systems, which take into account nonlinear data from a real system and nonlinear data with chaotic behaviors: the Logistic Map [20], the Duffing–Ueda oscillator [21,22], and the Buck converter [23].

2 Theoretical background

2.1 Coherence

The coherence function is a measurement of the correlation between two signals in a specific frequency band. The coherence function differs from the cross-spectrum in that it is normalized to yield values with magnitudes between zero (uncorrelated signals) and unity (correlated signals). The magnitude-squared coherence function is often called just coherence, and its estimation for two random, finite-length record, discrete-time signals, $y_1[k]$ and $y_2[k]$, can be obtained according to [7] as:

$$\hat{\gamma}^2_{y_1 y_2}[f] = \frac{|\hat{S}_{y_1 y_2}[f]|^2}{\hat{S}_{y_1 y_1}[f]\hat{S}_{y_2 y_2}[f]} \tag{1}$$

where the "^" superscript denotes an estimation, f is a discrete version of the frequency, $\hat{S}_{y_1 y_2}[f]$ is the cross-spectrum between $y_1[k]$ and $y_2[k]$, and $\hat{S}_{y_1 y_2}[f]\,(j = 1;2)$ is the power spectrum of $y_j[k]$. Such spectral estimations may be obtained using the well-known approach of dividing the signals into M segments as:

$$\hat{S}_{y_q y_p}[f] = \sum_{i=1}^{M} Y^*_{qi}[f] Y_{qi}[f] \tag{2}$$

where $Y_{qi}[f]$ ($Y_{pi}[f]$) is the ith window Discrete Fourier Transform of $y_{qi}[k]$ ($y_{pi}[k]$), the "*" superscript denotes complex conjugate, and M denotes the number of segments used for the estimation. This expression does not need to have the averaging factor (usually $1/MT$, with T denoting the window duration) because it would clearly cancel in coherence estimation according to (1).

2.2 Fourth-order cumulant

A fourth-order cumulant measurement for the independence of random variables $x_1, \ldots, x_n$ can be obtained as indicated by [9]:

$$\mu^4_{\text{cum}}(\mathbf{x}) = \sum_{i,j=1}^{n} \sum_{k,l=1}^{n} \mathbf{cum}\{x_i, x_j, x_k, x_l\}^2, \tag{3}$$

where $i < j$, $\mathbf{x}$ is a random vector whose components $x_1, \ldots, x_n$ have zero means and $\mathbf{cum}\{x_i, x_j, x_k, x_l\}$ denotes the fourth-order cross-cumulant of the random variables x_i, x_j, x_k and x_l:

$$\begin{aligned}\mathbf{cum}\{x_i, x_j, x_k, x_l\} &= E(x_i x_j x_k x_l) - E(x_i x_j)E(x_k x_l) \\ &\quad - E(x_i x_k)E(x_j x_l) - E(x_i x_l)E(x_k x_j)\end{aligned} \tag{4}$$

where E is the expectation operator.

To calculate $\mu^4_{\text{cum}}(\mathbf{x})$ in an efficient way, as proposed by [9], a fourth-order cumulant matrix $\mathbf{Cum}^{2,2}_{\mathbf{x}}(\mathbf{A})$ of $\mathbf{x}$ is defined as follows [24]:

$$\begin{aligned}\mathbf{Cum}^{2,2}_{\mathbf{x}}(\mathbf{A}) &= E\{\mathbf{x}\mathbf{x}^{\mathrm{T}}\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}\} - E\{\mathbf{x}\mathbf{x}^{\mathrm{T}}\}E\{\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}\} \\ &\quad - E\{\mathbf{x}\mathbf{x}^{\mathrm{T}}\}\mathbf{A}E\{\mathbf{x}\mathbf{x}^{\mathrm{T}}\} - E\{\mathbf{x}\mathbf{x}^{\mathrm{T}}\}\mathbf{A}^{\mathrm{T}}E\{\mathbf{x}\mathbf{x}^{\mathrm{T}}\}\end{aligned} \tag{5}$$

where $\mathbf{A} \in \Re^{n^2}$ is a matrix, and the (i, j)th element of $\mathbf{Cum}^{2,2}_{\mathbf{x}}(\mathbf{A})$ is

$$\mathbf{Cum}^{2,2}_{\mathbf{x}}(\mathbf{A})_{i,j} = \sum_{k,l=1}^{n} \mathbf{cum}\{x_i, x_j, x_k, x_l\}\mathbf{A}_{k,l} \tag{6}$$

