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Stability Criteria for Switched and Hybrid Systems*

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Abstract. The study of the stability properties of switched and hybrid systems gives rise to a number of interesting and challenging mathematical problems. The objective of this paper is to outline some of these problems, to review progress made in solving them in a number of diverse communities, and to review some problems that remain open. An important contribution of our work is to bring together material from several areas of research and to present results in a unified manner. We begin our review by relating the stability problem for switched linear systems and a class of linear differential inclusions. Closely related to the concept of stability are the notions of exponential growth rates and converse Lyapunov theorems, both of which are discussed in detail. In particular, results on common quadratic Lyapunov functions and piecewise linear Lyapunov functions are presented, as they represent constructive methods for proving stability and also represent problems in which significant progress has been made. We also comment on the inherent difficulty in determining stability of switched systems in general, which is exemplified by NP-hardness and undecidability results. We then proceed by considering the stability of switched systems in which there are constraints on the switching rules, through both dwell-time requirements and state-dependent switching laws. Also in this case the theory of Lyapunov functions and the existence of converse theorems are reviewed. We briefly comment on the classical Lur'e problem and on the theory of stability radii, both of which contain many of the features of switched systems and are rich sources of practical results on the topic. Finally we present a list of questions and open problems which provide motivation for continued research in this area.

Key words. hybrid systems, switched systems, stability, growth rates, converse Lyapunov theorem, common quadratic Lyapunov functions, dwell-time, Lur'e problem, stability radii

AMS subject classifications. 93-01, 93D09, 93D30, 34D08, 34D10, 37B25, 37C75

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1. Motivation. The past decade has witnessed an enormous interest in systems whose behavior can be described mathematically using a mixture of logic-based

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switching and difference/differential equations. This interest has been primarily motivated by the realization that many man-made systems, and some physical systems, may be modeled using such a framework. Examples of such systems include the multiple-models, switching, and tuning paradigm from adaptive control [118], hybrid control systems [66], and a plethora of techniques that arise in event driven systems. Due to the ubiquitous nature of these systems, there is a growing demand in industry for methods to model, analyze, and understand systems with logic-based and continuous components. Typically, the approach adopted to describe and analyze these systems is to employ theories that have been developed for differential equations whose parameters vary in time. Differential equations whose parameters are time-varying have been the subject of intense study in several communities for the better part of the last century; see, e.g., [127, 98, 149, 33, 34]. While major advances in this topic have been made in the mathematics, control engineering, and, more recently, computer science communities, many important questions that relate to their behavior still remain unanswered, even for linear systems. Perhaps the most important of these relate to the stability of such systems. The objective of this paper is to review the theory of stability for linear systems whose parameters vary abruptly with time and to outline some of the most pressing questions that remain outstanding.

Differential equations whose parameters vary discontinuously have been the subject of study in the mathematics community [50, 164] for a long time. While research has progressed at a steady pace, the past decade in particular has witnessed a developing interest in this topic in several other fields of research. The multidisciplinary research field of hybrid systems that has emerged as a result of this interest lies at the boundaries of computer science, control engineering, and applied mathematics.

A hybrid system is a dynamical system that is described using a mixture of continuous/discrete dynamics and logic-based switching. The classical view of such systems is that they evolve according to mode-dependent continuous/discrete dynamics, and experience transitions between modes that are triggered by "events." The following examples show that this feature occurs in diverse areas of application.

Example 1.1 (see [29]). *Automobile with a manual gearbox.* The motion of a car that travels along a fixed path can be characterized by two continuous variables: velocity v and position s. The system has two inputs: the throttle angle (u) and the engaged gear (g). It is evident that the manner in which the velocity of the car responds to the throttle input depends on the engaged gear. The dynamics of the automobile can therefore be thought of as hybrid in nature: in each mode (engaged gear) the dynamics evolves in a continuous manner according to some differential equation. Transitions between modes are abrupt and are triggered by driver interventions in the form of gear changes (see Figure 1.1).

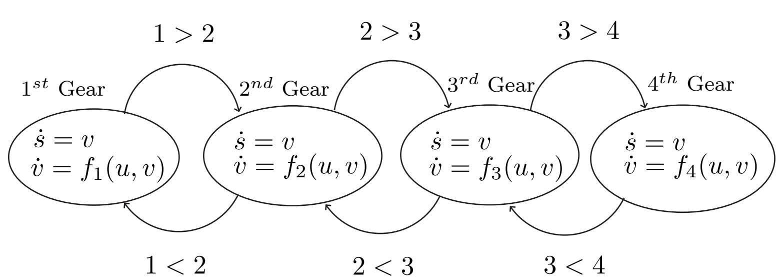

Fig. 1.1 *A hybrid model of a car with a manual gearbox* [29, 73].

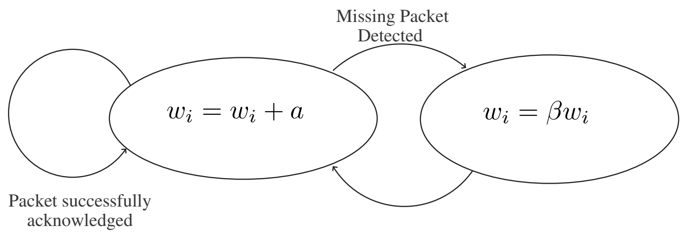

Fig. 1.2 *A hybrid model of TCP operation in congestion avoidance mode.*

Example 1.2 (see [63]). *Network congestion control.* The transmission control protocol (TCP) is the protocol of choice for end-to-end packet delivery in the Internet. TCP is an acknowledgment-based protocol. Packets are sent from sources to destinations, and destinations inform sources of packets that have been successfully received. This information is then used by the ith source to control the number of unacknowledged packets (belonging to source i) in the network at any one time (w_i). This basic mechanism provides TCP sources operating in congestion avoidance mode [93] with a method for inferring available network bandwidth and for controlling congestion in the network. Upon receipt of a successful acknowledgment, the variable w_i is updated according to the rule $w_i \leftarrow w_i + a$, where a is some positive number, and then a new packet is inserted into the network. This is TCP's self-clocking mechanism. If w_i exceeds an integer threshold, then another packet is inserted into the network to increase the number of unacknowledged packets by one. TCP deduces from the detection of lost packets that the network is congested and responds by reducing the number of unacknowledged packets in the network according to $w_i \leftarrow \beta w_i$, where β is some number between 0 and 1 (see Figure 1.2).

Example 1.3. *Biology.* A genetic regulatory network [7, 39, 38] consists of a set of interacting genes, each of which produces a protein through a process known as gene expression. These proteins can then regulate rates of gene expression in the network. The dependence of a gene's rate of expression on the concentrations of its regulatory proteins is typically nonlinear, and in a number of cases appears to be well described by a steep sigmoid function [110], with the regulatory dynamics changing rapidly when protein concentrations cross various threshold values. In order to facilitate analysis, several authors have suggested approximating these sigmoidal interactions by step functions [39, 110]. This approximation leads to a class of switched systems with piecewise linear dynamics (see Figure 1.3). While such models have a number of limitations, they can provide valuable qualitative insights into the dynamics of gene regulation.

It should be clear from the above examples that hybrid systems provide a convenient method for modeling a wide variety of complex dynamical systems. Unfortunately, while the modeling paradigm itself is quite straightforward, the analysis remains a highly nontrivial task. The basic difficulty in their analysis is that even simple hybrid dynamical systems may exhibit extremely complicated nonlinear behavior. Thus, while switched and hybrid systems provide an attractive paradigm for modeling a variety of practical situations, the analysis of such systems is far from straightforward. In fact, the study of switched systems has raised a number of challenging mathematical problems that remain to a large extent unanswered. Many of these problems are related to stability issues in hybrid dynamical systems [64, 89] and give rise to a number of basic questions in this area, some of which we now list.

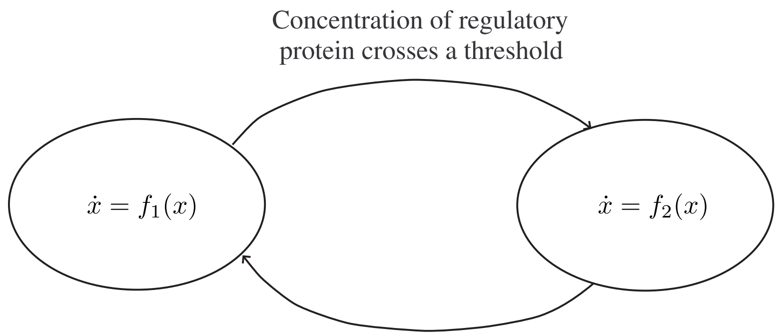

Fig. 1.3 *Switched system model of gene regulatory dynamics.*

(i) *Arbitrary switching.* Is it possible to determine verifiable conditions on a family of constituent systems that guarantee the stability of the associated switched system under arbitrary switching laws? Much of the work on this problem has been focused on the question of common Lyapunov function existence.

(ii) *Dwell-time.* If we switch between a family of individually stable systems sufficiently slowly, then the overall system remains stable [89, 117]. This raises the question of determining how fast we may switch while still guaranteeing stability. In other words, what is the minimum length of time that must elapse between successive switches to ensure that the system remains stable? This problem is usually referred to as the *dwell-time* problem.

(iii) *Stabilization.* While switching between stable systems can cause instability, on the other hand it is sometimes possible to stabilize a family of individually *unstable* systems by switching between them appropriately. Based on this observation, several authors have worked on the problem of determining such stabilizing switching laws [49, 172].

(iv) *Chaos.* Even though it is beyond the scope of the present paper, Chase, Serrano, and Ramadge presented an example in [31] to illustrate how chaotic behavior can arise when switching between low-dimensional linear vector fields. This raises the question as to whether it is possible to determine if a switched system can exhibit chaotic behavior for a given set of constituent vector fields.

(v) *Complexity.* Other problems that have been considered include questions relating to the complexity and decidability of determining the stability of switched systems [20, 18], and the precise nature of the connection between stability under arbitrary switching and stability under periodic switching rules (periodic stability) [17, 178, 84, 35, 59].

The objective of this article is to review the major progress that has been made on a number of these basic questions and other related problems over past decades; see also [102]. As part of this process we will attempt to outline the major outstanding issues that have yet to be resolved in the study of switched linear systems.

2. Definitions and Mathematical Preliminaries.

Throughout this paper our primary concern will be with the stability properties of the switched linear system

$$\Sigma_S : \dot{x}(t) = A(t)x(t), \qquad A(t) \in \mathcal{A} = \{A_1, \ldots, A_m\}, \tag{2.1}$$

where $\mathcal{A}$ is a set of matrices in $\mathbb{R}^{n\times n}$ and $t \to A(t)$ is a piecewise constant[1] mapping from the nonnegative real numbers, $\mathbb{R}_+$, into $\mathcal{A}$. For each such mapping, there is a corresponding piecewise constant function σ from $\mathbb{R}_+$ into $\{1,\dots,m\}$ such that $A(t) = A_{\sigma(t)}$ for all $t \geq 0$. This mapping σ is known as the *switching signal*, and the points of discontinuity, $t_1, t_2, \dots,$ of $A(t)$ (or $\sigma(t)$), are known as the *switching instances*. We denote the set of switching signals by $\mathcal{S}$. A function $x : \mathbb{R}_+ \to \mathbb{R}^n$ is called a solution of (2.1) if it is continuous and piecewise continuously differentiable and if there is a switching signal σ such that

$$\dot{x}(t) = A_{\sigma(t)}x(t)$$

for all t except at the switching instances of σ. By convention we consider only right continuous switching signals. This does not affect the set of solutions, as any other choice would change only the differential equation on a set of measure zero.

For $1 \leq i \leq m$, the ith constituent system of the switched linear system (2.1) is the linear time-invariant (LTI) system

$$\Sigma_{A_i} : \dot{x} = A_i x. \tag{2.2}$$

We can think of the system (2.1) as being constructed by switching between the constituent LTI systems $\Sigma_{A_1}, \dots, \Sigma_{A_m}$, with mode switches occurring at the switching instances and the precise nature of the switching pattern determined by the switching signal.

For a given switching signal $\sigma \in \mathcal{S}$, the system Σ_S evolves like an LTI system between any two successive switching instances. Thus for each switching signal σ and initial condition $x(0)$, there exists a unique continuous, piecewise differentiable solution $x(t)$ which is given by

$$x(t) = [e^{A(t_k)(t-t_k)}e^{A(t_{k-1})(t_k-t_{k-1})}\dots e^{A(t_1)(t_2-t_1)}e^{A(0)(t_1)}]x(0), \tag{2.3}$$

where $t_1 < t_2 < \cdots$ is the sequence of switching instances and t_k is the largest switching instance smaller than t.

It should be noted that the stability theory of switched linear systems has close links with the corresponding theory for linear differential inclusions (LDIs) [4, 164]. The LDI related to the set $\mathcal{A} = \{A_1, \dots, A_m\}$ is denoted by

$$\dot{x}(t) \in \{Ax(t) \mid A \in \mathcal{A}\}. \tag{2.4}$$

A solution of this inclusion is an absolutely continuous function x satisfying $\dot{x}(t) \in \{Ax(t) \mid A \in \mathcal{A}\}$ almost everywhere. By an application of Filippov's theorem this is equivalent to saying that there exists a measurable map $A : \mathbb{R}_+ \to \mathcal{A}$ such that

$$\dot{x}(t) = A(t)x(t) \quad \text{almost everywhere;}$$

see [50] for details. So studying the differential inclusion (2.4) amounts to extending the set of switching signals to the class of measurable functions. If we are studying the system for arbitrary switching sequences, the effect of this is often negligible. In fact, if we consider the convex hull of $\mathcal{A}$, denoted by $\operatorname{conv}\mathcal{A}$, and the convexified differential inclusion

$$\dot{x}(t) \in \{Ax(t) \mid A \in \operatorname{conv}\mathcal{A}\}, \tag{2.5}$$

[1]We recall that by definition piecewise constant maps have only finitely many discontinuities in any bounded time interval.

then the solution sets of the three systems we have now defined are closely related. To make this statement precise, we denote by $\mathcal{R}_t^{\text{switch}}(x)$ the set of points that can be reached from an initial condition x at time t by solutions of (2.1), i.e.,

$$\mathcal{R}_t^{\text{switch}}(x) := \{y \in \mathbb{R}^{n\times n} \mid \exists \text{ switching signal } \sigma \text{ such that } y = x(t;x,\sigma)\}\,.$$

Similarly, we introduce the notation $\mathcal{R}_t^{\text{ldi}}(x), \mathcal{R}_t^{\text{conv ldi}}(x)$ for reachable sets of (2.4) and (2.5), respectively; then we have for all $t \geq 0, x \in \mathbb{R}^n$ that

$$\mathcal{R}_t^{\text{switch}}(x) \subset \mathcal{R}_t^{\text{ldi}}(x) \subset \mathcal{R}_t^{\text{conv ldi}}(x) = \text{cl}\,\mathcal{R}_{\text{t}}^{\text{switch}}(\text{x}); \tag{2.6}$$

see, e.g., [4, 50]. For an in-depth investigation of the structure of the signal set and its interplay with the dynamics of (2.4), we refer to [32]. We also note that systems equivalent to (2.4) are often studied under the name of *linear parameter-varying systems*. We will make brief reference to the relation between this literature and switched systems where appropriate in what follows.

Notation. We will write $r(A)$ for the spectral radius of the matrix A, that is, the largest among the absolute values of its eigenvalues. We write $P > 0$ to indicate that the symmetric matrix P is positive definite, and $P \geq 0$ to indicate positive semidefinite. Similarly, $P < 0$ and $P \leq 0$ indicate negative definite and negative semidefinite, respectively.

2.1. Discrete-Time Systems. Thus far we have considered only switched linear systems in continuous time. However, as in the example of TCP congestion control discussed in the last section, it is also of interest to study discrete-time switched linear systems. In discrete time, a switched linear system has the form

$$\Sigma_S : x(k+1) = A(k)x(k), \qquad A(k) \in \mathcal{A} = \{A_1, \ldots, A_m\}, \tag{2.7}$$

where as before $\mathcal{A}$ is a set of matrices in $\mathbb{R}^{n\times n}$ and $k \to A(k)$ is a mapping from the nonnegative integers into $\mathcal{A}$. The notions of switching signal, switching instances, and constituent systems are defined analogously to the continuous-time case. The existence of solutions to (2.7) is straightforward. In the discrete-time case, the analysis of (2.7) is equivalent to that of the discrete linear inclusion

$$x(t+1) \in \{Ax(t) \mid A \in \mathcal{A}\}\,. \tag{2.8}$$

On the other hand the solution set is significantly enlarged when going over to the convexified inclusion

$$x(t+1) \in \{Ax(t) \mid A \in \text{conv}\,\mathcal{A}\}\,. \tag{2.9}$$

It is of interest to note that the exponential stability of (2.7), (2.8), and (2.9) is equivalent nonetheless, as we shall discuss in what follows.

