The Dirichlet energies of functions between spheres

WRITTEN BY

Samuel González-Castillo

UNDER THE SUPERVISION OF

Dr Stefan Bechtluft-Sachs

This thesis on

The Dirichlet energies of functions between spheres

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Author:	Samuel González-Castillo, BSc
Supervisor:	Dr Stefan Bechtluft-Sachs
Head of department:	Prof Stephen Buckley

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Introduction

1 The main focus of this thesis is the study of the p-energy of functions between spheres. In Theorems IV-2.4, IV-3.5 and IV-4.7, we introduce lower bounds for the p-energy of non-null-homotopic smooth functions from a sphere of dimension N to a sphere of dimension K. Since the infimum energy in every homotopy class is zero when $p < \mathrm{N}$, we always require at least $p \geqslant \mathrm{N}$.

We also prove in Theorem III-3.7 the precompactness in the uniform convergence topology of any set of smooth functions between compact manifolds with bounded p-energy, for p strictly bigger than the dimension of the domain. This implies, in particular, that any sequence of functions between compact manifolds with decreasing p-energy (for an appropriate p) must have a convergent subsequence in the uniform convergence topology (Corollary III-3.8).

In addition to this, we discuss the first variational form that characterises the critical functions of any Dirichlet energy (Theorem III-2.3). The thesis includes plenty of introductory material that helps contextualise the content and should make it accessible to readers from different backgrounds. Most proofs rely only on elementary tools from geometry, analysis and topology. Proofs are always either provided or referenced, except when the results are easily deducible.

2 **Bibliography and further reading.** The first two chapters are devoted to the introduction of background material. Chapter I is mostly based on the books *Introduction to smooth manifolds* [**10**] and *Introduction to Riemannian manifolds* [**9**], both written by Lee.

Chapter II covers a fairly wide range of topics. Our treatment of the calculus of variations is completely self-contained; similar expositions can be found in the literature, such as in chapter 9 of *A comprehensive introducion to differential geometry, Vol. 1* [**15**] by Spivak. Regarding algebraic topology, the book *Topology and geometry* [**5**] by Bredon may prove to be a useful reference. Functional analysis is not core to our discussion; we refer any curious reader to the works that we reference for proofs.

Our main reference for harmonic analysis is *Geometry of harmonic maps* [**18**] by Xin, but another valuable resource is *Two reports on harmonic maps* [**7**] by Eells and Lemaire.

Chapter I
Manifolds

1 Smooth manifolds

1.1 Definition. Let X be a set and N a natural number. An *(N-dimensional) chart* for X is a pair (U, φ) where $\mathrm{U} \subseteq \mathrm{X}$ and φ is a bijection from U to an open subset of $\mathbb{R}^{\mathrm{N}}$; we will often refer to these bijections as *charts* themselves. A collection of N-dimensional charts $\{(\mathrm{U}_i, \varphi_i)\}_{i \in \mathrm{I}}$ is said to be *compatible* if, whenever $\mathrm{U}_i \cap \mathrm{U}_j \neq \varnothing$ for $i, j \in \mathrm{I}$, the corresponding *transition function*

$$\begin{aligned} \varphi_i(\mathrm{U}_i \cap \mathrm{U}_j) &\longrightarrow \varphi_j(\mathrm{U}_i \cap \mathrm{U}_j) \\ x &\longmapsto \varphi_j \circ \varphi_i^{-1}(x) \end{aligned}$$

is a homeomorphism; if, in addition, $\cup_i \mathrm{U}_i = \mathrm{X}$, then the collection of charts is said to be an *(N-dimensional) atlas*. If any transition function between charts in a collection is smooth, we say that the collection is *smoothly compatible*, and a smoothly compatible atlas is said to be a *smooth atlas*.

Let us consider a set X together with an N-dimensional atlas $\mathcal{A}$. We shall endow X with the unique topology τ that makes every chart a homeomorphism. If the resulting topological space is Hausdorff and second-countable, we say that the space (X, τ) is an *(N-dimensional) manifold*. Moreover, if the atlas is smooth, we may consider its induced *smooth structure*: the maximal smooth atlas $\mathcal{S}_{\mathcal{A}}$ that contains $\mathcal{A}$. Then, the pair $(\mathrm{X}, \mathcal{S}_{\mathcal{A}})$ constitutes an *(N-dimensional) smooth manifold*.

When considering a smooth manifold $(\mathrm{X}, \mathcal{S}_{\mathcal{A}})$, we may refer to it without explicitly mentioning its smooth structure (i.e., as X), provided that doing so leads to no ambiguity.

A function $f : \mathrm{X} \longrightarrow \mathrm{Y}$ between smooth manifolds is said to be *smooth* if, for every $x \in \mathrm{X}$, given a pair of charts $(\mathrm{U}_{\mathrm{X}}, \varphi_{\mathrm{X}})$ of X and $(\mathrm{U}_{\mathrm{Y}}, \varphi_{\mathrm{Y}})$ of Y with $x \in \mathrm{U}_{\mathrm{X}}$ and $f(x) \in \mathrm{U}_{\mathrm{Y}}$, the *coordinate representation* of f, defined as $\varphi_{\mathrm{Y}} \circ f \circ \varphi_{\mathrm{X}}^{-1}$, is smooth.