Considering $\mathbf{I}_{k,l}$ as an $n \times n$ matrix that has a value of one at (k, l) position and the rest of its elements are zero, we then obtain

$$\mathbf{Cum}^{2,2}_{\mathbf{x}}(\mathbf{I}_{k,l})_{i,j} = \mathbf{cum}\{x_i, x_j, x_k, x_l\}. \tag{7}$$

Therefore,

$$\begin{aligned}&\mathbf{Cum}^{2,2}_{\mathbf{x}}(\mathbf{Cum}^{2,2}_{\mathbf{x}}(\mathbf{I}_{u,v}))_{i,j} \\ &\quad = \sum_{k,l=1}^{n} \mathbf{cum}\{x_i, x_j, x_k, x_l\}\mathbf{cum}\{x_i, x_j, x_u, x_v\}.\end{aligned} \tag{8}$$

In the particular case when $(i, j) = (u, v)$, using the symmetry of the cumulants, we obtain

$$\mathbf{Cum}^{2,2}_{\mathbf{x}}(\mathbf{Cum}^{2,2}_{\mathbf{x}}(\mathbf{I}_{u,v}))_{i,j} = \sum_{k,l=1}^{n} \mathbf{cum}\{x_i, x_j, x_k, x_l\}^2 \tag{9}$$

Hence,

$$\mu^4_{\text{cum}}(\mathbf{x}) = \sum_{i,j=1_{i<j}}^{n} \mathbf{Cum}^{2,2}_{\mathbf{x}}(\mathbf{Cum}^{2,2}_{\mathbf{x}}(\mathbf{I}_{i,j}))_{i,j} \tag{10}$$

which is an efficient way for calculating $\mu^4_{\text{cum}}(\mathbf{x})$.

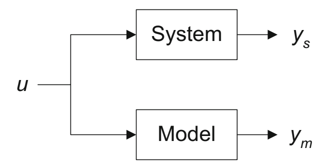


Fig. 1 Schematic representation, where y_{s} is the output of the system and y_{m} is the output of the simulated model for the same input u

It is important to mention that the presence of outliers in the data can lead to erroneous results, since the fourth-order cumulant is sensible to them as showed in [25].

3 Model validation indices

Consider the scheme of Fig. 1, where y_{s} is the output of the original system and y_{m} is the output of the simulated model for the same input u. The NRMSE, the coherence and the cross-cumulant between the signal y_{s}, generated by the real system, and y_{m}, obtained from the simulated model, are calculated. The results from these indices are combined to provide a criterion for evaluating models (model validation).

3.1 NRMSE

The NRMSE index assesses the quality of the estimator in terms of its variation and unbiasedness. In situations involving the presence of noise, NRMSE less than unity is a sign of quality. The NRMSE can be written as

$$\text{NRMSE} = \frac{\sqrt{\sum_{k=1}^{N} (y[k] - \hat{y}[k])^2}}{\sqrt{\sum_{k=1}^{N} (\hat{y}[k] - \bar{y})^2}}, \tag{11}$$

where $\hat{y}[k]$ is the free simulation of the signal, $\bar{y}$ is the average of the signal $y[k]$, and N is the number of samples. A model is worse than an average when it presents an NRMSE greater than 1.

3.2 Coherence weighted index (CWI)

In this paper, we propose a new index to be used as a quantitative measurement for model validation. The coherence estimate gives important information about the linear relation between signals for a specific frequency range, but it is also useful to have a value that represents an overall value of the coherence. The coherence weighted index is calculated as:

$$\text{CWI} = \frac{\sum_{j=1}^{P} w_j \hat{\gamma}^2_{y_{\mathrm{s}}y_{\mathrm{m}}}[f_j]}{\sum_{j=1}^{P} w_j}, \tag{12}$$

where P is the length of the windows (segments) used to compute the coherence, $\hat{\gamma}^2_{y_s y_m}[f_j]$ is the coherence between the signal y_s, generated by the real system, and y_m, obtained from the simulated model, at the discrete frequency f_j, which is calculated according to Eq. (1). The weights w_j, $j = 1, 2, \ldots, P$, are obtained from the power spectrum of the real system signal y_s by Eq. (13).

$$w_j = \left(\frac{\bar{L}_j}{\min(Q)} \right) \tag{13}$$

where $\bar{L}_j$ is the mean of the vector $L_j = [Y_{j1} Y_{j2} \ldots Y_{jM}]$, which contains the power of the frequency components j of y_s for all windows (segments) used to compute the coherence, and $Q = [\bar{L}_1 \;\; \bar{L}_2 \;\; \ldots \;\; \bar{L}_P]$ is the vector containing the mean values $\bar{L}_j$ of all frequency components ($j = 1, 2, \ldots, P$).