2.2. Exponential Growth Rates. One of the basic properties of switched linear systems is that a growth rate may be defined as in the case of LTI systems. The definition proceeds similarly in continuous and discrete time. There are several approaches to defining the exponential growth rate, all of which turn out to be equivalent. A trajectory-based definition considers Lyapunov exponents of individual trajectories, which are defined by

$$\lambda(x_0, \sigma) := \limsup_{t\to\infty} \frac{1}{t} \log \|x(t;x_0,\sigma)\|\,.$$

The exponential growth rate of the switched system is then defined by the maximal Lyapunov exponent

$$\kappa(\mathcal{A}) := \sup\{\lambda(x_0, \sigma) \mid x_0 \neq 0, \sigma \in \mathcal{S}\}\,.$$

A different point of view is to consider the evolution operators corresponding to the system equations (2.1), respectively, (2.7). In continuous time these are defined as the solution of

$$\dot{\Phi}_\sigma(t,s) = A_{\sigma(t)}\Phi_\sigma(t,s)\,, \quad \Phi_\sigma(s,s) = I\,.$$

Similarly, in discrete time we have for $t \geq s$

$$\Phi(t+1,s) = A_{\sigma(t)}\Phi_\sigma(t,s)\,, \quad \Phi_\sigma(s,s) = I\,.$$

The growth of the system can then also be measured by considering the maximal growth of the norms or the spectral radii of the operators $\Phi(t,0)$ as $t \to \infty$. Ultimately, these definitions coincide. More precisely, it is known that

$$\kappa(\mathcal{A}) = \limsup_{t\to\infty} \frac{1}{t} \log \max_{\sigma\in\mathcal{S}} r(\Phi_\sigma(t,0)) = \limsup_{t\to\infty} \frac{1}{t} \log \max_{\sigma\in\mathcal{S}} \|\Phi_\sigma(t,0)\|\,. \tag{2.10}$$

The previous equality has been obtained following two different avenues. In discrete time it was first shown by Berger and Wang [12], and alternative ways of proving the result were presented by Elsner [46] and Shih, Wu, and Pang [150]. On the other hand the result has also been implicit in the Russian literature. Namely, in the continuous-time setting Pyatnitskii and Rapoport [141] showed that if system (2.1) has an unbounded trajectory, then there exists a *periodic* switching signal $\sigma_j \in \mathcal{S}$ of period T such that $r(\Phi_{\sigma_j}(T,0)) = 1$, i.e., we can find a *periodic* switching signal for which the system is marginally stable. This implies, in particular, that absolute stability is equivalent to

$$\bar{\kappa} := \limsup_{t\to\infty} \frac{1}{t} \log \max_{\sigma\in\mathcal{S}} r(\Phi_\sigma(t,0)) < 0$$

(see Definition 3.2 and subsequent comments for the definition of absolute stability). On the other hand Barabanov [9] showed that absolute stability is equivalent to

$$\hat{\kappa} := \limsup_{t\to\infty} \frac{1}{t} \log \max_{\sigma\in\mathcal{S}} \|\Phi_\sigma(t,0)\| < 0\,.$$

As both $\bar{\kappa}, \hat{\kappa}$ are additive with respect to spectral shifts, i.e., $\bar{\kappa}(\mathcal{A} - \alpha I) = \bar{\kappa}(\mathcal{A}) - \alpha$ for all $\alpha \in \mathbb{R}$, they have to satisfy $\bar{\kappa} = \hat{\kappa}$.

In the area of discrete-time systems the growth rate of discrete inclusions is often defined as $\rho(\mathcal{A}) := e^{\kappa(A)}$. This quantity has become known familiarly as joint spectral radius or generalized spectral radius.

There are numerous approaches to the computation of growth rates, either in their guise as maximal Lyapunov exponents or as joint spectral radii. We cannot cover these methods here but refer the reader to the various methods presented in [55, 97, 10, 56].

3. Stability for Switched Linear Systems. As with general linear and nonlinear systems, numerous different concepts of stability have been defined for switched linear systems, including uniform stability, uniform attractivity, uniform asymptotic stability, and uniform exponential stability. We now recall the definitions of uniform stability and uniform exponential stability.

Definition 3.1. *The origin is a uniformly stable equilibrium point of Σ_S if, given any $\epsilon > 0$, there is some $\delta > 0$ such that $\|x(0)\| < \delta$ implies $\|x(t)\| < \epsilon$ for $t \geq 0$, for all solutions $x(t)$ of the system.*

Definition 3.2. *The origin is a uniformly exponentially stable equilibrium of Σ_S if there exist real constants $M \geq 1$, $\beta > 0$ such that*

$$\|x(t)\| \leq Me^{-\beta t}\|x(0)\| \tag{3.1}$$

for $t \geq 0$, for all solutions $x(t)$ of Σ_S.

Uniform exponential stability is often called *absolute stability*, especially in the Russian literature. It is known that the related concepts of attractivity and asymptotic stability together are equivalent to exponential stability for switched linear systems [37, 9, 36]. For a switched linear system the exponential growth rate κ defined in (2.10) is negative if and only if (3.1) is satisfied for some $\beta > 0$, that is, if and only if the origin is uniformly exponentially stable.

In a slight abuse of notation we shall often speak of the stability or exponential stability of the system Σ_S itself. One of the major topics discussed here is the problem of establishing when the system (2.1) is exponentially stable for arbitrary switching signals. In this case, Definition 3.2 requires the existence of constants $M \geq 1$, $\beta > 0$ such that (3.1) is satisfied for every piecewise continuous switching signal $\sigma(t)$. When considering the question of exponential stability under arbitrary switching, it is necessary to assume that the matrices $A_1, \ldots, A_m$ in the set $\mathcal{A}$ are all *Hurwitz* (all of their eigenvalues lie in the open left half of the complex plane), thus ensuring that each of the constituent LTI systems is exponentially stable.

In certain situations, it is not necessary to guarantee stability for every possible switching signal, and a number of authors have considered questions related to the stability of switched linear systems under restricted switching regimes. One important example of this is *state-dependent* switching, where the rule that determines when a switch in system dynamics may occur is determined by the value of the state vector x. The example of a genetic regulatory network discussed in the previous section was of this type.

All of the above definitions concern the stability properties of the system (2.1). In practice, however, it is often necessary to consider systems with inputs and outputs of the form

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u, \\ y &= \phantom{A(t)x + {}} C(t)x. \end{aligned} \tag{3.2}$$

In this context the notion of bounded-input bounded-output (BIBO) stability arises. Formally, the input-output system (3.2) is uniformly BIBO stable if there exists a positive constant η such that for any essentially bounded input signal u, the zero-state response y satisfies

$$\operatorname*{ess\,sup}_{t\geq 0} \|y(t)\| \leq \eta \operatorname*{ess\,sup}_{t\geq 0} \|u(t)\|\,.$$

Essentially, if a system is BIBO stable, this means that an input signal cannot be amplified by a factor greater than some finite constant η after passing through the system. While we shall not consider BIBO stability explicitly here, it should be noted that if the system (2.1) is uniformly exponentially stable, then the corresponding input-output system (3.2) is BIBO stable provided the matrices $B(t), C(t)$ are uniformly bounded in time [111], which is the case when the system switches between a finite family of matrices.

4. Stability for Arbitrary Switching.

The arbitrary switching problem is concerned with obtaining verifiable conditions on the matrices in $\mathcal{A}$ that guarantee the exponential stability of the switched system (2.1) for any switching signal. A number of general approaches to this problem have been investigated, many of which rely on the construction of common quadratic and nonquadratic Lyapunov functions for the constituent systems of (2.1). In this context, it has been established that the existence of a common Lyapunov function is necessary and sufficient for the exponential stability of a switched linear system. In particular, a number of authors have derived *converse theorems* that prove the existence of common Lyapunov functions under the assumption of exponential stability. We begin by describing some of these theorems and then proceed by reviewing results on the common Lyapunov function existence problem for switched linear systems. Many of the mature results in this area concern the existence of common quadratic Lyapunov functions (CQLFs), and this part of our review reflects this fact. Nevertheless, some results are also presented concerning the existence of common nonquadratic Lyapunov functions.

4.1. Converse Theorems.

Lyapunov theory played a key role in the stability analysis of both linear and nonlinear systems for much of the last century [96, 135, 119, 76]. The key idea of this approach is that the stability of a dynamical system can be established by demonstrating the existence of a positive definite, norm-like function that decreases along all trajectories of the system as time evolves. Much of the recent research on the stability of switched linear systems has been directed toward applying similar ideas to the class of systems (2.1): relating the stability of such systems to the existence of positive definite functions, V, on $\mathbb{R}^n$ such that $V(x(t))$ is a decreasing function of t for all solutions $x(t)$ of (2.1). Before discussing results that have been derived for specific forms of Lyapunov functions, we first present a number of more general facts about Lyapunov theory as it relates to the stability of switched linear systems.

First of all, note that if a positive definite function $V(x(t))$ decreases along all trajectories of the system (2.1) for arbitrary switching signals, then this certainly must be true for constant switching signals. Hence, any such function $V(x)$ would have to be a *common* Lyapunov function for each of the constituent LTI systems of (2.1). It is well established [115, 89, 36] that if a common Lyapunov function exists for the constituent systems of a switched linear system, then the system is uniformly exponentially stable for arbitrary switching signals. We shall now discuss the work of a variety of authors who have considered the problem of deriving *converse theorems* to establish the necessity of common Lyapunov function existence for uniform exponential stability under arbitrary switching.

In [115], it was established that the uniform exponential stability of the system (2.1) under arbitrary switching is equivalent to the existence of a common Lyapunov function $V(x)$ for its constituent LTI systems. Formally the authors derived the following result.

THEOREM 4.1. *The system* (2.1) *is uniformly exponentially stable for arbitrary switching signals if and only if there exists a strictly convex, positive definite function* $V(x)$*, homogeneous of degree* 2*, of the form*

$$V(x) = x^T\mathcal{L}(x)x, \qquad \text{where } \mathcal{L}(x) \in \mathbb{R}^{n\times n}, \text{ and}$$
$$\mathcal{L}(x)^T = \mathcal{L}(x) = \mathcal{L}(cx) \quad \text{for all nonzero } c \in \mathbb{R}, x \in \mathbb{R}^n,$$

such that

$$\max_{y\in\mathcal{A}x} \frac{\partial V(x)}{\partial y} \le -\gamma\|x\|^2$$

for some $\gamma > 0$*, where* $\mathcal{A}x = \{A_1 x, \ldots, A_m x\}$ *and*

$$\frac{\partial V(x)}{\partial y} = \inf_{t>0} \frac{V(x+ty) - V(x)}{t}$$

is the usual directional derivative of the convex function $V(x)$ [145].

A number of points about the results presented in [115] are worth noting.

(i) While we have stated Theorem 4.1 for switched linear systems of the form (2.1), the result was originally derived in [115] for the associated LDI (2.4). However, the statement given here follows from combining (2.6) with Theorem 1 of [115] and shows that common Lyapunov function existence is a necessary condition for the uniform exponential stability of the switched linear system (2.1) under arbitrary switching.

(ii) Theorem 4.1 establishes that any switched linear system which is uniformly exponentially stable under arbitrary switching must have a common Lyapunov function that is strictly convex and homogeneous of degree two.

(iii) Furthermore, it was shown in [115] that the uniform exponential stability, under arbitrary switching, of the switched linear system (2.1) is equivalent to the existence of

  (a) a *piecewise quadratic* (convex but not necessarily continuously differentiable) common Lyapunov function;

  (b) a *piecewise linear* (convex but not necessarily continuously differentiable) common Lyapunov function;

  (c) a smooth (C^∞) common Lyapunov function of the form $V(x) = \sum_{i=1}^{M} (l_i^T x)^{2p}$ for some integer $p > 0$ and vectors $l_1, \ldots, l_M$ in $\mathbb{R}^n$. Such a Lyapunov function is homogeneous of degree $2p$.

(iv) It has been shown in [36] that the uniform exponential stability of the switched linear system (2.1) under arbitrary switching is equivalent to the existence of a C^1 common Lyapunov function that is homogeneous of degree two.

Note that an example was presented in [36] to show that the uniform exponential stability of (2.1) under arbitrary switching does not in general imply that there exists a CQLF for its constituent systems. This had already been noted by Brockett in [30].

In the context of converse Lyapunov theorems, the work of Brayton and Tong, described in [27], is also worthy of mention. These authors established that the existence of a common Lyapunov function for the constituent systems of a discrete-time switched linear system is equivalent to the uniform stability of the system under arbitrary switching. Independently, Barabanov [8] showed that for an exponentially stable discrete linear inclusion there is always a norm that is a Lyapunov function. In particular, this implies, by the convexity of norms, that if the set $\mathcal{A}$ generates an exponentially stable discrete linear inclusion, then so does $\operatorname{conv} \mathcal{A}$.

More recently, Lin, Sontag, and Wang derived a general converse Lyapunov theorem for nonlinear systems with time-varying parameters in [91]. An earlier result on the existence of differentiable Lyapunov functions (on compact sets) for exponentially stable time-varying systems can be found in [109]. The more general results presented in [91] can be applied to stable invariant sets as well as to equilibria, and establish (among other things) that for nonlinear systems, with time-varying parameters, the existence of a globally uniformly asymptotically stable equilibrium is equivalent to the existence of a smooth Lyapunov function. The subsequent converse Lyapunov theorems for switched nonlinear systems presented in [100] and [36] follow from Theorem 2.9 of [91]. Related results for *input to state stability*, a notion that we shall not discuss here, have appeared in [101]. Finally, we note the recent result of Mason, Boscain, and Chitour that shows that while a common Lyapunov function always exists for systems that are exponentially stable under arbitrary switching, its level curves may in fact be arbitrarily complex [107]. Thus searching for such a function using numerical techniques is not easy.

4.2. The CQLF Existence Problem. Quadratic Lyapunov functions (QLFs) play a central role in the study of LTI systems. Their existence is well understood in this context and, consequently, studying the existence of such functions is a natural starting point in the study of switched linear systems. At the heart of the CQLF existence problem is the desire to find useful criteria to determine whether a given collection of Hurwitz matrices $\{A_1, \ldots, A_m\}$ has a CQLF. The main purpose of this section is to survey the known results in this area and indicate the different lines of attack that have been used. Despite the considerable work done so far, there are still some open questions that remain.

4.2.1. Definitions. Recall that $V(x) = x^T P x$ is a QLF for the LTI system $\Sigma_A : \dot{x} = Ax$ if (i) P is symmetric and positive definite, and (ii) $PA + A^T P$ is negative definite. Let $\{A_1, \ldots, A_m\}$ be a collection of $n \times n$ Hurwitz matrices with associated stable LTI systems $\Sigma_{A_1}, \ldots, \Sigma_{A_m}$. Then the function $V(x) = x^T P x$ is a CQLF for these systems if V is a QLF for each individual system. Given a set of matrices $\{A_1, \ldots, A_m\}$, the CQLF existence problem is to determine whether such a matrix P exists. A secondary question is to construct a CQLF when one is known to exist. It is a standard fact that an LTI system Σ_A has a QLF if and only if the matrix A is Hurwitz. This property is also equivalent to the exponential stability of the system Σ_A, so for a single LTI system there is no gap between the existence of a QLF and exponential stability. Therefore a simple spectral condition determines completely the stability of the LTI system Σ_A.

For a collection of Hurwitz matrices the situation is more complicated in several respects. First, in general, CQLF existence is only a sufficient condition for the exponential stability of a switched linear system under arbitrary switching. Second, no correspondingly simple condition is known which can determine the existence of a CQLF for a family of LTI systems, although progress has been made in some special cases. The rest of this section will describe a variety of approaches which have been used to attack this problem, and to outline some of the open problems.

In many cases it is useful to analyze the mapping $P \mapsto PA + A^T P$ as a linear function on the space of real symmetric $n \times n$ matrices, denoted $S^{n\times n}$. The Lyapunov map defined by the real $n \times n$ matrix A is

$$\mathcal{L}_A : S^{n\times n} \to S^{n\times n}, \qquad \mathcal{L}_A(H) = HA + A^T H. \tag{4.1}$$

The following properties of $\mathcal{L}_A$ are well known [70]. (i) If A has eigenvalues $\{\lambda_i\}$ with associated eigenvectors $\{v_i\}$, then $\mathcal{L}_A$ has eigenvalues $\{\lambda_i + \lambda_j\}$ with eigenvectors $\{v_i v_j^T + v_j v_i^T\}$ for all $i \le j$. It follows immediately that $\mathcal{L}_A$ is invertible whenever A is Hurwitz, since in this case $\lambda_i + \lambda_j$ cannot be zero. (ii) A is Hurwitz if and only if there exists $P > 0$ such that $\mathcal{L}_A(P) < 0$. Note that in this case $x^T P x$ is a QLF for the system Σ_A.