The weights obtained with Eq. (13) take into account the power spectrum of the signal generated by the real system, being 1 for the minimum power and greater than 1 for the frequencies where the power is above the minimum one. Therefore, the multiplication of w_j by $\hat{\gamma}^2_{y_s y_m}[f_j]$ in Eq. (12) weighs the values of coherence in accordance with the spectrum content of the real system output signal. Observe that the CWI index is normalized with respect to the weights w_j and, therefore, the CWI values will fall between 0 and 1.

According to [26], the number of windows M used to properly estimate the coherence must be large because the maximum bias, the maximum standard deviation, and the maximum RMS error only decay as $M^{(-1/2)}$. On the other hand, a large number of windows imply a large computation time for processing. For the examples in this paper, $M = 12$ was used due to its compromise between performance and computational cost.

3.3 Forth-order cumulant for model validation

The fourth-order cumulant measurement $\mu^4_{\text{cum}}(\mathbf{x})$ can be applied directly to the output signals from the model and from the system (as depicted in Fig. 1). Values near 0 indicate that the signals are statistically independent, while greater values indicate a level of dependency. Therefore, values near 0 indicate that the model cannot be validated, whereas larger values indicate that the model can be validated.

The $\mu^4_{\text{cum}}(\mathbf{x})$, as used in this paper, normalizes each random variable such that the normalized inputs $\tilde{x}_i$ satisfy $\mu^4_{\text{cum}}(\tilde{x}_i, \tilde{x}_i, \tilde{x}_i, \tilde{x}_i) = 1$ for $i = 1, \ldots, 4$. Note that for different random variables, $\mu^4_{\text{cum}}(\mathbf{x})$ can reach values greater than 1, indicating a certain level of dependence between the random variables.

Table 1 Values achieved for the proposed indices in validating the models y_{m1}, y_{m2} and y_{m3}

Case	CWI	Cumulant ($\mu^4_{\text{cum}}(\mathbf{x})$)
y_{m1}	0.8141	0.0557
y_{m2}	0.9791	1.0000
y_{m3}	0.0213	0.0000

3.4 Example of the use of CWI and fourth-order cumulant for model validation

To illustrate the application of the proposed indices CWI and $\mu^4_{\text{cum}}(\mathbf{x})$ for model validation, consider a toy simulation where the output of the original system is:

$$y_s(t) = \sin(2\pi 60 t) + n_1(t) \tag{14}$$

and the outputs of three different models for the original system are:

$$y_{m1}(t) = \cos(2\pi 60 t) + n_2(t), \tag{15}$$

$$y_{m2}(t) = \sin(2\pi 60 t) + n_3(t), \tag{16}$$

$$y_{m3}(t) = \cos(2\pi 120 t) + n_4(t), \tag{17}$$

where $n_1(t)$, $n_2(t)$, $n_3(t)$ and $n_4(t)$ are uncorrelated white Gaussian noise with zero mean and variance of 0.01. The signals in (14)–(17) were simulated with a sampling frequency of 15.36 kHz (256 samples by cycle) and length of six cycles.

The achieved values of the proposed indices are shown in Table 1. As the outputs y_{m1} and y_{m2} have power spectra similar to the output of the original system y_s, the CWI indicates suitable models for both, despite the fact that the output signal y_{m1} has a phase shift of $\pi/2$ radians with respect to the original system y_s, indicating that CWI is blind to phase shift. On the other hand, the cumulant index indicates no statistical dependence between signals y_{m1} and y_s.

For the output signal y_{m2}, the cumulant indicates high statistical dependence with the original system as expected, since both were modeled as sine waves with 60 Hz frequency.

The third model has the output signal different from the original system on both time and frequency domains and, therefore, both proposed indices indicate a not suitable model for the system.

3.5 Proposed scale for model validation

It is known that the usage of more than one validation index can be a good approach, depending on the system to be modeled and on the use purposes. On the one hand, models for chaotic and nonlinear systems often present a complex behavior that is difficult to reproduce with a model. On the other hand, the system data can be corrupted by noise or