Now define $\mathcal{P}_A$ to be the collection of all positive definite matrices which provide QLFs for the system Σ_A, that is,

$$\mathcal{P}_A = \{P > 0 : \mathcal{L}_A(P) < 0\}. \tag{4.2}$$

Clearly $\mathcal{P}_A$ is an open convex cone in $S^{n\times n}$. The above results concerning the Lyapunov map show that $\mathcal{P}_A$ is nonempty if and only if A is Hurwitz. In this language, the CQLF existence problem for a collection of matrices $\{A_1,\ldots,A_k\}$ is the problem of determining whether the intersection of the cones $\mathcal{P}_{A_1}\cap\cdots\cap\mathcal{P}_{A_k}$ is nonempty.

There are some straightforward observations that can be made at this point. First, for $A\in\mathbb{R}^{n\times n}$, the cones $\mathcal{P}_A$ and $\mathcal{P}_{A^{-1}}$ are identical. Thus, there exists a CQLF for the systems $\Sigma_{A_1},\ldots,\Sigma_{A_m}$ if and only if there is a CQLF for the systems $\Sigma_{A_1^{\epsilon_1}},\ldots,\Sigma_{A_m^{\epsilon_m}}$, where $\epsilon_i=\pm 1$ for $i=1,\ldots,m$. Second, CQLF existence is invariant under a change of coordinates. That is, if $R\in\mathbb{R}^{n\times n}$ is nonsingular, then

$$\mathcal{P}_{R^{-1}AR}=R^T\ \mathcal{P}_A\ R\equiv\{R^TPR\,:\,P\in\mathcal{P}_A\}. \tag{4.3}$$

Therefore CQLF existence for the family of systems $\Sigma_{A_1},\ldots,\Sigma_{A_m}$ is equivalent to CQLF existence for the transformed family $\Sigma_{R^{-1}A_1R},\ldots,\Sigma_{R^{-1}A_mR}$.

4.2.2. Dual Formulation. The QLF and CQLF existence problems have dual formulations which will play an important role in some of our later discussions. To set up the notation, define $\widehat{\mathcal{L}}_A$ to be the adjoint of the Lyapunov map with respect to the standard inner product $\langle X,Y\rangle=\operatorname{Tr}X^TY$ on $S^{n\times n}$, that is,

$$\langle X,\mathcal{L}_A(Y)\rangle=\langle\widehat{\mathcal{L}}_A(X),Y\rangle \tag{4.4}$$

for all $X,Y\in S^{n\times n}$. It follows that

$$\widehat{\mathcal{L}}_A\,:\,S^{n\times n}\to S^{n\times n},\qquad \widehat{\mathcal{L}}_A(H)=AH+HA^T=\mathcal{L}_{A^T}(H). \tag{4.5}$$

We will use the following formulation of duality, which can be found, for example, in [77]: Given a collection of Hurwitz matrices $\{A_1,\ldots,A_k\}$, there exists a CQLF if and only if there do not exist positive semidefinite matrices $X_1,\ldots,X_k$ (not all zero) satisfying $\sum_{i=1}^k\widehat{\mathcal{L}}_{A_i}(X_i)=0$. That is,

$$\begin{aligned}&\exists\ P>0\ \text{ such that }\ \mathcal{L}_{A_i}(P)<0\text{ for all }i=1,\ldots,k\\ \iff&\nexists\ X_1,\ldots,X_k\geq 0\ \text{ (not all zero) such that }\ \sum_{i=1}^k\widehat{\mathcal{L}}_{A_i}(X_i)=0.\end{aligned} \tag{4.6}$$

4.3. Numerical Approaches to the CQLF Problem. While we shall concentrate here on theoretical results obtained on the CQLF existence problem, it should be noted that numerical methods are also available for testing for CQLF existence. Recent advances in computational technology along with the development of efficient numerical algorithms for solving problems in the field of convex optimization have resulted in the widespread use of linear matrix inequality (LMI) techniques throughout systems theory. For details on the various applications of LMI methods in systems and control consult [24, 45, 52]. In this section, we focus on one specific aspect of this development: the use of LMI methods to test for the existence of a CQLF for a number of stable LTI systems.

The conditions for $V(x)=x^TPx$ to be a CQLF for the asymptotically stable LTI systems Σ_{A_i}, $i\in\{1,\ldots,m\}$, are equivalent to a system of LMIs in P, namely,

$$P=P^T>0,\ (A_i^TP+PA_i)<0\text{ for }i\in\{1,\ldots,m\}. \tag{4.7}$$

The system of LMIs (4.7) is said to be *feasible* if a solution P exists; otherwise the LMIs (4.7) are *infeasible*. Thus, determining whether or not the LTI systems Σ_{A_i},

$i \in \{1, \ldots, m\}$, possess a CQLF amounts to checking the feasibility of a system of LMIs. Solvers for LMIs are built on convex optimization algorithms developed over the past two decades which are capable of solving this type of problem with considerably more speed than was possible using previous techniques. Conversely, it is also possible to verify that no CQLF exists for the LTI systems Σ_{A_i} via the use of LMI techniques. More specifically [24], there is no CQLF for the LTI systems Σ_{A_i} if there exist matrices $R_i = R_i^T$, $i \in \{1, \ldots, m\}$, satisfying[2]

$$R_i > 0, \qquad \Sigma_{i=1}^{m}(A_i^T R_i + R_i A_i) > 0. \tag{4.8}$$

While LMIs provide an effective way of verifying that a CQLF exists for a family of LTI systems, there are a number of drawbacks associated with this approach which should be noted.

(i) LMIs provide little insight into why a CQLF may or may not exist for a set of LTI systems, or into the relationship between CQLF existence and the dynamics of switched linear systems. For instance, the problem of identifying specific classes of systems for which CQLF existence is equivalent to uniform exponential stability under arbitrary switching is of considerable interest and importance. We shall describe some system classes of this type later in this section. A purely numerical approach to CQLF existence cannot answer this and similar questions.

(ii) LMI-based methods are not effective when the number of constituent systems is very large, and they cannot be directly applied to check for the existence of a CQLF for an infinite family of systems. Recently, an alternative numerical technique based on iterative gradient descent methods was presented in [90]. The methods described in this paper can be combined with randomization algorithms and applied to compact, possibly infinite, families of system matrices $\mathcal{A}$. In this case, it has been shown that the algorithm will converge to a CQLF with probability 1, provided such a CQLF exists.

4.4. Special Structures of Matrices That Guarantee Existence of a CQLF. Some special cases are known where the structure of the matrices $\{A_1, \ldots, A_m\}$ by itself guarantees the existence of a CQLF for the associated LTI systems, provided of course that the matrices are Hurwitz. We now review these cases.

4.4.1. Matrices with Lyapunov Function $x^T x$. The condition that a system Σ_A have the Lyapunov function $x^T x$ is

$$\mathcal{L}_A(I) = A^T + A < 0, \tag{4.9}$$

where I is the $n \times n$ identity matrix. If $\{A_1, \ldots, A_m\}$ is a collection of matrices which all satisfy the condition (4.9), then $x^T x$ must be a QLF for every individual system, and hence must be a CQLF for the collection. The condition (4.9) is satisfied in the following cases:

(i) A is normal (i.e., $AA^T = A^T A$) and Hurwitz;

(ii) if A satisfies (4.9) and if the matrix S is skew-symmetric (i.e., if $S^T = -S$), then $A + S$ also satisfies (4.9).

4.4.2. Triangular and Related Systems. If the Hurwitz matrices $\{A_i\}$ are all in upper triangular form, then it was shown in [155], and independently in [116], that

[2] This follows easily from (4.6).

the collection of systems Σ_{A_i} always has a CQLF, and furthermore that the matrix P which defines the CQLF can be chosen to be diagonal. This result extends, by the remarks in section 4.2.1, to the case where there is a nonsingular matrix R for which the matrices $\{R^{-1}A_iR\}$ are all upper triangular.

One interesting application of this result arises when the matrices $A_1, \dots, A_m$ all commute with each other. In this case there is a unitary matrix U such that U^*A_iU is in upper triangular form for each $i = 1, \dots, m$ [70], and it then follows that the systems $\Sigma_{A_1}, \dots, \Sigma_{A_m}$ have a CQLF [118]. This result has an extension to a class of systems with noncommuting matrices [59]. To explain this class, let $\mathfrak{g} = \{A_1, \dots, A_m\}_{LA}$ denote the Lie algebra generated by the matrices $\{A_1, \dots, A_m\}$, that is, the collection of all matrices of the form $\{A_i, [A_i, A_j], [A_i, [A_j, A_k]], \dots\}$. If $\mathfrak{g}$ is solvable, then it follows from a well-known theorem of Lie [62] that the matrices $\{A_1, \dots, A_m\}$ can be put into upper triangular form by a nonsingular transformation. Recall that a Lie algebra is said to be *solvable* if $\mathfrak{g}^k = \{0\}$ for some finite k, where the sequence of Lie algebras $\mathfrak{g}^0, \mathfrak{g}^1, \mathfrak{g}^2, \dots$ is defined recursively by $\mathfrak{g}^{k+1} = [\mathfrak{g}^k, \mathfrak{g}^k]$, and $\mathfrak{g}^0 = \mathfrak{g}$. The basic example of a solvable Lie algebra is the Lie algebra generated by a set of upper triangular matrices, where it can be seen that the nonzero entries of $\mathfrak{g}^k$ retreat further from the main diagonal at each step.

Using this result, and the results of [116, 155] concerning CQLF existence for upper triangular matrices, it follows that if $\mathfrak{g} = \{A_1, \dots, A_m\}_{LA}$ is solvable, then the systems $\Sigma_{A_1}, \dots, \Sigma_{A_m}$ have a CQLF. The most general result along these lines is the following theorem due to Agrachev and Liberzon [2]. The theorem describes the type of Lie algebra which can be generated by a collection of Hurwitz matrices that share a CQLF. Their result also shows that if $\mathfrak{g}$ is not of this type, then it could be generated by a collection of matrices whose LTI systems are individually stable, but for which the corresponding switching system can be made unstable with some switching sequence. So the theorem describes the most general conclusions about the CQLF existence question which can be reached using only the Lie algebra structure generated by the collection $\{A_1, \dots, A_m\}$.

THEOREM 4.2 (see [2]). *Let $A_1, \dots, A_m$ be Hurwitz matrices, and let $\widehat{\mathfrak{g}} = \{I, A_1, \dots, A_m\}_{LA}$, where I is the identity matrix. Let $\widehat{\mathfrak{g}} = \mathfrak{r} \oplus \mathfrak{s}$ be the Levi decomposition, where $\mathfrak{r}$ is the radical, and suppose that $\mathfrak{s}$ is a compact Lie algebra. Then the systems $\Sigma_{A_1}, \dots, \Sigma_{A_m}$ have a CQLF. Furthermore, if $\mathfrak{s}$ is not compact, then there is a set of Hurwitz matrices which generate $\widehat{\mathfrak{g}}$ such that the corresponding switched linear system is not uniformly exponentially stable.*

Given the body of literature that has been dedicated to, and continues to be dedicated to, triangular systems, a few further comments are in order.

(i) In essence, establishing the stability of triangular switched linear systems is as straightforward as for LTI systems; uniform exponential stability under arbitrary switching is equivalent to the exponential stability of all of the constituent systems. Thus, the system is exponentially stable if and only if all the eigenvalues of the system matrices $A_1, \dots, A_m$ lie in the open left half of the complex plane.

(ii) It is important to appreciate that the property that a family of matrices is simultaneously triangularizable is not robust, and that this requirement is satisfied only by a very limited class of systems.

(iii) From a practical viewpoint, the requirement of simultaneous triangularizability imposes unrealistic conditions on the matrices in the set $\mathcal{A}$. It is therefore of interest to extend the results derived by [116] with a view to relaxing this requirement. In this context several authors have recently published new conditions for exponential stability of the switching system. Typically, the approach adopted is to bound the maximum allowable perturbations of the matrix parameters from a nominal (triangularizable) set of matrices, thereby guaranteeing the existence of a CQLF; see [116]. An alternative approach was presented in [160, 161]; rather than assuming maximum allowable perturbations from nominal matrix parameters, it was explicitly assumed that every pair of matrices, A_i, A_j, belonging to $\mathcal{A}$ can be simultaneously triangularized by some nonsingular matrix T_{ij}. By placing additional assumptions on the eigenvectors of the matrices in $\mathcal{A}$, the authors showed asymptotic stability of the origin.

(iv) Several papers in the area of switching systems are related to the simultaneous triangularization of a set of matrices. For example, matrices that commute are simultaneously triangularizable. Hence, the commuting vector field result of Narendra and Balakrishnan [118] is a special case of the above discussion [152].[3]

(v) To apply the above results, it is necessary to be able to determine if a nonsingular T exists such that for all A_i in a set of matrices $\mathcal{A}$, TA_iT^{-1} is upper triangular. McCoy's theorem, and the Lie algebraic conditions already discussed, provide verifiable tests for the existence of such a T [83, 155, 88].

4.5. Necessary and Sufficient Conditions for Special Classes. One longstanding goal in the field of switched systems has been to find simple algebraic conditions for existence of a CQLF for a set of matrices $\{A_1, \dots, A_m\}$. In the discrete-time case it is known from the work of Kozyakin [81] that exponential stability is not a property that can be described by finitely many algebraic constraints in the set of pairs of 2×2 matrices. In this section we describe several cases where such conditions are known.

4.5.1. Two Second Order Systems. For a pair of second order systems there is a complete solution to the CQLF existence problem. We quote the following result from [157].

THEOREM 4.3. *Let A_1 and A_2 be 2×2 Hurwitz matrices. Then the two LTI systems Σ_{A_1} and Σ_{A_2} have a CQLF if and only if the matrix products A_1A_2 and $A_1A_2^{-1}$ have no negative real eigenvalues.*

Theorem 4.3 provides an extremely simple and elegant solution to the CQLF problem for the case of two matrices in $\mathbb{R}^{2\times 2}$. It is known that CQLF existence is a conservative criterion for the stability of second order systems; however, the simplicity of Theorem 4.3 demonstrates the usefulness of using CQLF methods to analyze stability, and it provides insights into the precise relationship between CQLF existence and stability. In particular, using this result, it can be shown that if a CQLF fails to exist for a pair of LTI systems Σ_{A_1}, Σ_{A_2}, with $A_1, A_2 \in \mathbb{R}^{2\times 2}$ Hurwitz, then at least one of the related switched linear systems

$$\dot{x} = A(t)x, \quad A(t) \in \{A_1, A_2\}, \tag{4.10}$$

$$\dot{x} = A(t)x, \quad A(t) \in \{A_1, A_2^{-1}\}, \tag{4.11}$$

fails to be exponentially stable for arbitrary switching signals. Moreover, Theorem 4.3 has been used in [60, 105] to show that for second order positive switched linear

[3]It is worth noting that it has been shown by Shim, Noh, and Seo [151] that a common Lyapunov function exists for commuting vector fields even if the vector fields are themselves nonlinear.

systems[4] with two stable constituent systems, CQLF existence is in fact equivalent to exponential stability under arbitrary switching.

No simple spectral condition is known when there are more than two matrices in $\mathbb{R}^{2\times 2}$, although the following result provides some useful information in this case [157]. Suppose that Σ_{A_i} are stable LTI systems in the plane. If any subset of three of these systems has a CQLF, then there is a CQLF for the whole family. This can be viewed as a consequence of Helly's theorem from convex analysis [145] in combination with the discussion of intersection of the cones $\mathcal{P}_{A_j}$ in section 4.2.1.

4.5.2. Two Systems with a Rank One Difference. Suppose that $A \in \mathbb{R}^{n\times n}$ is Hurwitz, and $b, c \in \mathbb{R}^n$. Then there is again a simple spectral condition which is equivalent to CQLF existence for the systems $\{\Sigma_A, \Sigma_{A-bc^T}\}$. This condition was originally derived as a frequency domain condition using the single-input single-output (SISO) circle criterion [119]; however, it was later realized [158] that the condition has the following natural and elegant formulation similar to Theorem 4.3 [153].

THEOREM 4.4. *Let A_1 and A_2 be Hurwitz matrices in $\mathbb{R}^{n\times n}$, where the difference $A_1 - A_2$ has rank one. Then the two LTI systems Σ_{A_1} and Σ_{A_2} have a CQLF if and only if the matrix product A_1A_2 has no negative real eigenvalues.*

Theorem 4.4 provides a simple spectral condition for CQLF existence for a pair of exponentially stable LTI systems whose system matrices differ by a rank one matrix. Further, it follows from this result that for a switched linear system

$$\dot{x} = A(t)x, \quad A(t) \in \{A_1, A_2\}, \tag{4.12}$$

where $A_1, A_2 \in \mathbb{R}^{n\times n}$ are Hurwitz and $\mathrm{rank}(A_2 - A_1^{-1}) = 1$, CQLF existence is *equivalent* to exponential stability under arbitrary switching signals.