Table 2 Proposed conditions of the validation indices

Case	Condition
C_1	NRMSE <1
C_2	CWI >0.5
C_3	Cumulant >0.5

other unwanted external influences, which can lead to errors concerning the model validation. To address these issues, we proposed a scale to make the model validation process more objective, i.e., providing a quantitative measurement based on multiple indices that allows one to make a decision on the validation of the model. For this, the conditions summarized in Table 2 are defined. From this table, a logical vector including such conditions is constructed:

$$\Delta = [C_1 \quad C_2 \quad C_2], \tag{18}$$

which a scale is defined from the sum of the vector elements ($\sum_i \Delta_i$). The element of the vector is assigned 1 in (18) if the corresponding condition presented in Table 2 is true. This scale, varying from 0 to 3, gives a simple measurement of the validation of the model based on the time and frequency domains. Larger scales indicate better models, whereas smaller scales indicate worse models. A user, according to a specific purpose, may specify a value from 0 to 3 to accept or decline the validation of a model. Moreover, this idea may be extended to more than three indices, according to the user needs. Thus, the proposed scale combines three different indices to provide an objective model validation decision such that each index validates the model under different points of view.

The imposed condition for each index (Table 2) follows a usual process of validation in many works, such as [4]. The values adopted reflect in some sense the meaning of each index, for instance, NRMSE = 1 means the model presents an equivalent performance of a prediction made by a simple average of data, whereas an NRMSE <1 indicates a performance better than the average [27]. This idea is adapted to indices CWI and cumulant in a similar way. The difference here is that we present a simple framework to address multiple indices simultaneously. Moreover, other indices may be added in this strategy to increase the generality of validation procedure.

4 Application and analysis

In this section, the proposed validation method is used for the validation of three nonlinear systems: the Logistic Map, the Duffing–Ueda oscillator and the Buck converter.

4.1 Logistic map

The logistic map [20] is one of the simplest forms of a chaotic process and has been used as a model for population dynamics. Basically, this map, similar to any one-dimensional map, is considered as something even simpler, i.e., an "iterative map", given by equation:

$$y_s[k+1] = r y_s[k](1 - y_s[k]), \tag{19}$$

where r is a parameter, and $k = 1, \ldots, N$ is the sampling index.

Here, we have considered Eq. (19) as the system, and for the model, we have considered 27 different models in which each one undergoes a different interference (see Table 3). The system and the models were simulated with a sampling frequency (f_s) of 100 Hz, $N = 256$ samples, $r = 3.7$ and initial condition $y_s[0] = y_m[0] = 0.75$. An additional model with the initial condition different from that of the system ($y_m[0] = 0.5$) was also tested (Case 27). The goal is to test the sensitivity of the validation indices to the interferences included, which are changes due to phase and frequency noise, offset, Gaussian noise, exponential noise, multiplicative factor, power of second and third order, saturation, delay, exponential decay, polynomial relation, only a noise signal, and different initial conditions.

The results of the validation with the proposed indices and the achieved scale for each model are shown in Table 3. Note the following: (i) the NRMSE index focuses on detecting only time series similarities and changes, while the CWI focuses on detecting only frequency similarities and changes, discarding any similarity in time; (ii) the CWI index is not sensitive to phase shift and noise with spectral content similar to the original system, offset, multiplication factor, saturation and delay; (iii) the cumulant index is immune to Gaussian noise and it is also not sensitive to offset and multiplication factor; (iv) all indices are shown to be sensitive to different initial conditions. Thus, our aim in this case study was to provide the reader with a set of typical interferences, from nonlinearities to noise interference, to clarify the differences among these investigated indices.

The coherence was revealed to be an important tool for system validation because it indicates the linear relationship between the model and the original system in the frequency domain. Thus, the model studied can be evaluated within specific frequency bands. However, non-expected (or undesired) similarities in frequency domain may lead to errors, as indicated in Case 22 (Table 3), which stresses the need for multiple indices.

The fourth-order cross-cumulant was shown to have an important advantage, which is its immunity to Gaussian noise, indicating that it is a good approach for system validation with noisy signals. Examples of this feature are shown in Case 23 (Table 3).

Table 3 Model validation based on (18)