4.6. Sufficiency. In addition to the results discussed above, several authors have developed tests for CQLF existence which provide sufficient conditions. In some cases these tests allow explicit computations and therefore can be useful in practical applications.

4.6.1. Lyapunov Operator Conditions. In a series of papers [124, 122, 123, 125], Ooba and Funahashi derived conditions involving the Lyapunov operators $\mathcal{L}_A$ defined in (4.1). The key idea in their work is the observation that Σ_{A_1} and Σ_{A_2} have a CQLF if and only if there is some positive definite Q such that $\mathcal{L}_{A_1}\mathcal{L}_{A_2}^{-1}(Q)$ is also positive definite. This leads to their following result [122]. Recall that for an operator $\mathcal{L}$ on the space of symmetric matrices $S^{n\times n}$, $\widehat{\mathcal{L}}$ denotes the adjoint of $\mathcal{L}$ with respect to the usual inner product on $S^{n\times n}$.

THEOREM 4.5. *Let A_1 and A_2 be $n \times n$ Hurwitz matrices, and suppose that*

$$\widehat{\mathcal{L}}_{A_2-A_1}\mathcal{L}_{A_2-A_1} - (\widehat{\mathcal{L}}_{A_1}\mathcal{L}_{A_1} + \widehat{\mathcal{L}}_{A_2}\mathcal{L}_{A_2}) < 0. \tag{4.13}$$

Then Σ_{A_1} and Σ_{A_2} have a CQLF.

A second similar but independent condition involving the Lyapunov operators of the commutators of the matrices A_1 and A_2 was presented in [123]. In one of their other papers [124], Ooba and Funahashi derived sufficient conditions which involve minimal eigenvalues computed using the Lyapunov operators. Given a collection of Hurwitz matrices $\{A_1, \ldots, A_m\}$ in $\mathbb{R}^{n\times n}$, define

$$\mu_{ij} = \lambda_{\min}\left(\mathcal{L}_{A_i}\mathcal{L}_{A_j}^{-1}(I)\right), \quad i, j = 1, \ldots, m, \tag{4.14}$$

[4] A positive dynamical system is one where nonnegative initial conditions imply that the state vector remains in the nonnegative orthant for all time.

where I is the $n \times n$ identity matrix and where $\lambda_{\min}$ is the smallest eigenvalue, and define the $m \times m$ matrix

$$M = (\mu_{ij})_{i,j=1,\ldots,m}. \tag{4.15}$$

Then we have the following result.

THEOREM 4.6. *Suppose that the matrix M defined in (4.15) is semipositive, meaning that there is a vector $x \in \mathbb{R}^n$ with $x_i \geq 0$ for all i such that $(Mx)_i > 0$ for all i. Then the systems Σ_{A_i}, $1 \leq i \leq m$, have a CQLF.*

4.7. Necessary and Sufficient Conditions for the General Case. In this section we review a new approach to deriving necessary and sufficient conditions for existence of a CQLF, based on the duality condition (4.6). This relation states that $\Sigma_{A_1}, \ldots, \Sigma_{A_m}$ do *not* have a CQLF if and only if there are positive semidefinite matrices $X_1, \ldots, X_m$ (not all zero) which satisfy the equation

$$\sum_{i=1}^{m} A_i X_i + X_i A_i^T = 0. \tag{4.16}$$

In the statement of the next theorem, for matrices $A \in \mathbb{R}^{n\times m}$, $B \in \mathbb{R}^{p\times q}$, the notation $A \otimes B$ denotes the usual Kronecker product in $\mathbb{R}^{np\times mq}$ (see Chapter 4 of [71]). The main idea is to rewrite (4.16) in the following form.

THEOREM 4.7. *Suppose that (4.16) holds, and let $d = \mathrm{rk}(X_1 + \cdots + X_m)$. Then there are positive semidefinite $d \times d$ matrices $Y_1, \ldots, Y_m$, with $rk(Y_i) = rk(X_i)$ for all i, and a skew-symmetric $d \times d$ matrix S such that*

$$\det\Big(\sum_{i=1}^{m} A_i \otimes Y_i + I \otimes S\Big) = 0, \tag{4.17}$$

where I is the $n \times n$ identity matrix. Conversely, if (4.17) holds for some positive semidefinite matrices $\{Y_i\}$ and skew-symmetric matrix S, then (4.16) holds with $\{X_i\}$ positive semidefinite and not all zero, and $rk(X_i) \leq rk(Y_i)$ for all i.

The key to deriving (4.17) is to select a basis $v_1, \ldots, v_d$ for the range of $X_1 + \cdots + X_M$, and to express the matrices X_i with respect to this basis as

$$X_i = \sum_{p,q=1}^{d} (Y_i)_{pq} v_p v_q^T. \tag{4.18}$$

The $d \times d$ matrix Y_i is also positive semidefinite and has the same rank as X_i.

Using (4.17), four necessary and sufficient conditions were derived for nonexistence of a CQLF for a pair of 3×3 Hurwitz matrices [80]. Three of the conditions can be expressed as singularity conditions for some convex combinations of A_i and A_i^{-1}. For example, one of the conditions says that some convex combination of A_1, A_2, and $(xA_1 + (1-x)A_2)^{-1}$ is singular for some $0 \leq x \leq 1$. Testing this condition involves searching over a three-parameter space, so it is quite infeasible. The main importance of the conditions lies in the possibility that they can lead to new insights into the CQLF problem.

4.8. Stability Radii. The existence of a Lyapunov function or a CQLF for a switched system implies exponential stability. This is a robust property, meaning that small perturbations of the systems data do not destroy stability. One is often interested in quantifying this robustness, and this is the aim in the study of stability radii.

We assume we are given a nominal asymptotically stable system, which we take for the sake of simplicity to be time-invariant. It is thus of the form

$$\dot{x} = A_0 x \,. \tag{4.19}$$

Due to imprecise modeling it may be expected that the system of interest does not have the same dynamics as the nominal system, but can be interpreted as a particular system in the class

$$\dot{x}(t) = \left(A_0 + \sum_{k=1}^{m} \delta_k(t) A_k \right) x(t). \tag{4.20}$$

Here the matrices $A_k, k = 1, \dots, m$, are prescribed, modeling the expected perturbations of the systems, while $\delta(t) = (\delta_1(t), \dots, \delta_m(t))$ is an *unknown*, essentially bounded perturbation. The question is, how large can this perturbation be without destroying stability? To measure this size we prescribe a norm $\|\cdot\|$ in $\mathbb{R}^m$, and denote by $\|\cdot\|_\infty$ the corresponding norm on bounded functions $\delta : \mathbb{R}_+ \to \mathbb{R}^m$.

The stability radius in a switched systems sense is then given by

$$r_{Ly}(A, (A_i)) = \inf\{\|\delta\|_\infty \mid (4.20) \text{ is not exponentially stable for } \delta\} \,. \tag{4.21}$$

Stability radii of this type were discussed in [32, 69]. In particular, the interested reader will find an in-depth discussion of related literature in these references. The calculation of stability radii has been studied in [175] in the discrete-time case and in [57] in continuous time. Of course, this is again a difficult problem, as already the determination of the growth rate is NP-hard. We note that if the set

$$\left\{ A_0 + \sum_{k=1}^{m} \delta_k A_k \mid \|\delta\| \le 1 \right\}$$

is a polytope, then we are in the case of the switched system (2.1) again, as by (2.6) the exponential stability of the inclusion (2.4) and (2.1) are equivalent.

We note that in the theory of stability radii there is an elegant interpretation of the CQLF problem. This applies to the special case that perturbations are measured in the spectral norm $\|\cdot\|_2$ and the perturbation is determined by structure matrices $B \in \mathbb{R}^{n\times l}, C \in \mathbb{R}^{q\times n}$. We are thus considering perturbed systems of the form

$$\dot{x}(t) = (A + B\Delta(t)C)\, x(t), \tag{4.22}$$

where $\Delta(t) \in \mathbb{R}^{l\times q}$ is an unknown perturbation. In this case three different stability radii may be defined corresponding to real constant, real time-varying, and complex constant perturbations. They are given by

$$\begin{aligned}
r_{\mathbb{R}}(A,B,C) &:= \inf\left\{\|\Delta_0\|_2 \mid \Delta_0 \in \mathbb{R}^{l\times q} : (4.22) \text{ is not exp. stable for } \Delta(t) \equiv \Delta_0\right\},\\
r_{Ly}(A,B,C) &:= \inf\left\{\|\Delta\|_\infty \mid \Delta : \mathbb{R}_+ \to \mathbb{R}^{l\times q} : (4.22) \text{ is not exp. stable for } \Delta\right\},\\
r_{\mathbb{C}}(A,B,C) &:= \inf\left\{\|\Delta_0\|_2 \mid \Delta_0 \in \mathbb{C}^{l\times q} : (4.22) \text{ is not exp. stable for } \Delta(t) \equiv \Delta_0\right\}.
\end{aligned}$$

The relation among these stability radii is

$$r_{\mathbb{R}}(A,B,C) \le r_{Ly}(A,B,C) \le r_{\mathbb{C}}(A,B,C)\,. \tag{4.23}$$

In particular, we have the following theorem.

THEOREM 4.8 (see [68]). *Let $A \in \mathbb{R}^{n\times n}$ be Hurwitz and $B \in \mathbb{R}^{n\times l}, C \in \mathbb{R}^{q\times n}$. The following statements are equivalent:*

(i) $\rho < r_{\mathbb{C}}(A,B,C)$.

(ii) *There exists a CQLF for the set of matrices*

$$\{A + B\Delta C \mid \|\Delta\|_2 \le \rho\}\,.$$

In view of (4.23) it is of course interesting to find conditions that guarantee $r_{\mathbb{R}}(A,B,C) = r_{\mathbb{C}}(A,B,C)$, because in this case the intrinsically difficult problem of calculating the stability radius of switched systems reduces to the calculation of the complex stability radius. For the latter problem there exist quadratically convergent algorithms if stability radii are constructed with respect to the spectral norm. One interesting case where this can be done concerns the area of positive systems. In fact, for this system class the problem turns out to be particularly simple.

A matrix $B \in \mathbb{R}^{n\times m}$ is called nonnegative if all its entries are nonnegative numbers. We denote this property by $(B)_{ij} \ge 0$. Recall that a matrix $A \in \mathbb{R}^{n\times n}$ is called *Metzler* if all offdiagonal entries are nonnegative numbers. Metzler matrices are precisely the matrices for which $(e^{At})_{ij} \ge 0$ for all $t \ge 0$. For this system class the following result was shown in [51] for the more general case of positive systems in infinite dimensions.

THEOREM 4.9. *Let $A \in \mathbb{R}^{n\times n}$ be Metzler and Hurwitz. Assume $B \in \mathbb{R}^{n\times l}, C \in \mathbb{R}^{q\times n}$ are nonnegative. Then*

$$r_{\mathbb{R}}(A,B,C) = r_{Ly}(A,B,C) = r_{\mathbb{C}}(A,B,C) = \|CA^{-1}B\|_2^{-1}\,.$$

The previous result shows that determining the stability radius or stability margin of switched systems is quite easy for positive systems subjected to a particular perturbation structure.

4.9. Decidability and Computability Issues. At this point some readers may wonder whether Lyapunov theory is not overkill for analyzing switched *linear* systems. After all, explicit solutions to any given differential equation of this form can be constructed by piecing together solutions of the constituent LTI systems as in (2.3), and the stability properties of such solutions, for any given switching sequence, can be easily deduced. We shall see that this comment is naive and that determining the properties of all such solutions is a computational impossibility.

In the discrete-time case, the computational complexity and decidability of problems regarding stability properties of linear inclusions have been actively investigated. The problem can be described as follows. Consider a finite set of matrices $\mathcal{A} = \{A_1, \dots, A_m\} \subset \mathbb{R}^{n\times n}$ and the associated switched system

$$x(t+1) = A(t)x(t)\,, \quad A(t) \in \mathcal{A}\,, \quad t \in \mathbb{N}\,. \tag{4.24}$$

One might be tempted to ask for a good algorithmic procedure for deciding whether the set $\mathcal{A}$ defines an exponentially stable or stable system.

An easy answer would be possible if the question could be decided by checking a finite number of algebraic inequalities, as one does in the Schur–Cohn test for single matrices. To formulate the problem we consider m-tuples of matrices $M = (A_1, \dots, A_m) \in (\mathbb{R}^{n\times n})^m$ and we denote by $\Sigma(A_1, \dots, A_m)$ the system (4.24) corresponding to the set $\mathcal{A}$ of distinct matrices in M.

DEFINITION 4.10. *A set Y is called* semialgebraic *in $\mathbb{R}^p$ if it can be represented as a finite union of sets that are each described by a finite number of polynomial equalities and inequalities.*

The first negative result is then as follows.

THEOREM 4.11 (Kozyakin [81]). *Given $n, m \geq 2$, the sets*

$$\{(A_1, \ldots, A_m) \in (\mathbb{R}^{n\times n})^m \mid \Sigma(A_1, \ldots, A_m) \text{ is exponentially stable}\},$$

$$\{(A_1, \ldots, A_m) \in (\mathbb{R}^{n\times n})^m \mid \Sigma(A_1, \ldots, A_m) \text{ is stable}\}$$

are not semialgebraic.

In fact, the proof of Kozyakin even shows that both sets are not subanalytic, a notion of real analytic geometry that we cannot discuss here; see [82]. Summarizing, this shows that there is no simple description of the set of stable systems in algebraic or even analytic terms, which suggests that the problem of deciding whether a given system is stable is a computationally difficult one, in general.

To investigate the problem further, recall that a computational problem is said to be of class P if there exists an algorithm on a Turing machine that solves the problem in a time that depends in a polynomial manner on the amount of information needed to describe a particular instance of the problem. A problem is in class NP if a nondeterministic Turing machine can solve the problem in polynomial time. In particular, any problem in P is in NP. A problem is termed NP-hard if by its solution any other problem in the class NP can be solved, so that it is at least as hard as any other NP problem. Whether $P = NP$ is one of the fundamental open questions, but assuming this is not the case, this means that for any NP-hard problem there is no algorithm that computes the answer to this question in a time that is a polynomial function of the size of the data.

THEOREM 4.12 (Tsitsiklis and Blondel [167]). *Unless $P = NP$, there is no algorithm that approximates the joint spectral radius ρ in polynomial time for all finite sets of matrices $\{A_1, \ldots, A_m\}$, $n, m \geq 2$.*

DEFINITION 4.13. *A problem is said to be algorithmically decidable if there exists an algorithm to solve it that terminates after a finite number of steps with the correct answer. If a problem is not algorithmically decidable, then it is said to be algorithmically undecidable.*

A more fundamental question is whether checking exponential stability is algorithmically decidable. As we have seen, exponential stability is equivalent to $\rho(\mathcal{M}) < 1$ and stability implies $\rho(\mathcal{M}) \leq 1$. The following theorem states that determining the maximum spectral radius of a switched linear system is algorithmically undecidable.

THEOREM 4.14 (Blondel and Tsitsiklis [19]). *The problem of whether $\rho(\mathcal{M}) \leq 1$ is algorithmically undecidable, even when restricted to sets $\mathcal{M}$ containing only 2 elements. Furthermore, the problem of determining whether $\mathcal{M}$ is stable is undecidable.*

It is an open question whether it is algorithmically decidable if a discrete linear inclusion is exponentially stable, that is, if $\rho(\mathcal{M}) < 1$; see [15].

4.10. Periodic Systems and Switched Systems. One class of switching system for which easily verifiable conditions for stability are known is the class of periodic switched linear systems. For this system class, necessary and sufficient conditions are available from Floquet theory [113, 147]; the growth rate of these systems is determined by the spectral radius of the evolution operator evaluated at the period (and suitably normalized). Since any general system may be thought of as a periodic system whose period is infinite, and notwithstanding the decidability issues that we have just discussed, it is natural to question the precise relationship between switched linear systems with arbitrary switching signals and those with periodic switching signals. In view of the equality (2.10) we already know that periodic switching signals can have growth rates arbitrarily close to the uniform exponential growth rate of the system. However, this does not answer the question posed below, which asks whether it is possible to realize the growth rate with one particular periodic switching signal.

> *Consider the system $\dot{x} = A(t)x$, $A(t) \in \{A_1, \ldots, A_m\}$. Suppose that the switching system is exponentially stable for all periodic switching signals σ. Does this imply that the system is exponentially stable for arbitrary switching signals?*

The above question has been studied extensively for both discrete- and continuous-time switched systems; see, for example, [141] and [16] and the references therein. In principle, if it were true, it would provide a method for testing the exponential stability of any given switching system, through considering the stability of the system under periodic switching signals.