Case	Model	Type	Δ	Scale
(1)	$y_m[k] = y_s[k] + \exp(\mathcal{N}(0, 0.05))$	Exponential noise	[1 1 1]	3
(2)	$y_m[k] = y_s[k] + \mathcal{N}(0, 0.05)$	Normalnoise	[1 1 1]	3
(3)	$y_m[k] = y_s[k] + \mathcal{N}(0, 0.2)$	Normal noise	[1 1 1]	3
(4)	$y_m[k] = (y_s[k])^2$	Square	[1 1 1]	3
(5)	$y_m[k] = 1.2y_s[k]$	Amplification	[1 1 1]	3
(6)	$y_m[k] = y_s[k]$ (saturated at 0.7)	Saturation	[1 1 1]	3
(7)	$y_m[k] = 0.1y_s[k] + 0.1(y_s[k])^2$	Polynomial order 2	[1 1 1]	3
(8)	$y_m[k] = 0.1y_s[k] + 0.1(y_s[k])^2 + 0.1(y_s[k])^3$	Polynomial order 3	[1 1 1]	3
(9)	$y_m[k] = y_s[k + 2]$	Delay	[1 1 1]	3
(10)	$y_m[k] = y_s[k] + \exp(\mathcal{N}(0, 0.2))$	Exponential noise	[1 1 1]	3
(11)	$y_m[k] = y_s[k] + 0.5$	Offset	[0 1 1]	2
(12)	$y_m[k] = 1.5y_s[k]$	Amplification	[0 1 1]	2
(13)	$y_m[k] = 10y_s[k]$	Amplification	[0 1 1]	2
(14)	$y_m[k] = (y_s[k])^3$	Cube	[0 1 1]	2
(15)	$y_m[k] = -0.1(y_s[k])^2$	Minus and square	[0 1 1]	2
(16)	$y_m[k] = y_s[k] + 0.3\sin(20\pi k)$	Additional frequency	[0 1 0]	1
(17)	$y_m[k] = y_s[k] + 0.3\sin(20\pi k + \pi/2)$	Additional frequency	[0 1 0]	1
(18)	$y_m[k] = y_s[k]$ (saturated at 0.4)	Saturation	[0 1 0]	1
(19)	$y_m[k] = y_s[k + 8]$	Delay	[0 1 0]	1
(20)	$y_m[k] = y_s[k + 15]$	Delay	[0 1 0]	1
(21)	$y_m[k] = y_s[k]\exp(-2k/0.6)$	Exponential decay	[0 1 0]	1
(22)	$y_m[k] = 0.6 + 0.01[\sin(2\pi k15) + 2$ $\sin(2\pi k40) + \sin(2\pi k45) + 5\sin(2\pi k47) + \sin(2\pi k50)]$	Similar band frequency	[0 1 0]	1
(23)	$y_m[k] = y_s[k] + \mathcal{N}(0, 0.5)$	Normal noise	[0 0 1]	1
(24)	$y_m[k] = y_s[k] + \exp(\mathcal{N}(0, 0.5))$	Exponential noise	[0 0 0]	0
(25)	$y_m[k] = y_s[k] + \exp(-5k/0.6)$	Exponential decay	[0 0 0]	0
(26)	$y_m[k] = \mathcal{N}(0, 1)$	Normal noise	[0 0 0]	0
(27)	$y_m[1] \neq y_s[1]$	Different initial conditions	[0 0 0]	0

From this case study, one can observe and understand, with a set of typical interferences, the different contributions that each proposed index can provide to the validation process. Note that the goal of the proposed scale is mainly to summarize these contributions and provide an objective answer about the validation of a model.

4.2 Duffing–Ueda system

The classical model for the Duffing–Ueda oscillator is given by the following equation [21]:

$$A\cos(\omega t) = A\cos(\omega t)|_{t=\frac{k}{fs}} = \ddot{y}_s + \alpha\dot{y}_s + y_s^3, \qquad (20)$$

where $k = 0, 1, \ldots, N - 1$, for $N = 1910$, $f_s = 60/\pi$ is the sampling frequency, $\alpha = 1$, $A\cos(\omega k)$ is related to the system input, and $\omega = 1$ rad/s in this case. Such an equation is called the Duffing–Ueda equation, which was originally proposed as a model for nonlinear oscillators and has become a test-bench for the study of nonlinear dynamics because this model can produce a variety of dynamic regimes, in spite of its simple form. This variety can be seen in the bifurcation diagram shown in Fig. 2.

The described model in Eq. (20) has been identified by [22] using the NARX polynomial, as written in Eq. (21).

$$\begin{aligned} y_m[k] = {} & 2.15790y_m[k-1] - 1.32030y_m[k-2] \\ & +0.16239y_m[k-3] + 0.22480 \times 10^{-3}y_m[k-3]^3 \\ & -0.48196 \times 10^{-2}y_m[k-1]^3 \\ & +0.19463 \times 10^{-2}u[k-2] \\ & +0.34160 \times 10^{-3}u[k-1] \\ & +0.35230 \times 10^{-2}y_m[k-1]^2y_m[k-2] \\ & -0.12162 \times 10^{-2}y_m[k-1]y_m[k-2]y_m[k-3] \end{aligned} \qquad (21)$$