For discrete-time systems, this question is equivalent to the *finiteness conjecture* that was introduced by Lagarias and Wang [84]. The conjecture has been disproved by Bousch and Mairesse [23], so in discrete time the answer to the above question is no. Blondel, Theys, and Vladimirov presented an alternative analysis of this example in [16]. In particular, in the latter paper the existence of a switching system is shown where all periodic switching signals result in transition matrices with spectral radius strictly less than one, whereas the joint spectral radius $\rho = \exp(\kappa) = 1$. Hence, it would appear that periodic stability does not generally imply stability of switched linear systems under arbitrary switching signals in discrete time. Note that the counterexample relies crucially upon the switched system operating at the boundary of stability; namely, the system is characterized by a limiting generalized spectral radius of 1.

While the counterexample is certainly interesting, it merely states that switching systems that are marginally stable (neither divergent nor convergent) need not be characterized by periodic motions at the boundary of stability. However, by introducing a robustness margin, namely, by insisting that $r(\Phi_\sigma(T, 0)) < 1 - \varepsilon$ for a suitable $\varepsilon > 0$ and for all periodic switching signals σ, we can conclude that robust periodic stability does indeed imply asymptotic stability.

The following sufficient condition for stability of discrete-time systems is due to Gurvits [59, Theorem 2.3], and a continuous-time version of the result has been given in [178]. Note that the result is an improvement on (2.10).

THEOREM 4.15. *The switched linear system* (2.1) *is asymptotically stable under arbitrary switching if and only if there exists an $\varepsilon > 0$ such that $r(\Phi_\sigma(T, 0)) < 1 - \varepsilon$ for all periodic switching signals σ.*

As mentioned above, this result holds for both continuous- and discrete-time systems. Thus if the switched system is periodically stable with some finite robustness margin ε, then it is exponentially stable for arbitrary switching signals. In principle, the above theorem gives a practical method for testing the stability of any given switching system.

In continuous time, there are several positive results available for low-dimensional systems, showing that periodic stability is sufficient for stability under arbitrary switching. Results of this type have been derived by Pyatnitskii and Rapoport [139, 140] as well as by Barabanov [11]. In particular, for second order systems of the form $\dot{x} = A(t)x, A(t) \in \{A_1, A_2\}$, where $\operatorname{rank}(A_1 - A_2) = 1$, stability under arbitrary switching may be tested by considering all periodic switches with two switches per period (*worst-case switching*). Similar results have been obtained for third order systems, as well as for higher order systems that leave a proper cone invariant [142]. A

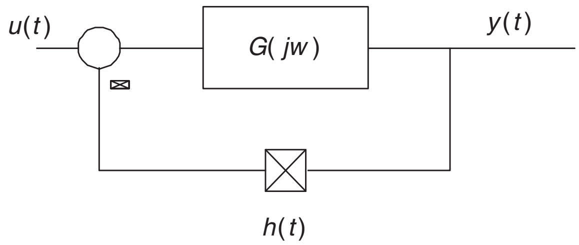

Fig. 4.1 $G(s) = c^T(sI - A)^{-1}b$.

numerical approach for determining stability based on finding periodic solutions is described in [104]. However, the extension of these results to general higher-dimensional continuous-time systems remains an open question.

4.10.1. Describing Functions for Switched Systems. A simple argument that identifies the existence (or nonexistence) of unstable periodic switching sequences was given in [137, 152]. Here, the authors considered systems of the form

$$\begin{aligned} \dot{x} &= A(t)x, \ A(t) \in \{A, A + bc^T\}, \\ &= (A + \sigma(t)bc^T)x, \ \sigma(t) \in \{0, 1\}, \end{aligned} \tag{4.25}$$

where $A, A+bc^T$ are $n \times n$ companion matrices, $b, c \in \mathbb{R}^{n\times 1}$, and where the switching signal $\sigma(t)$ is assumed to be periodic. By introducing the output y and setting $x^T = [y, \dot{y}, \ldots, y^{(n-1)}]$, systems of this form can be conveniently represented in the frequency domain. The key to the analysis in [137, 152] revolves around finding conditions under which a sinusoid, at a particular critical frequency ω_c, undergoes an amplitude magnification of unity and an effective net phase shift of 2π as it traverses the feedback loop in Figure 4.1 (i.e., by assuming that the system destabilizes via a sinusoidal limit cycle). Clearly, the existence of such an output signal, $y_c(t) = A\sin(\omega_c t + \theta)$, constitutes the existence of a marginally stable (unstable) limit cycle as a result of switching. If $\sigma(t)$ is assumed to be periodic with period T_σ, then

$$\sigma(t) = \sum_{-\infty}^{\infty} k_n e^{jn\omega_\sigma t},$$

where the k_i are the Fourier coefficients of $\sigma(t)$ and $\omega_\sigma = \frac{2\pi}{T_\sigma}$. Then, by applying the principle of harmonic balance, the condition for the existence of $y_c(t)$ is that

$$Y_c(j\omega) = G(j\omega) \sum_{-\infty}^{+\infty} k_n Y_c(j(\omega + n\omega_\sigma)). \tag{4.26}$$

Clearly, finding conditions under which (4.26) is satisfied is not simple. However, assuming the typical low-pass characteristic of $G(j\omega)$ enables us to neglect the effect of frequency components in $\sigma(t)$ with the exception of $n = 0$ and $n = -1$, and assuming that the system destabilizes via a sinusoidal limit cycle whose frequency is half that of the switching signal $\sigma(t)$, one obtains using "Describing Function"-like [108] arguments the following approximate condition for the existence of $y_c(t)$:

$$-\frac{1}{G(j\omega)} = k_0 + k_1 e^{-2j\omega t_0}, \tag{4.27}$$

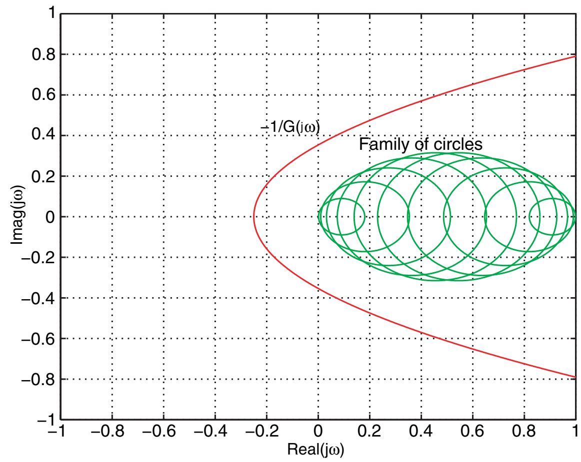

Fig. 4.2 *Example.*

where $t_0 \in \mathbb{R}$ is some arbitrary time origin. Equation (4.27) suggests that the intersection of the inverse Nyquist plot of $-G(j\omega)$ and one of the family of circles centered at $(k_0, 0)$ with radius k_1 for some frequency ω_c, for all k_0, k_1, implies the existence of a periodic switching signal $\sigma(t)$ with fundamental frequency $2\omega_c$ such that the system is marginally unstable.

Example 4.16. Consider the system (4.25) with

$$A_1 = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ -10 & -2 \end{pmatrix}.$$

It is easy to verify that $G(s) = \frac{8}{s^2+2s+10}$. The plot of $-\frac{1}{G(j\omega)}$ is depicted in Figure 4.2, and 10 circles centered at radius $(k_0, 0)$ with radius k_1 are depicted for $\sigma(t)$ with a single switch per period and with duty cycle increasing in steps of 0.1. For $\sigma(t)$ of this form, the above analysis does not indicate the existence of marginally stable sinusoidal signals of the form discussed above.

Remark 4.17. Note that due to the approximation implied in condition (4.27) the method does not yield reliable results in terms of either stability or instability. However, it can serve as a convenient engineering tool for the system design. Including higher harmonics can improve accuracy and even approximate the smallest feedback gain for which a periodic solution is obtained [177].

To conclude we state a convenient result, formulated in terms of matrix cones, that identifies unstable periodic switching sequences in switched linear systems arising due to unstable chattering.

Theorem 4.18 (see [154, 162]). *A sufficient condition for the existence of a switching sequence, such that the system* (4.25) *is unstable, is that there exist nonnegative constants* $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ *with* $\sum_{i=1}^m \alpha_i > 0$ *such that* $\sum_{i=1}^m \alpha_i A_i$ *has an eigenvalue with a positive real part.*

Remark 4.19. We note that similar results, albeit in another context, were obtained in [172, 49]. In these papers the authors considered the quadratic stabilization of switched linear systems. In [172] it was shown that a switched linear system is quadratically stabilizable if $\sum_{i=1}^m \alpha_i A_i$ is Hurwitz for some $\{\alpha_1, \ldots, \alpha_m\}$; in [49] it was shown that this condition is both necessary and sufficient for the quadratic stabilization of switched linear systems where switching takes place between two LTI systems. In both cases the authors used arguments from Lyapunov theory to obtain their results. While it may be possible to use the Lyapunov-based arguments in [172, 49] to obtain the previous result, it was shown in [162] that this result follows immediately from the nature of the solution to (2.1) using only arguments from Floquet theory. A direct consequence of this result is the existence of a periodically destabilizing switching sequence; this is entirely consistent with the more general, but also more abstract, results presented in [141].

Remark 4.20. The conditions of Theorem 4.18 guarantee the existence of a periodic switching sequence such that the system (2.1) is unstable. More specifically, given a positive sum that has an eigenvalue with a positive real part for some nonnegative constants $\{\alpha_1, \dots, \alpha_m\}$, an unstable switching sequence for (2.1) is constructed as follows: (a) scale the positive constants α_i such that $\sum_{i=1}^m \alpha_i = 1$; (b) let the matrix A_i describe the dynamics of (2.1) for $\alpha_i T$ seconds. Then, for sufficiently small T, the periodic switching sequence so defined results in an unbounded solution to (2.1).

Theorem 4.18 has a number of interesting consequences for the switched system (2.1):

(i) It is well known that a necessary condition for the existence of a CQLF, $V(x) = x^T P x$, $P = P^T > 0$, for the LTI systems Σ_{A_i}, $i \in \{1, \dots, m\}$, with $\dot{V}(x) < 0$, is that $\sum_{i=1}^m \alpha_i A_i$ is Hurwitz for all $\alpha_i \geq 0$ with $\sum_{i=1}^m \alpha_i > 0$. Hence the condition that this sum has no eigenvalues with positive real part is necessary for the existence of a CQLF. It follows from Theorem 4.18 that this condition is a much stronger necessary condition; namely, it is also necessary for the existence of any type of common Lyapunov function for the systems Σ_{A_i}.

(ii) Often design laws based upon the existence of a CQLF place unnecessarily conservative restrictions on the switching system. However, this is not necessarily true for second order systems. It follows from Theorem 4.3 that if a CQLF does not exist for Σ_{A_1} and Σ_{A_2}, then one of the following positive sums is singular (and hence non-Hurwitz) for some $\alpha_0 \in [0, 1]$:[5]

$$\alpha_0 A_1 + (1 - \alpha_0) A_2,$$
$$\alpha_0 A_1 + (1 - \alpha_0) A_2^{-1}.$$

Hence, as we mentioned in section 4, from Theorem 4.18, the nonexistence of a CQLF for (2.1) implies that an unstable or a marginally unstable[6] switching sequence exists for at least one of the dual switching systems

$$\dot{x} = A(t)x, \quad A(t) \in \{A_1, A_2\}, \tag{4.28}$$
$$\dot{x} = A(t)x, \quad A(t) \in \{A_1, A_2^{-1}\}. \tag{4.29}$$

Although this observation is not true for $m > 2$ matrices [156], it is somewhat surprising since it implies a strong connection between the stability problem for (2.1) and the CQLF existence problem for the constituent systems Σ_{A_i},

[5] The stated implication can be obtained as follows. Suppose that $A_1 A_2$ (respectively, $A_1 A_2^{-1}$) has a negative real eigenvalue, $-\lambda$. Then $\det[\lambda I + A_1 A_2] = 0$. Since A_2 is Hurwitz, this implies that $\det[\lambda A_2^{-1} + A_1] = 0$; hence, the matrix $\lambda A_2^{-1} + A_1$ is singular.

[6] By marginally unstable we mean a switching sequence for which all solutions are bounded and for which there is one bounded solution that does not converge to 0.

namely,

> *given two stable second order LTI systems for which a CQLF does not exist, it follows that an unstable, or marginally unstable, switching sequence exists for one of the associated switching systems* (4.28) *and* (4.29).

4.11. Common Piecewise Linear Lyapunov Functions. Most of the available results for the arbitrary switching problem are related to the existence of CQLFs. However, it is not difficult to construct a switched linear system that is asymptotically stable for arbitrary switching sequences where the constituent systems do not have a CQLF (see, for example, [36]). A rapidly maturing area of research is concerned with determining conditions for the existence of nonquadratic Lyapunov functions.

It follows from the converse theorem of Molchanov and Pyatnitski that a common piecewise quadratic or a common piecewise linear Lyapunov function always exists provided that the underlying switched linear system is asymptotically stable for arbitrary switching. Motivated by this result, a number of authors have sought to develop verifiable conditions for the existence of piecewise linear Lyapunov functions (PLLFs) of the form

$$V(x) = \max_{1 \le i \le N} \{w_i^T x\}, \tag{4.30}$$

where $w_i \in \mathbb{R}^n$, $i = 1, \dots, N$, and the linear functions $w_i^T x$ are called generators of the PLLF. The function (4.30) is induced by a polyhedral set of the form

$$\mathcal{P} = \left\{ x \in \mathbb{R}^n : \; w_i^T x \le c, \; i = 1, \dots, N \right\}, \quad c \in \mathbb{R}_+.$$

Such functions can be shown to be proper and locally Lipschitz [121] and decompose the state space into a number of convex cones whose interiors are pairwise disjoint. The polyhedral set $\mathcal{P}$ is called positively invariant with respect to the trajectories of a dynamical system if for all $x(0) \in \mathcal{P}$ the solution of $x(t) \in \mathcal{P}$ for $t > 0$. A complete survey of properties of positively invariant sets and their usage for a series of problems in control theory can be found in [13].

If the polyhedron $\mathcal{P}$ is bounded and centrally symmetric, then it describes a polytope, and the Lyapunov function V can be expressed as

$$V(x) = ||Wx||_\infty \;=\; \max_{1 \le i \le N} \left\{ |w_i x| \right\}, \tag{4.31}$$

where $W \in \mathbb{R}^{N \times n}, N \ge n$, has full rank n. Functions of the form (4.31) are radially unbounded, have a unique minimum, and the one-sided derivative exists [115].

The existence of PLLFs has been considered in a number of papers for establishing the stability of nonlinear time-varying systems, and numerical techniques for the calculation of such functions have been developed. The existence question for PLLFs can be traced back to a series of papers in the 1960s by Rosenbrock [146] and Weissenberger [170] on Lur'e-type systems. However, despite several decades of research, powerful algebraic tools for the existence of PLLFs remain scarce. Although the class of PLLFs appears powerful in theory, the computational requirements necessary to establish their existence represents a serious bottleneck in practice. The main reason is that a complex representation (with a large number of parameters) is usually required for a solution to be found, rendering the techniques applicable to low-dimensional problems only. Indeed few theoretical tools exist to support the development of numerical or analytical tests for checking the existence of such functions. One notable exception is the following result that was obtained in [114, 115].

THEOREM 4.21. *The function* $V(x) = ||Wx||_\infty$ *is a common PLLF for the switched system* (2.1) *if and only if there exist* $Q_i \in \mathbb{R}^{N\times N}$, $i = 1,\dots,m$, *such that*

$$q_{kk}^{(i)} + \sum_{\substack{l=1\\ l\neq k}}^{N} |q_{lk}^{(i)}| < 0 \tag{4.32}$$

and

$$WA_i - Q_iW = 0 \tag{4.33}$$

for $i = 1,\dots,m$. *Here,* $q_{jk}^{(i)}$ *denotes the* (j,k) *entry in the matrix* Q_i.

A generalization of this result for norm-based Lyapunov functions of the form $V(x) = ||Wx||_p$ can be found in [78, 92].

A particular problem for the lack of results in this area is that it appears to be difficult to specify a priori the number N of generators that are necessary for the construction of a common Lyapunov function (4.30) for a given switched system. We note that recently some progress on this question has been made in the context of LTI systems [22, 21]. In this work, the authors related the number of faces of the PLLF (4.31) to the location of the spectrum of the system matrix A. The results in these papers may serve as a starting point for the derivation of conditions for the existence of a common PLLF for a set of LTI systems.

In [179] the existence of a PLLF with four faces ($N = 4$) is considered for second order switched systems with two subsystems.

THEOREM 4.22. *Given the switched linear system* (2.1) *with* $A_1, A_2 \in \mathbb{R}^{2\times 2}$ *and* $\mathrm{spec}(A_i) \subset (-\infty, 0), i = 1, 2$, *where* $\mathrm{spec}\big(\alpha A_1 + (1-\alpha)A_2\big) \cap R = \emptyset$ *for some* $\alpha \in (0,1)$, *there exists a common PLLF* (4.31) *with* $N = 4$ *if and only if for all* $\alpha \in [0,1]$ *the absolute value of the real part of the eigenvalues of*

$$\alpha A_1 + (1-\alpha)A_2$$

is greater than the imaginary part.