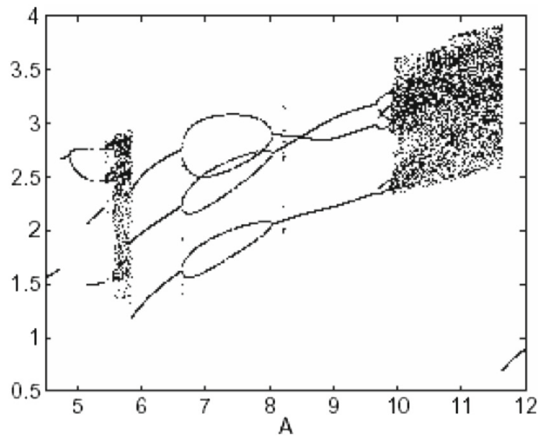


Fig. 2 Bifurcation diagram of the Duffing–Ueda system described in (20)

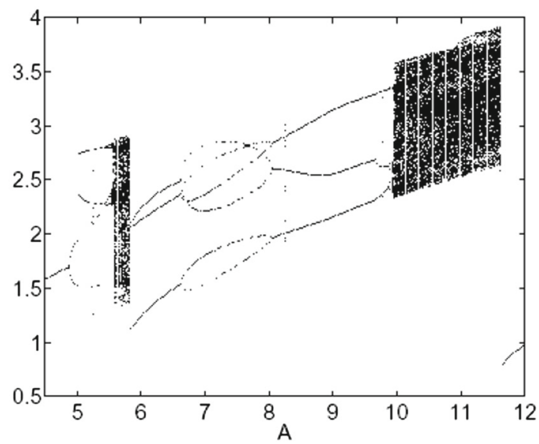


Fig. 3 Bifurcation diagram of the Duffing–Ueda system of the NARX model described in (21)

The bifurcation diagram of Eq. (21) is shown in Fig. 3. The similarities with the original diagram can be noted.

The coherence between the output signal ($y_s[k]$) of the Duffing–Ueda system (Eq. (20)) and that ($y_m[k]$) of the NARX model (Eq. (21)), for the same input ($u[k] = 9\cos[k]$), was estimated according to Eq. (1), as shown in Fig. 4. The coherence values indicate the similarities in frequency between the output signals of the Duffing–Ueda system and its model. Coherence values near 1 indicate strongly linear related signals in frequency, while values near 0 indicate no linear related signals. One can see that for frequencies smaller than 2 Hz, there are several points with low values of coherence, indicating that the NARX model could not reproduce well the system in this low-frequency range. In contrast, coherence values of approximately 0.7300 were achieved for frequencies higher than approximately 2 Hz. The CWI was found to be 0.6167 for this model.

The cross-cumulant ($\mu^4_{\text{cum}}\mathbf{x}$) between the output signals of the Duffing–Ueda system and its model was found to be 0.4019, which shows a certain level of statistical dependency between these random variables. The NRMSE index was also calculated and found to be 0.6353.

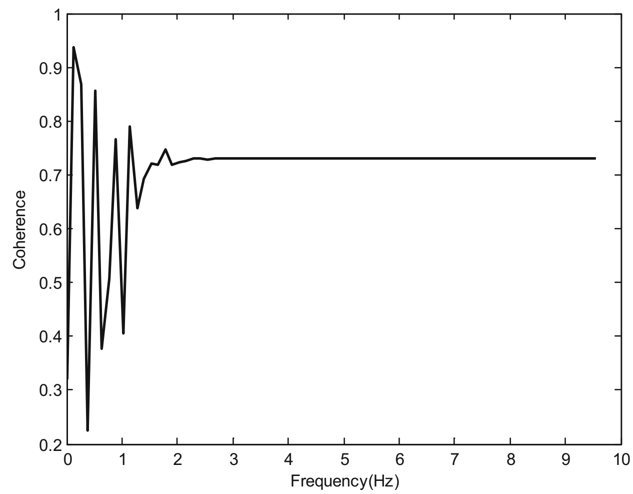


Fig. 4 Coherence between the output signal of the Duffing–Ueda system (y_s) and that (y_m) of the simulated NARX model of this system

Table 4 Values of the validation for the Duffing–Ueda system

Indices	Values of the indices
(NRMSE)	0.6353
(CWI)	0.6167
(Cumulant)	0.4019
Δ	[1 1 0]
Scale	$\sum_i \Delta_i = 2$

According to the proposed decision test, based on a scale (Eq. (18)), the logical vector $\Delta = [110]$ was achieved for the model described in (21), obtaining a scale of $\sum_i \Delta_i = 2$. Table 4 summarizes these results.