Finally we note that a number of attempts have been made to develop numerical techniques for the construction of such Lyapunov functions. In [27] and [28] Brayton and Tong developed an algorithm for difference inclusions which calculates a series of balanced polytopes converging to the level set of a common PLLF after a finite number of steps. Barabanov [10] proposed another technique for checking asymptotic stability of an LDI. An algorithm is constructed which calculates the Lyapunov exponent and a common PLLF in a finite number of steps. This idea has been developed initially for difference inclusions and requires a sufficiently dense discretization and progressive refinements. Again convex hull computations increase the computational load significantly, rendering the techniques applicable to planar systems, as evidenced by the examples in [27, 28, 10].

In a series of publications, Polański described an algorithm to construct a common PLLF (4.31) with a given number of generators for the LTI systems $\Sigma_{A_1},\dots,\Sigma_{A_m}$ [132, 133, 134]. Here the algebraic stability condition (4.33) and a scaling idea are used to formulate the search for PLLFs as a linear program. Similar numerical difficulties with high complexity arise and the technique is applicable to planar systems.

In [134] an improved formulation using polytope vertices and scaling makes the technique applicable to three-dimensional problems. Thus there are cases, even in three dimensions, in which instability cannot be inferred even when a solution cannot be found. Furthermore, the question of the number N of generators required remains unsolved.

This problem is partially avoided in the ray-gridding method developed by Yfoulis and his coauthors in [183, 185, 186]. The approach is based on uniform partitions of the state space in terms of ray directions which allow refinable families of polytopes of adjustable complexity. The technique provides two important advantages. First, the optimization problem can be solved much more efficiently such that a complete treatment of the three-dimensional case is feasible; and second, by applying a refinement technique, no prior knowledge about the numbers of generators is required.

5. Restricted Switching. In earlier sections we dealt with the problem of determining conditions on the set of matrices $\{A_1, \ldots, A_m\}$ such that the resulting switched system

$$\dot{x} = A(t)x,\ A(t) \in \{A_1, \ldots, A_m\}, \tag{5.1}$$

is exponentially stable for all switching sequences. While it is true that this problem has received most of the recent attention in the switching systems literature, a number of other stability problems stand out as being worthy of attention. Among these, the problem of determining the stability of (5.1) in the case where the switching action is constrained in some manner is a problem that arises in a number of important applications [89, 165].

Example 5.1. An example of how constrained switching can arise in practical situations is given in [169], where the problem of designing a controller to deliver prescribed handling behavior for a four-wheel steering vehicle is considered. The controller described in [169] operates by manipulating the front and rear steering angles of the vehicle to achieve the desired behavior. Due to physical considerations, the steering angle of the rear tires is subject to a tight constraint, and when the maximal allowed steering angle is reached, a change of control action is required, leading to an abrupt switch in the overall system dynamics.

Roughly speaking, research on systems in which switching is constrained has proceeded along two distinct lines of enquiry (see Figure 5.1). The first of these involves the study of systems in which constraints on the switching action are induced by the evolution of the state vector x, as in the example above. The second body of research is concerned with systems in which one seeks to impose constraints on the rate at which switching takes place between the constituent subsystems so as to ensure the stability of the overall system. It should be noted that the classical Lur'e system studied by Popov [120] may be viewed as an example of the former system class, whereas classical Floquet theory developed for the study of periodic systems may be viewed as an example of the latter [113].

One further important problem that arises in the context of this discussion is the following: namely, given a set of non-Hurwitz matrices, determine whether or not it is possible to develop a state dependent switching law such that the system (5.1) is globally uniformly asymptotically stable [87]. We shall briefly discuss this problem later in the paper. The interested reader is referred to [87] and the references therein for a discussion of some of the approaches that have been employed in the study of this and the other problems concerned with constrained switching.

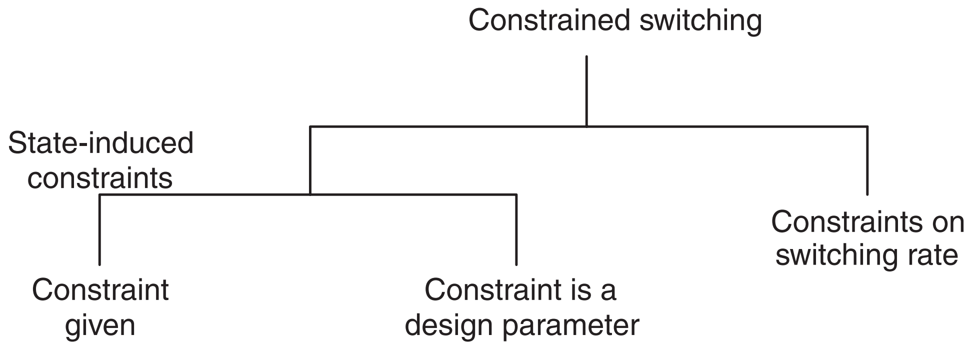

Fig. 5.1 *Types of constraints on switching laws that arise in practical applications.*

5.1. Constraints on the Rate of Switching. If all of the matrices in the switching set $\{A_1, \ldots, A_m\}$ are Hurwitz, then it is possible to ensure the stability of the associated switched system by switching sufficiently slowly between the asymptotically stable constituent LTI systems. This means that instability arises in (5.1) as a result of rapid switching between these vector fields. Given this basic fact, a natural and obvious method to ensure the stability of (5.1) is to somehow constrain the rate at which switching takes place.

The basic idea of constraining the switching rate has appeared in many studies on time-varying systems over the past few decades [58, 191, 72]. One of the best known and most informative of these studies was described by Charles Desoer in 1969 in his study of slowly varying systems [43]. The basic problem considered by Desoer was to find conditions on the switching rate that would ensure the stability of an unforced system of the form $\dot{x} = A(t)x$, where $A(t)$ is a matrix valued continuous function such that $\sup_{t\geq 0} \| A(t) \| < \infty$ and $A(t)$ is Hurwitz for all fixed t (here $\| \cdot \|$ can be any norm on $\mathbb{R}^{n\times n}$). Using an argument based on QLFs, Desoer demonstrated that there exists some constant $K > 0$ such that the solution $x(t)$ satisfies $\| x(t, t_0) \| < me^{\lambda(t-t_0)} \| x(t_0) \|$ for some $m > 0$ and $\lambda < 0$ provided $\sup \| \dot{A}(t) \| = K$. There are two key points to emphasize here; first, the stability of the time-varying system can be ensured by suitably constraining the rate of variation of $A(t)$, and second the constraint on $\dot{A}(t)$ is determined by a Lyapunov function associated with the system.

Recently, similar ideas have been exploited in the hybrid and switched systems community [89, 42, 86, 112, 181, 53]. However, when dealing with switched linear systems, one may approach the problem of constraining the rate at which switching takes place in at least two ways. The first, an indirect method, is close to the method suggested by Desoer; one constrains switching indirectly by ensuring negative definiteness of the derivative of a certain Lyapunov function. An alternative to this approach is to use knowledge of the form of the solutions to (5.1) to ensure stability. While the latter approach is difficult for general time-varying linear systems, the explicit form of the solution to (5.1) makes such an approach possible in the case of switched linear systems and gives rise to the following basic problem in the study of switched linear systems:

> *Given the system* (5.1)*, let $\sigma_\tau(t)$ denote any switching signal with the property that $t_{k+1} - t_k > \tau$ for all $k > 0$. Let $\mathcal{S}[\tau]$ denote the class*

of all such signals. One may then pose the following question: find the minimum τ for which (5.1) *is uniformly exponentially stable for all $\sigma_\tau \in \mathcal{S}[\tau]$.*

The above problem, often referred to in the literature as the *dwell-time problem*, poses a fundamental question in the study of switched systems. However, rather surprisingly, very little progress has been made on this and related problems, and to the best of our knowledge few papers have appeared in the recent literature that deal with this topic; the most notable of those that have appeared are [42, 86, 112, 181]. For convenience we report here on the work developed by Hespanha and his coauthors in [67] as this appears at the present time to be the most complete treatment of the dwell-time problem to have appeared.

DEFINITION 5.2 (dwell-time [67]). *Given a positive constant τ_D, then $\mathcal{S}[\tau_D] \subset \mathcal{S}$ denotes the set of all switching signals $\sigma(t) \in \mathcal{S}$, where the intervals between consecutive discontinuities are no shorter than τ_D. The constant τ_D is called the (fixed) dwell-time.*

For switched linear systems all of whose subsystems, Σ_{A_i}, are Hurwitz stable, Morse [117] established the (unsurprising) fact that the switching system (2.1) is asymptotically stable provided the dwell-time τ_D is chosen to be sufficiently large. This result was extended by Hespanha in [67], where the notion of *average* dwell-time $\bar{\tau}_D$ was introduced. This concept allows some switching intervals to be of length less than $\bar{\tau}_D$ provided that, in a sense to be made precise below, the average dwell-time is at least $\bar{\tau}_D$.

Formally, for a switching signal σ and real numbers t_1, t_2 with $t_2 > t_1 > 0$, let $N_\sigma(t_1, t_2)$ denote the number of discontinuities of σ in the interval (t_1, t_2). Then, given a positive real number $\bar{\tau}_D > 0$ and a positive integer $N_0 > 0$, define $\mathcal{S}[\bar{\tau}_D, N_0]$ to be the set of switching signals $\sigma \in \mathcal{S}$ such that

$$N_\sigma(t_1, t_2) \le N_0 + \frac{t_2 - t_1}{\bar{\tau}_D} \tag{5.2}$$

for all $t_2 > t_1 > 0$. The parameter N_0 is referred to as the *chatter bound* and $\bar{\tau}_D$ is known as the *average dwell-time*. Note that for $\bar{\tau}_D > 0$, $\mathcal{S}[\bar{\tau}_D] = \mathcal{S}[\bar{\tau}_D, 1]$. Using these concepts, Hespanha and Morse derived the following sufficient condition for stability in [67].

THEOREM 5.3. *Consider the switching system* (2.1) *and suppose that $A_i \in \mathcal{A}$ is Hurwitz for $1 \le i \le m$. Further, let $\lambda_0 > 0$ be such that $A_i + \lambda_0 I$ is Hurwitz for $1 \le i \le m$. Then, for any chosen $\lambda \in [0, \lambda_0)$, there is a finite constant $\bar{\tau}_D^*$ such that* (2.1) *is exponentially stable, with decay rate λ, for all switching signals $\sigma(t) \in \mathcal{S}[\bar{\tau}_D, N_0]$ for $\bar{\tau}_D \ge \bar{\tau}_D^*$ and any chatter bound $N_0 > 0$.*

In other words, the system is stable if we switch *on average* more slowly than the rate corresponding to $\bar{\tau}_D^*$. A conservative estimate (upper bound) for the average dwell-time $\bar{\tau}_D^*$ can be calculated by first selecting $\lambda_0 > 0$ such that $A_i + \lambda_0 I$ is Hurwitz for all $i \in \{1, \ldots, m\}$. We then calculate a QLF, $V_i(x) = x^T P_i x$, for each subsystem by solving the Lyapunov equations

$$P_i(A_i + \lambda_0 I) + (A_i + \lambda_0 I)^T P_i = -I$$

for $i = 1, \ldots, m$ and calculate

$$\mu = \sup_{1 \le i,j \le m} \frac{\rho_{max}[P_i]}{\rho_{min}[P_j]}.$$

Finally, one chooses a stability margin λ for the switched system and obtains the average dwell-time by

$$\bar{\tau}_D^* = \frac{\log \mu}{2(\lambda_0 - \lambda)}.$$

Note that the results on average dwell-time in [67] were derived for a compact (not necessarily finite) set of Hurwitz matrices $\mathcal{A}$ and that a version of Theorem 5.3 was also derived for certain classes of switched nonlinear systems. A number of authors have further extended this work and derived analogous results for other classes of switched nonlinear systems [128, 40]. Further developments have also included the work of Zhai et al. [189] in which the authors modify this result so that the lowest average dwell-time $\bar{\tau}_D^*$ ensures that the switched system achieves a chosen L_2 gain, while several other authors have extended the results to allow for switching between stable and unstable subsystems, e.g., [190, 182].

Remark 5.4. The principal difficulty associated with the dwell-time problem is to obtain a tight lower bound on τ_D. It is quite remarkable that a nonconservative estimate of τ_D, even for simple classes of switching systems, has to date eluded the research community. Morse [117] provided an upper bound for the dwell-time of switched linear systems by considering the explicit solution in the time domain (2.3) and requiring that the norm of the state transition matrix between consecutive switches be at most 1. This is equivalent to the requirement that once the system switches into the state with matrix dynamics $\dot{x} = A_p x$, it must remain in the same state for at least time τ_p, where

$$\tau_p = \sup\{t : ||e^{A_p t}|| \geq 1\}.$$

Morse's upper bound is then $\tau_D \leq \sup_p\{\tau_p\}$. Because this selects the worst case among the τ_p, the upper bound is generally not sharp, and so it gives a conservative estimate for τ_D. For example, if $\{A_p\}$ is any set of matrices with a common Lyapunov function, then $\tau_D = 0$; however, in general some of the values τ_p may be positive.

Remark 5.5. A basic tool used to analyze systems with smooth dynamics is the invariance principle of LaSalle [85]. Recently, a number of authors have explored the use of this theorem for analyzing linear and nonlinear switching systems. Examples given in each of these papers indicate that application of the principle to switched systems is not straightforward (essentially, due to the effects of fast switching) but that these complications may be overcome by imposing mild restrictions on the switching signal. In this context the notion of dwell-time arises in several recent papers on the application of LaSalle's results to switched systems [6, 99, 65]. Briefly summarized, these papers prove that the trajectories of a switched system approach an invariant set under the assumption of a common Lyapunov function or multiple Lyapunov functions which decrease along all trajectories, together with a bound on the rate at which switching takes place.

5.2. Converse Lyapunov Theorems for Systems with Dwell-Time. In the present context, it is important to note that converse theorems for the existence of Lyapunov functions also exist for switched systems with a restriction on the dwell-time. Before we discuss these results, we note that in general it is unreasonable to expect one Lyapunov function as a function of the state to suffice for capturing stability. Consider again a set of matrices $\mathcal{A} = \{A_1, \ldots, A_m\}$ and assume we are given a dwell-time restriction τ_D. If there exists a Lyapunov function V such that $t \to V(x(t))$

is strictly decreasing for all nonzero solutions of the switched system (5.1), then this implies that the system (2.1) with arbitrary switching is also exponentially stable. In general, there is of course a distinction between stability under arbitrary switching and switching with a dwell-time restriction.

A converse theorem for the case of dwell-times is presented in [176, Cor. 6.5]. In the following statement we tacitly use the same symbol v for a norm defined on $\mathbb{R}^n$ and the induced matrix norm on $\mathbb{R}^{n\times n}$.

THEOREM 5.6. *The system* (5.1) *with fixed dwell-time τ_D is exponentially stable if and only if there are norms $v_1, \ldots, v_m$ on $\mathbb{R}^n$ and a constant $\beta > 0$ such that*

(i) *for all $i = 1, \ldots, m$ it holds that*

$$v_i(e^{A_i t}) \le e^{-\beta t} \text{ for all } t \ge 0;$$

(ii) *for all $i, j = 1, \ldots, m$ it holds that*

$$v_j(e^{A_j t}x) \le e^{-\beta t} v_i(x) \text{ for all } x \in \mathbb{R}^n, t \ge \tau_D\,.$$

Additionally, if the set of matrices $\mathcal{A}$ is irreducible,[7] then the norms in the previous theorem may be chosen such that β is equal to the exponential growth rate of (5.1) with fixed dwell-time τ_D. In this manner the result is an extension of the results of Molchanov and Pyatnitski and of Barabanov on the existence of nonquadratic Lyapunov functions in the case of arbitrary switchings.

Finally for this subsection, we note that the work described in [41, 131] also contains results relating dwell-time conditions for stability to the existence of Lyapunov functions.

5.2.1. Indirectly Induced Constraints: Multiple Lyapunov Functions and Slowly Varying Systems. An important paper in the recent evolution of stability theory of switched systems was published in 1994 by Michael Branicky [25]. Branicky made the observation, as Desoer had done in the 1960s, that Lyapunov functions could be used to derive laws to constrain the rate of switching in such a way as to guarantee stability. Thus, the multiple Lyapunov function method, which we shall describe here, is closely related to the dwell-time approach to stability described above. Rather than use a single Lyapunov function to constrain the rate of switching as Desoer had done, Branicky suggested the use of multiple Lyapunov functions (one for each mode of the system) to guarantee exponential stability. It is interesting to note that such an approach is suggested by the converse theorem for dwell-time systems described in the previous subsection (Theorem 5.6).

> *Branicky's basic idea was to define a Lyapunov function for each mode i of the system Σ. One then uses these functions to construct a stabilizing switching signal $\sigma(t)$ by only allowing the system to switch into mode i if the value of the corresponding Lyapunov function $V_i(x)$ is less than it was when the system last left mode i.*

It is important to note that the use of multiple Lyapunov functions to select a stable switching sequence for switched systems had been suggested by a number of authors prior to Branicky's original paper in 1994; in particular, Peleties and DeCarlo [126] deserve credit for promoting the original idea for switched linear systems. However, Branicky's paper, which extended the basic idea to the nonlinear case, has had a

[7] A set of matrices is called irreducible if only the trivial subspaces $\{0\}$ and $\mathbb{R}^n$ are invariant under all matrices in the set.

great impact on the community, and in the following discussion we adopt the notation and arguments given in the original papers [26].

Branicky considered a general autonomous switched nonlinear system

$$\dot{x}(t) = f(x,t), \quad f(x,t) \in \{f_1(x), \ldots, f_m(x)\}, \; t \geq 0, \tag{5.3}$$

where each mode, f_i, is assumed to be globally Lipschitz continuous and exponentially stable, and where the switching strategy is chosen in such a way that there are finite switches in finite time. While there are several similar versions of Branicky's result, we quote the following from [25], which can be stated with a minimum of mathematical formalism.

THEOREM 5.7. *Suppose that we have a finite number of Lyapunov functions $V_i(x)$ associated with the continuous-time vector fields $\dot{x} = f_i(x)$. Let $S_k = i_0, i_1, \ldots, i_k, \ldots$ denote the switching sequence of the system, and let $T_k = t_0, t_1, \ldots, t_k, \ldots$ denote the sequence of corresponding switching instances for the system. If, for each instant t_j when we switch into mode i, with corresponding Lyapunov function V_i, we have that*

$$V_i(x(t_j)) \leq V_i(x(t_k)), \tag{5.4}$$

where $t_k < t_j$ and t_k is the last time that we switched out of mode i, then the system is stable in the sense of Lyapunov.

Theorem 5.7 gives a simple rule for the construction of a stable slowly varying switching system. It states that when the system enters mode i, the value of the Lyapunov function associated with this mode must be less than the value it attained when the system last left mode i. For the purpose of illustration, consider a general nonlinear system with two modes, $\dot{x} = f_i(x)$, $i \in \{1, 2\}$. The profile of typical Lyapunov functions associated with modes 1 and 2, for a switching strategy constructed according to Theorem 5.7, is depicted in Figure 5.2.

The approach of multiple Lyapunov functions can be varied in several ways. For instance, we can relax the requirement that the functions V_i, $1 \leq i \leq m$, are proper Lyapunov functions in the sense that $\dot{V}_i(x) \leq 0$ along every entire trajectory $x(t)$ of the ith subsystem. Clearly, we may achieve less conservative results if we only demand that $\dot{V}_i(x)$ is nonpositive during time intervals where the system actually is in mode i (in which case V_i is often referred to as a Lyapunov-like function). Further improvement can be achieved if we compare the values of V_i in (5.4) at consecutive *starting* points of mode i. For details see, e.g., [26, 54].

Remark 5.8. The multiple Lyapunov function approach to stability analysis offers several advantages over other more conventional methods: (i) the underlying idea of the paradigm is easy to understand and is suitable for use in industry; (ii) the approach can be used for the stability analysis of heterogeneous systems; and (iii) the analysis can be based upon the existence of any type of Lyapunov function (not just quadratic functions).

Remark 5.9. There are several disadvantages associated with the approach: (i) no constructive procedure for choosing the best Lyapunov functions is currently known; (ii) a poor choice of Lyapunov functions V_i may lead to very conservative switching rules; (iii) in order to choose a Lyapunov function for each subsystem, the subsystems must be individually stable; (iv) the technique places conditions on all of the candidate Lyapunov functions.

Remark 5.10. A number of interesting research questions in the multiple Lyapunov function framework remain unanswered. In our context, namely, for switched linear systems, the most important of these pertains to developing a constructive

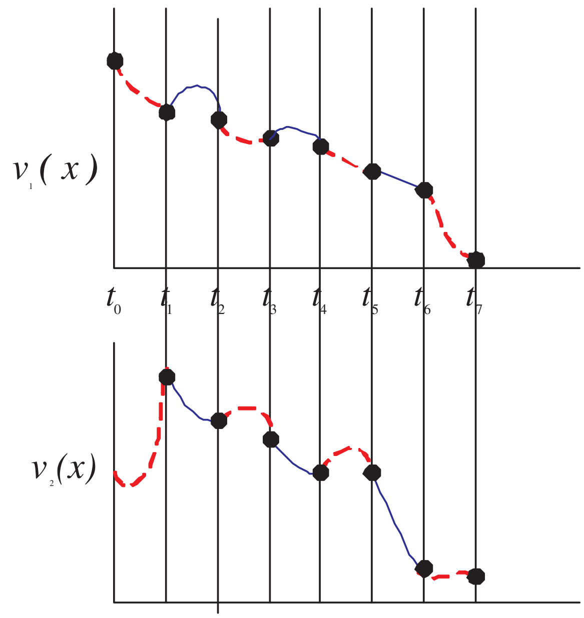

Fig. 5.2 *Multiple Lyapunov functions.*

method of choosing the candidate Lyapunov functions that minimize the dwell-time for each mode.

5.3. State-Dependent Switching and Stability. In the previous subsection, we considered a variety of results concerned with switched systems for which the rate of switching is constrained in some way. An alternative type of constraint that arises in the study of switched systems is where the switching action is constrained by the state vector of the system. Internet congestion control is an example of one such system. If the rule for switching between the constituent subsystems of a switched system is determined by the state vector of the system, we say that the switching is *state dependent*. Loosely speaking, the stability problems associated with this type of switching regime can be divided into two classes. In the first of these, the state space is partitioned by a number of hypersurfaces that determine the mode switches in the system dynamics, and the problem is to analyze the stability of the time-varying system defined in this way. On the other hand, in the second class of problem we are concerned with finding state-dependent rules for switching between a family of unstable systems that result in stability. Thus, in the former case, a partition of state space is specified and the problem is to determine the stability of the piecewise linear system defined by that partition, while in the latter case the aim is to find stabilizing state-dependent rules for switching between individually unstable systems.

5.3.1. Switched Systems and State-Dependent Constraints. We shall now consider switched systems that are constrained in the sense that a mode switch occurs in the systems dynamics when its state vector crosses certain threshold surfaces in state space. We first turn our attention to the class of piecewise affine systems.

Piecewise Affine Systems. A piecewise affine system is a dynamical system of the form

$$\dot{x} = A_i x + a_i \quad \text{for } x \in \Omega_i, \tag{5.5}$$

where $\Omega_1, \ldots, \Omega_m$ are closed sets with pairwise disjoint interiors such that $\cup_i \Omega_i = \mathbb{R}^n$, $A_1, \ldots, A_m$ are in $\mathbb{R}^{n\times n}$, and $a_1, \ldots, a_m$ are in $\mathbb{R}^n$. It is typical to assume that $a_i = 0$ for any region Ω_i that contains the origin. (If $a_i = 0$ for all i, the system is piecewise linear.)

Note that, in general, there can be an issue with the definition of solutions to (5.5) on common boundaries of the closed regions Ω_i. In particular, so-called *sliding motions* can arise if the vector fields corresponding to two adjacent sets Ω_i, Ω_j both point toward their common boundary. In the interests of simplicity, we shall not explicitly discuss the stability of systems with sliding motions in this subsection. However, it should be noted that the piecewise quadratic methods described below can be adapted to deal with sliding-mode dynamics. The interested reader should consult [74] for details.

For such systems, requiring the existence of a common Lyapunov function can be an unduly restrictive criterion for asymptotic stability of the origin. For instance, in order for the quadratic form $V(x) = x^T P x$ to define a CQLF for $\Sigma_i : \dot{x} = A_i x + a_i$ for each $i \in \{1, \ldots, m\}$, $V(x)$ must decrease along all trajectories of Σ_i everywhere in the state space. However, it is clear that this may well lead to unnecessarily conservative stability conditions as it fails to take into account that the system Σ_i is active only within the region Ω_i.

The S-procedure [24, 74] is a numerical technique that seeks to exploit the structure of (5.5) to obtain less conservative stability conditions. The key idea of the S-procedure is to require only that the function $V(x)$ decreases along trajectories of Σ_i in the corresponding region Ω_i. This can lead to less restrictive conditions for stability than are obtained through requiring the existence of a common Lyapunov function.

Piecewise QLFs. Extending the ideas of the S-procedure, a number of authors have studied piecewise QLFs [74, 130, 129, 61] in order to find less conservative stability criteria for piecewise linear systems. Here, rather than looking for a single QLF, $V(x) = x^T P x$, for the system (5.5), the idea is to search for a family of such functions satisfying certain local conditions and then to piece these together appropriately to form a Lyapunov function for the overall system.

For convenience, and to illustrate the main ideas behind the use of piecewise QLFs, we shall focus mainly on the results of [74]. In this paper, under the assumption that the regions Ω_i are polyhedral, a numerical procedure is described for searching for a piecewise QLF of the form[8]

$$V(x) = x^T P_i x \quad \text{for } x \in \Omega_i, \tag{5.6}$$

where $P_i = P_i^T \in \mathbb{R}^{n\times n}$ for $i = 1, 2, \ldots, m$. Extending the basic idea of the S-procedure, the authors of [74] relaxed the conditions for stability given by CQLF existence in a number of ways.

(i) The use of different quadratic forms for the different operating regions Ω_i can lead to greater flexibility in the definition of the Lyapunov function V.

[8]The function takes a slightly different form in regions Ω_i that do not contain the origin. For details, consult [74].

(ii) The matrices P_i are not required to be globally positive definite. In fact, by applying the S-procedure, the inequality $x^T P_i x > 0$ is required to hold only when $x \in \Omega_i$ for $1 \leq i \leq m$.
(iii) Similarly, $V_i(x) = x^T P_i x$ is required to decrease only along trajectories of $\dot{x} = A_i x + a_i$ for $x \in \Omega_i$.

A few specific points relating to the results described in [74] are worth noting.
(i) The matrices P_i are parameterized so as to ensure that the piecewise quadratic function $V(x)$ is continuous.
(ii) The conditions for (5.6) to define a piecewise QLF for the system (5.5) are expressed in the form of LMIs. Hence, modern convex optimization algorithms can be used to search for piecewise QLFs.
(iii) It is possible to use a partition other than that dictated by the system dynamics to define the piecewise quadratic function. Thus, if an initial search is unsuccessful, it may be possible to find a piecewise QLF defined with respect to an alternative, possibly finer, partition of the state space. However, the problem of selecting an initial partition, and of devising automatic methods of successively refining the partition to systematically search for piecewise QLFs, is in general far from straightforward.

The ideas and techniques of [74] were subsequently developed and extended in [75, 144], and similar LMI conditions for the stability of piecewise linear systems based on piecewise QLFs were presented in [61]. In this context, the work of [48] on applying piecewise quadratic methods to the problem of controller design for uncertain piecewise affine systems should also be noted.

The paper [184] described closely related work on the stability of the class of *orthogonal* piecewise linear systems. For such systems, the hyperplanes that partition the state space, $\mathbb{R}^n$ take the general form $x_i = c_{i,j}$ for $i = 1, \dots, n$, $j = 1, \dots, n_i$, and divide the state space into a family of hyperrectangles. The conditions for stability derived in [184] are based on the existence of PLLFs as opposed to piecewise QLFs.

Switching Rules Specified by Switching Surfaces. Ideas similar to those used to analyze piecewise linear systems have also been used to study more general state-dependent switching rules. Typically, these rules are defined by specifying a set of surfaces S_{ij}, $1 \leq i, j \leq m$, in the state space such that the system switches from mode i to mode j if the ith subsystem is currently active and the state vector crosses the surface S_{ij} [130, 74, 172]. For instance, in [74] piecewise QLFs are employed to obtain LMI-based stability conditions for such systems under the assumption that the switching surfaces are given by hyperplanes. The Lyapunov function used to derive the conditions in this paper is not required to be continuous provided that the value of the function decreases whenever the system switches from one mode to another. Similar results based on piecewise QLFs were also presented in [130]. The results of this last paper are based on a Lyapunov function that need not be continuous and, moreover, is not required to decrease at each switching instant. In fact, all that is needed for stability is that the values of the function are bounded by some continuous function of its initial value. The conditions in [130] are again formulated as LMIs, and can also be applied to systems with nonlinear constituent systems.

Piecewise Linear Systems and the Popov Criterion. The classical Popov criterion for the absolute stability of nonlinear systems [135, 168] can also be used in the analysis of certain piecewise linear systems. An example illustrating this is described in [74], where the stability of a two-dimensional piecewise linear system is established using the Popov criterion. For the system considered in this example, the Popov criterion ensures the existence of a Lyapunov function of Lur'e Postnikov form, meaning that it is the sum of an integral of a nonlinear function and a quadratic form. For the system considered in [74], this function actually reduces to a piecewise QLF of the form considered above. It should be noted that the connection between Lyapunov functions of Lur'e Postnikov form and piecewise QLFs was pointed out earlier by Weissenberger in [170], and that the idea of using piecewise QLFs for stability analysis had been suggested by Power and Tsoi in [136].

Positive Switched Linear Systems. The classes of switched systems considered above are constrained in the sense that the system dynamics must undergo a mode switch when the state vector crosses some surface in state space. The nature of these systems has led naturally to the consideration of both piecewise quadratic and piecewise linear Lyapunov functions in their stability analysis. A different type of constraint arises in the study of positive switched linear systems, where any trajectory starting from nonnegative initial conditions must remain within the nonnegative orthant for all subsequent times [47, 94]. In view of the considerable restriction that this imposes on the possible trajectories of a positive system, it is natural to consider so-called copositive Lyapunov functions when analyzing the stability of such systems [14]. These functions are only required to satisfy the requirements of a traditional Lyapunov function within the nonnegative orthant and may lead to less conservative stability conditions for positive switched linear systems than can be obtained using traditional Lyapunov functions. Some initial results on common copositive Lyapunov function existence can be found in [60, 107, 106]

5.3.2. Stabilizing Switching Rules. The results discussed in the last subsection were concerned with establishing the stability of systems subject to some specified state-dependent switching rule. Another problem of interest in this context is that of determining stabilizing switching rules for systems with unstable constituent systems.

In [172], the following problem was addressed. Given two LTI systems Σ_{A_1}, Σ_{A_2}, where both A_1 and A_2 have some eigenvalues in the right half plane, determine if there exists some rule for switching between these systems that results in stability. It has been established [172] that if some convex combination of the matrices A_1 and A_2 is Hurwitz, then such a stabilizing switching rule does indeed exist. Formally, this amounts to testing for the existence of some α with $0 < \alpha < 1$ such that the matrix

$$A(\alpha) = \alpha A_1 + (1 - \alpha)A_2$$

is Hurwitz. Moreover, the authors of [172] described how to construct a state-dependent stabilizing switching rule when such a stable convex combination exists.

The basic idea behind this construction is the following. As the matrix $A(\alpha)$ is Hurwitz, there exists some positive definite matrix $P = P^T > 0$ such that

$$A(\alpha)^T P + PA(\alpha) < 0. \tag{5.7}$$

It follows that the two cones Ω_1, Ω_2 defined by

$$\Omega_i = \{x \in \mathbb{R}^n : x^T(A_i^T P + PA_i)x < 0\} \tag{5.8}$$

cover the space (meaning that $\Omega_1 \cup \Omega_2 = \mathbb{R}^n$). Using this fact, it is possible to define two switching surfaces close to the boundaries of these cones such that the associated switching rule asymptotically stabilizes the overall system. In fact, the quadratic function $V(x) = x^T Px$, where P is a solution of (5.7), is a Lyapunov function for the

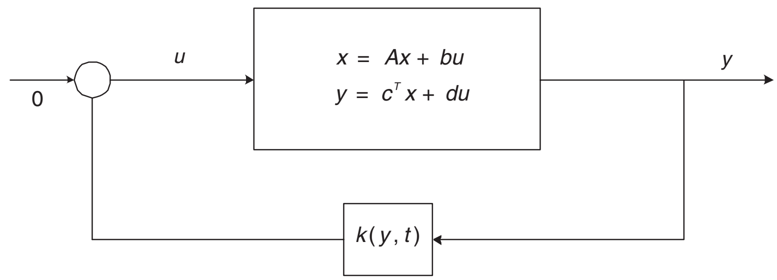

Fig. 6.1 *Graphical depiction of the Lur'e system. For the present discussion we consider SISO systems: A is a stable $n \times n$ matrix, b, c are appropriately dimensioned vectors, and d is a scalar.*

system defined by this switching rule. For this reason, we say that the switching rule defined in the above manner quadratically stabilizes the system.