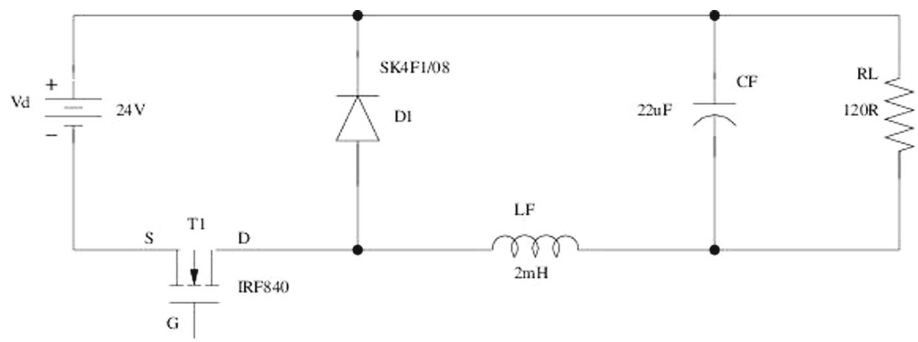
Although the obtained scale has not reached the maximum value, the model in (21) could be considered a validated model because two of the three indices have assigned it as a good model and the cross-cumulant index achieved a slightly lower value than the threshold defined by condition C_3 in Table 2.

4.3 Buck converter

The Buck converter consists of a voltage regulation system controlled by a MOSFET IRF840. The experimental data were obtained from an independent test described in [23], with a sampling time $T_s = 120$ s. The used Buck converter circuit is shown in Fig. 5.

The Buck converter model used in this paper was the NARX model, as described in [23], where only the first half of the data was used for the identification. The NARX poly-

Fig. 5 Buck converter circuit



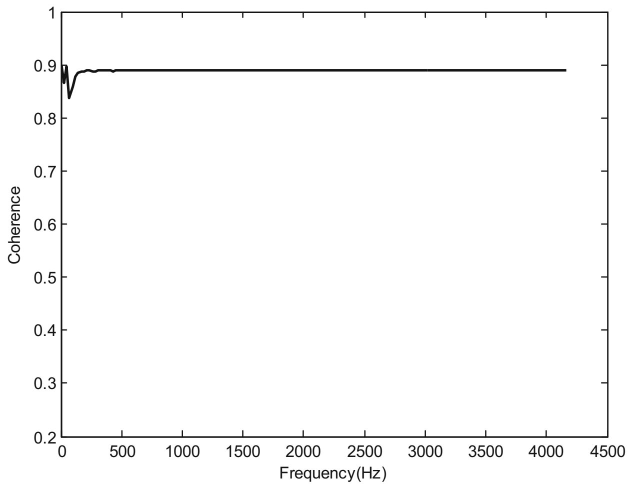


Fig. 6 Coherence between the real output signal of the Buck converter and its model

nomial for the Buck converter is described by Eq. (22).

$$y_m[k] = 1.4628y_m[k-1] - 6.8343 \times 10^{-1}y_m[k-2] \\ +6.6153 - 1.8636u[k-1] \\ +2.5723 \times 10^{-5}y_m[k-3] \\ +2.9910 \times 10^{-1}u[k-3] \quad (22)$$

where $u[k]$ is the input signal.

For the validation process, the second half of the time series was used. Thus, the coherence between the output signal of the system and the output signal of the model described in Eq. (22) was estimated and is displayed in Fig. 6. One can see that high coherence values were achieved, showing that the model in (22) describes well the Buck converter behavior in all frequency ranges. The CWI obtained for this model was 0.8885, indicating its good performance for representing the system.

The cross-cumulant index between the output signal of the system and the output signal of the model described in Eq. (22) was found to be 0.9173, in agreement with the coherence analysis. Table 5 presents the proposed indices. An NRMSE of 0.3684 was achieved for this model. One can see that for this model, $\Delta = [111]$, and hence, the model was assessed as having the maximum scale of 3.