The switching rule described in the previous paragraph is constructed so as to ensure that only a finite number of switches occur in any finite time interval. Two other switching rules for stabilizing the system are also described in [172], but both of these allow for the possibility of infinitely many switches occurring in a finite time interval and for practical reasons this may be undesirable. For second order systems a characterization of the existence of stabilizing switching rules is provided in [103].

The result that a stabilizing switching rule exists if some convex combination of the matrices A_1 and A_2 is Hurwitz extends to the case of an arbitrary finite family of matrices. Formally, given a family of unstable LTI systems $\Sigma_{A_1}, \ldots, \Sigma_{A_m}$, there is some rule for switching between them that results in quadratic stability if there are nonnegative real numbers $\alpha_1, \ldots, \alpha_m$ with $\alpha_1 + \cdots + \alpha_m = 1$ such that the matrix $\alpha_1 A_1 + \cdots + \alpha_m A_m$ is Hurwitz. In general this condition is not known to be necessary for the existence of such a quadratically stabilizing switching rule. However, for the case of switching between two unstable systems, the existence of a Hurwitz convex combination of the system matrices is known to be *equivalent* to the existence of a quadratically stabilizing switching rule [49].

The work of [172] was extended in the paper [171], where conditions for the existence of stabilizing switching rules were derived using piecewise QLFs as opposed to QLFs. Note also the related work on piecewise QLFs described in [180], and the recent paper [5] where it was shown that, under an additional assumption, there exists a stabilizing rule for switching between a pair of unstable LTI systems $\Sigma_{A_1}, \Sigma_{A_2}$, provided that there is some convex combination of the system matrices A_1, A_2 with the property that all of its eigenvalues have nonpositive real parts and any eigenvalues on the imaginary axis are simple.

6. The Lur'e Problem. The study of stability reached a peak in the 1960s and early 1970s in the engineering community. During this period a vast number of researchers studied the Lur'e problem. Consequently, it is impossible to write a review on this topic without mentioning this work, however briefly.

The SISO version of the Lur'e problem is depicted in Figure 6.1 (although our comments are equally valid for multiple-input multiple-output (MIMO) versions of the Lur'e problem).

As can be seen the Lur'e system is composed of a feedback connection of a stable LTI system and a nonlinear and possibly time-varying gain that is constrained to lie in some interval, say, $k(y, t) \in [0, 1]$.

Remark 6.1. Note that the Lur'e system reduces to an example of a classical switched system when $k(y,t) \in \{0,1\}$. Evidently, tools developed for the study of Lur'e systems can therefore be applied to the analysis of special classes of switched systems.

Three problems were widely studied in the context of the Lur'e system.

(i) When is the equilibrium solution of the Lur'e system globally uniformly exponentially stable for arbitrary time-varying gains $k(t)$?

(ii) Can one impose constraints on $k(y,t)$ such that the equilibrium solution of the Lur'e system is globally uniformly exponentially stable?

(iii) Can one impose conditions on the rate of change of $k(y,t)$ so that the equilibrium solution of the Lur'e system is globally uniformly exponentially stable?

Problems (i), (ii), and (iii) are clearly analogues of the *arbitrary switching*, *restricted switching*, and the *dwell-time* problems, respectively. Their study has led to many classical stability criteria: the Kalman–Yacubovich–Popov–Meyer lemma; the circle criterion; the off-axis circle criterion; the passivity theorem; and the Popov criterion [1]. The main tool used in developing these results was the Lur'e–Postnikov Lyapunov function. That is, one sought conditions on $A, b, c, d, k(y,t)$ to ensure the existence of a Lyapunov function of the form

$$V(x) = x^T P x + \lambda \int_0^y k(\delta, t) d\delta, \tag{6.1}$$

where $\lambda \in \mathbb{R}^+$ and where $P = P^T > 0$. Note that $V(x)$ is rarely quadratic. Due mainly to the remarkable results of Popov, and later of Kalman and Meyer, it was found that the existence of a function $V(x)$ for the Lur'e system could be deduced by testing whether or not

$$1 + \mathrm{Re}(H(j\omega)G(j\omega)) > 0 \quad \text{for all } \omega \in \mathbb{R}, \tag{6.2}$$

where $G(j\omega) = d + c^T(j\omega I - A)^{-1}b$ and $H(j\omega)$ is some rational function of ω that is referred to as a multiplier. Recently, a number of results have been obtained that relate results derived in the context of the Lur'e problem to more general results derived in the context of switched linear systems. In particular, the following result has proved to be a useful bridge between these areas.

THEOREM 6.2. *Let $G(j\omega) = \frac{N(j\omega)}{D(j\omega)}$ be a proper real rational transfer function and $K \in \mathbb{R}_+$. Let $\{A, b, c, d\}$ be a realization of $G(j\omega)$ so that $G(j\omega) = c^T(j\omega I - A)^{-1}b + d$. Assume that A and $\left(A - \frac{1}{K+d} bc^T\right)$ are Hurwitz. Then a necessary and sufficient condition for*

$$K + \mathrm{Re}\{G(j\omega)\} > 0 \quad \textit{for all } \omega \in \mathbb{R} \cup \{\infty\} \tag{6.3}$$

is that the matrix product $A\left(A - \frac{bc^T}{K+d}\right)$ has no negative real eigenvalues.

Theorem 6.2 has a number of implications for classical frequency domain stability criteria.

(i) *The Kalman–Yacubovich–Popov (KYP) lemma.* The SISO version of the KYP lemma [76] is expressed in the form of a strictly positive real (SPR) condition, namely, A and $A - \frac{1}{\gamma} bc^T$ are Hurwitz and

$$\gamma + \mathrm{Re}\left\{c^T(j\omega I_n - A)^{-1}b\right\} > 0 \quad \text{for all } \omega \in \mathbb{R}$$

for some $\gamma \in \mathbb{R}_+$. Hence it follows from Theorem 6.2 that a time-domain version of the SPR condition for SISO systems is that the matrices A and

$\left(A - \frac{1}{\gamma}bc^T\right)$ are Hurwitz and $A(A - \frac{1}{\gamma}bc^T)$ does not have any negative real eigenvalues.

(ii) *The circle criterion* [158]. The SISO version of the circle criterion is derived directly from the SISO KYP lemma. Here, conditions are derived for the existence of a Lyapunov function $V(x) = x^T Px, P = P^T \in \mathbb{R}^{n\times n}$, for the nonlinear Lur'e type system. In this case a necessary and sufficient condition for the existence of a QLF $V(x)$ is that [119] A and $A - bc^T$ are Hurwitz and that

$$1 + \mathrm{Re}\left\{c^T(j\omega I_n - A)^{-1}b\right\} > 0 \quad \text{for all } \omega \in \mathbb{R}.$$

It follows from Theorem 6.2 that a time-domain version of the circle criterion with $0 \le k(y,t) \le 1$ is that matrices A and $A - bc^T$ are Hurwitz and that matrix $A(A - bc^T)$ does not have any negative real eigenvalues.

(iii) *The Popov criterion.* The SISO Popov criterion [163] considers the stability of the Lur'e system where the function k is nonlinear but time-invariant. A sufficient condition for the absolute stability of this system is that A and $A - \frac{1}{\gamma}bc^T$ are Hurwitz and there exists a strictly positive $\alpha \in \mathbb{R}$ such that

$$\frac{1}{k} + \mathrm{Re}\left\{(1 + j\alpha\omega)c^T(j\omega I_n - A)^{-1}b\right\} > 0 \quad \text{for all } \omega \in \mathbb{R}.$$

It follows from Theorem 6.2 that a time-domain version of the Popov criterion is that A and $A - \frac{1}{\gamma}bc^T$ are Hurwitz and there exists a positive $\alpha \in \mathbb{R}$ such that the matrix $\bar{A}(\bar{A} - \frac{1}{\bar{d}+\frac{1}{k}}\bar{b}\bar{c})$ does not have any negative real eigenvalues, where $\{\bar{A}, \bar{b}, \bar{c}, \bar{d}\}$ is a realization of $(1 + \alpha s)c^T(sI_n - A)^{-1}b$.

Remark 6.3. The most interesting observation arising from this theorem is that all stability problems have been reduced to a CQLF existence problem. For example, the Popov criterion, which searches for the existence of a nonquadratic Lyapunov function for the original nonlinear system, has been reduced to a CQLF existence problem for $\dot{x} = \bar{A}x$ and $\dot{x} = (\bar{A} - \frac{1}{\bar{d}+\frac{1}{k}}\bar{b}\bar{c})x$. This appears to be an unexplored (and perhaps important) observation.

6.1. Passivity. Apart from the circle criterion, one of the big successes of this period was the derivation of the passivity theorem [168]. The passivity theorem gives a sufficient condition for stability of the following time-varying, nonlinear system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t) + Du(t),\\ u(t) &= -\Phi(t, y(t)), \end{aligned} \tag{6.4}$$

where $x \in \mathbb{R}^n, u, y, z \in \mathbb{R}^m, m < n$, $\Phi : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$, and A, B, C, D are matrices with the appropriate dimensions. These equations describe the dynamics of a control system where $x(t)$ is the state variable, $y(t)$ is the output variable, and $u(t)$ is the feedback. The dynamics of $x(t)$ is separated into a linear part Ax and a nonlinear part Bu, and it is assumed that the matrix A is Hurwitz so that the linear part is stable. The question of finding conditions on the nonlinearity Φ which are sufficient to guarantee stability of the system is known as the Lur'e problem. The passivity theorem provides the following solution: the system (6.4) is globally exponentially stable if $\Phi(t, 0) = 0$, $y^T\Phi(t, y) \ge 0$ for all $t \in \mathbb{R}$, all $y \in \mathbb{R}^m$, and if $H(j\omega) + H^*(j\omega)$ is positive definite for every real ω, where the transfer matrix $H(s)$ is defined by

$$H(s) = C(sI - A)^{-1}B + D, \quad s \in \mathbb{C}. \tag{6.5}$$

When these conditions are satisfied the KYP lemma guarantees the existence of a QLF $x^T Px$ for the system (6.4) [168].

This problem is relevant to the study of switching systems because in many cases the dynamics of a switching system has the following form:

$$\dot{x}(t) = Ax(t) - z(t, x(t)), \tag{6.6}$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$ is Hurwitz, and $z : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a time-varying, nonlinear vector field. The Lur'e system (6.4) can be rewritten in the form (6.6) with $z = \Phi(t, y(t))$, where $y(t)$ depends on the state $x(t)$ through the implicit relation $y(t) = Cx(t) - D\Phi(t, y(t))$. Thus (6.6) is more general than the Lur'e system, because z is allowed to depend on x in an arbitrary way.

One example of such a system arises in state-dependent switching, where $\mathbb{R}^n$ is partitioned into disjoint sets $\{\Omega_i\}$, and the dynamics switches between different linear systems as the state crosses from one region to another. So the dynamics is given by

$$\dot{x}(t) = A(x)\, x(t), \quad A(x) = A_i \quad \text{for all } x \in \Omega_i. \tag{6.7}$$

This system can be presented in the form (6.6) by defining $z(t, x) = (A - A(x))x(t)$ for some Hurwitz matrix A.

The passivity theorem provides a method to analyze the stability of systems of the form (6.6). The crucial ingredient is the existence of a QLF $x^T Px$ for the system. Instead of using the KYP lemma to provide the matrix P, one can instead postulate the existence of such a function and use it to derive conditions which must be satisfied by the nonlinearity Φ. To see how this might be achieved consider the following construction (which closely resembles the result of [172]). Let $\{A_1, \dots, A_m\}$ be a collection of real matrices such that the convex combination $\tilde{A} = \sum_{i=1}^m \alpha_i A_i$ is Hurwitz for some $\alpha_1, \dots, \alpha_m$ satisfying $\alpha_i \geq 0$, $\sum \alpha_i = 1$. Choose $P = P^T > 0$ such that $\tilde{A}^T P + P\tilde{A} < 0$ and define $K_i = \{x : x^T P(\tilde{A} - A_i)x \geq 0\}$. Then

$$\bigcup_{i=1}^m K_i = \mathbb{R}^n.$$

For $i = 1, \dots, m$ let $\Omega_i \subset K_i$ such that the Ω_i are disjoint and their union is $\mathbb{R}^n$. Then the system (6.7) is globally exponentially stable. The statement follows by choosing $z(t, x) = (\tilde{A} - A(x))x(t)$ in (6.7), where $A(x) = A_i$ for all $x \in \Omega_i$. Then the definition of K_i implies that $x^T Pz(t, x) \geq 0$ for all $x \in \mathbb{R}^n$, and this is sufficient to guarantee that $x^T Px$ is a Lyapunov function for (6.7). This observation can be used to design stable switching systems of the form (6.7). In particular, given a collection of real matrices $\{A_1, \dots, A_m\}$ for which some convex combination is Hurwitz, one would like to determine all matrices $P = P^T > 0$ which satisfy $\tilde{A}^T P + P\tilde{A} < 0$, as these would describe possible stable state-dependent switching rules for this collection. In general it is difficult to find a compact parameterization of these matrices. However, the procedure simplifies when the matrices A_i have the special form $A_i = \tilde{A} - BD_i^T$, where $\tilde{A}$ is Hurwitz and B is some fixed matrix. In this case, if there is a matrix C such that $H(j\omega) + H^*(j\omega)$ is positive definite for all real ω, where now $H(s) = C^T(sI - A)^{-1}B$, then the cones K_i are given as $\{x : x^T CD_i^T x \geq 0\}$. Therefore the search for matrices P can be replaced by a simpler search for matrices C that satisfy this positivity condition. In the case $m = 1$, where B, C are vectors in $\mathbb{R}^n$, the set of all possible vectors C can be described in the following compact and useful way [79]: A vector C satisfies the positivity condition above if and only if $C^T AB < 0$ and $C^T(A^2 + \omega^2)^{-1}AB < 0$ for all $\omega \in \mathbb{R}$. This allows a constructive procedure to find the vectors [79].

7. Some Open Problems and Future Directions. In this section, we shall briefly review and summarize some of the open questions arising from the results and issues discussed throughout the paper. Broadly speaking, the major open problems in the stability theory of switched linear systems can be divided into three categories, corresponding to the first three problems discussed at the end of section 1, namely, the problem of stability under arbitrary switching, the dwell-time problem, and the problem of determining stabilizing switching signals.

(i) In the context of stability under arbitrary switching, Theorems 4.3 and 4.4 provide simple conditions for CQLF existence that are related to the dynamics of switched linear systems via Theorem 4.18. While Theorem 4.7 gives necessary and sufficient conditions for a general family of stable LTI systems to have a CQLF, the conditions described by this result are extremely complicated and difficult to check, even for the case of a pair of third order systems. Hence, an open problem of some interest is to determine system classes, such as the class of second order systems or pairs of systems with system matrices differing by a rank one matrix, for which simple conditions for CQLF existence can be given. In this context, the work reported in [153] should be noted. In this paper, it was shown that the system classes covered by Theorems 4.3 and 4.4 can be treated within a common framework provided by the main result of [159]. This may provide some insights into how to obtain further system classes for which similarly simple conditions for CQLF existence can be derived.

(ii) A closely related problem to that described above is that of determining classes of switched linear systems for which CQLF existence is equivalent to exponential stability under arbitrary switching. Two examples of such system classes have been described in section 4, and, for such systems, the problem of determining stability under arbitrary switching is simplified considerably.

(iii) Theorem 4.22 gives a simple spectral condition for the existence of a common PLLF for a pair of stable second order LTI systems, and to date there are very few results of this kind available in the literature. This gives rise to the question of whether or not it is possible to extend this result to higher-dimensional systems.

(iv) For the class of positive switched linear systems, as mentioned in section 5, it is natural to consider copositive Lyapunov functions. In particular, given that the trajectories of positive systems are constrained to remain within the nonnegative orthant, such Lyapunov functions may lead to less conservative stability criteria than those obtained through requiring CQLF existence. This raises the problem of determining verifiable conditions for common copositive Lyapunov function existence for families of positive LTI systems.

(v) Apart from the problems above on stability for arbitrary switching signals, the important issue of determining nonconservative estimates of the *dwell-time* for constrained switching regimes is still unresolved.

(vi) On the question of determining stabilizing switching signals for unstable constituent systems, the work of Feron, Peleties, de Carlo, and others discussed in section 5 provides sufficient conditions for quadratic stabilization laws. Also, it is known that for the case of two constituent systems, the conditions discussed above are both necessary and sufficient for the existence of a quadratically stabilizing switching rule. To the best of the authors' knowledge, necessary and sufficient conditions for the existence of a general (not necessarily quadratic) stabilizing switching law are not known. Some results related to this topic were previously mentioned in section 5, where appropriate references were also given.

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