Table 5 Values of the validation for the Buck converter

Indices	Values of the indices
(NRMSE)	0.3684
(CWI)	0.8885
(Cumulant)	0.9173
Δ	[1 1 1]
Scale	$\sum_i \Delta_i = 3$

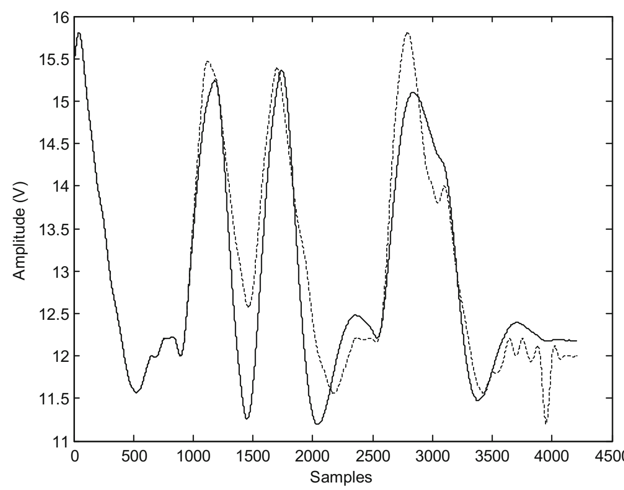


Fig. 7 The Buck converter output signal (*continuous line*), and its model output (*dashed line*)

The validation can also be obtained through the comparison between the system output signal (continuous line) and that from the model (dotted line), as shown in Fig. 7. Again, there is an agreement between the different methods applied to the system validation, but using the proposed indices, quantitative information is available, taking into account two important informative details: linear dependency on the frequency (information available from the coherence estimation) and statistical dependency between the system and model (measured by the cross-cumulant).

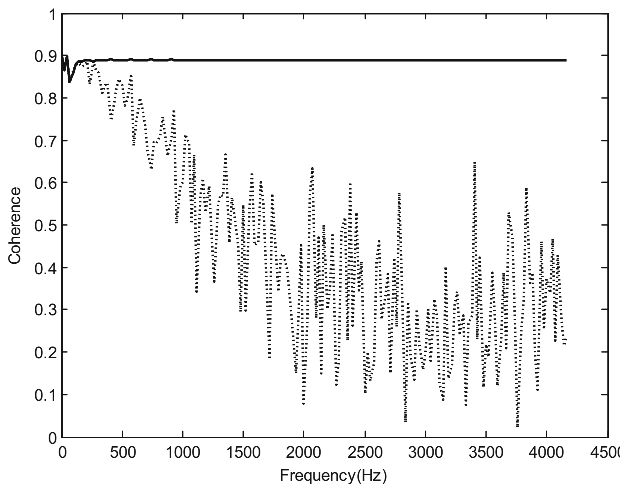


Fig. 8 Coherence between the real output signal of the Buck converter and its model. The *solid line* is the coherence for the system output without noise, and the *dotted line* is the coherence for the noisy output

Table 6 Values of the validation for the Buck converter (noisy simulation)

Indices	Values of the indices
(NRMSE)	0.3772
(CWI)	0.4751
(Cumulant)	0.9173
Δ	[1 0 1]
Scale	$\sum_i \Delta_i = 2$

To evaluate the performance of the proposed indices for signals corrupted by noise, a simulation (noisy simulation) was performed by adding zero mean Gaussian white noise to the Buck converter output with a 10 dB signal-to-noise ratio (SNR). For this scenario, the coherence between the system output (noisy) and the model output was estimated, as shown in Fig. 8. It is possible to verify that the coherence was seriously affected by the noise (CWI $= 0.4751$). For the same simulation, the cross-cumulant was also obtained, and the resulting value was the same for the case without noise (0.9173), as expected. Table 6 presents the CWI, NRMSE and cumulant values for the noisy simulation.

Now, the model with interference from noise is assessed as having a scale of 2 ($\Delta = [101]$). The coherence is found to be more sensitive than the other validation methods, while the fourth-order cross-cumulant is immune to Gaussian noise. Therefore, for Gaussian noise scenarios, it can be concluded that the fourth-order cumulant measurement of independence is a suitable method for model validation and the proposed scale decision method could still be used. Moreover, the condition statements presented in Table 2 could be adjusted to increase its robustness to noise, if desired.

5 Conclusions

In this paper, the use of multiple indices for model validation was exploited, in which a scale combining three validation indices was proposed. For this, two new indices were introduced: the coherence function and the fourth-order cross-cumulant. These indices were combined with the NRMSE, aiming at providing more complete and objective information concerning model validation.

Finally, a scale combining information from the NRMSE, the coherence, and the fourth-order cross-cumulant retains the properties of the three indices, being sensitive to both frequency and time domains and also robust to noise, leading to a quantitative method for model validation. Note that other validation indices can be added to the proposed scale to increase its generalization capability.

The proposed scale is a simple framework that may indicate the quality of a model for multiple indices, which although does not find the best model, can be very useful to provide the end-user a set of suitable models.

In future works, we intend to increase the number of indices and consider the possibility of selecting a set with a minimum number of indices. In addition, we plan to apply this method to investigate structure selection, evaluating the impact of terms on time and frequency domains.

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