We will denote the set of smooth maps between two manifolds X and Y as $C^\infty(\mathrm{X}, \mathrm{Y})$, and we will denote the set of smooth maps from a manifold X to the real numbers as $C^\infty(\mathrm{X})$.

1.2 Example (Spheres)**.** The N-dimensional sphere $\mathrm{S}^{\mathrm{N}} := \{x \in \mathbb{R}^{\mathrm{N}+1} \mid \|x\| = 1\}$ is an N-dimensional smooth manifold when considered with the smooth atlas given by the charts

$$\varphi_\pm : \mathrm{S}^{\mathrm{N}} \setminus \{(0, \ldots, 0, \mp 1)\} \longrightarrow \mathbb{R}^{\mathrm{N}}$$

$$(x_1, \ldots, x_{N+1}) \longmapsto \frac{1}{1 \pm x_{N+1}}(x_1, \ldots, x_N).$$

These charts are known as the *stereographic projections* of the sphere. As the name suggests, the charts are obtained by finding, for each $x \in S^N$, the point of intersection with the plane $\mathbb{R}^N \times \{0\} \subseteq \mathbb{R}^{N+1}$ of the line that connects x with $(0, \ldots, 0, \pm 1)$; this is depicted in Figure 1.1. Following a straightforward geometrical argument, it is easy to deduce that

$$\varphi_\pm^{-1}(y_1, \ldots, y_n) = \left(2\frac{y_1}{\|y\|^2 + 1}, \ldots, 2\frac{y_n}{\|y\|^2 + 1}, \mp\frac{\|y\|^2 - 1}{\|y\|^2 + 1}\right).$$

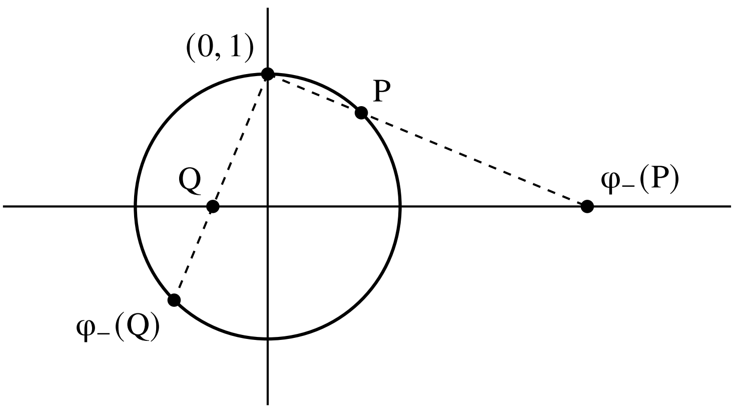

Figure 1.1: Visual representation of the stereographic projection defined by the chart φ_+ on the one-dimensional sphere.

1.3 Definition. Let X be an N-dimensional manifold with $x \in X$. A *derivation* at x is an $\mathbb{R}$-linear map $v_x : C^\infty(X) \longrightarrow \mathbb{R}$ satisfying the following product rule for every pair of functions $\alpha, \beta \in C^\infty(X)$:

$$v_x(\alpha \cdot \beta) = v_x\alpha \cdot \beta(x) + \alpha(x) \cdot v_x\beta.$$

The *tangent space* to X at x, denoted as T_xX, is defined to be the collection of all derivations at x. It can be shown that it is an N-dimensional vector space with the structure that it inherits from $\mathrm{Hom}(C^\infty(X), \mathbb{R})$ [**10**, Ch. 3].

If φ is a chart defined at x, we can obtain a basis of T_xX by considering the tangent vectors $\partial/\partial\varphi_i|_x$, with $i = 1, \ldots, N$, that act on any function $\alpha \in C^\infty(X)$ as

$$\left.\frac{\partial}{\partial\varphi_i}\right|_x \alpha := \left.\frac{\partial}{\partial y_i}\alpha \circ \varphi^{-1}(y)\right|_{\varphi(x)}.$$

Whenever there be is risk of ambiguity, we may write ∂_i in lieu of $\partial/\partial\varphi_i$.

1.4 Definition. Given any smooth function $f : \mathrm{X} \longrightarrow \mathrm{Y}$ between manifolds, we define the *differential* of f at a point x to be the operator

$$\begin{aligned} \mathrm{D}f_x : \mathrm{T}_x\mathrm{X} &\longrightarrow \mathrm{T}_{f(x)}\mathrm{Y} \\ v(-) &\longmapsto v(- \circ f). \end{aligned}$$

1.5 Scholium. The N-dimensional Euclidean space $\mathbb{R}^{\mathrm{N}}$ is trivially a smooth manifold when considered with the atlas $\{(\mathbb{R}^{\mathrm{N}}, \mathrm{id})\}$. At any $x \in \mathbb{R}^{\mathrm{N}}$, the tangent vectors $\partial/\partial\mathrm{id}_i|_x$ act on any real-valued function by returning its i-th partial derivative at x.

There exists a canonical linear correspondence between $\mathbb{R}^{\mathrm{N}}$ and $\mathrm{T}_x\mathbb{R}^{\mathrm{N}}$ obtained by identifying the vectors $\partial/\partial\mathrm{id}_i|_x$ with the canonical basis vectors $e_i \in \mathbb{R}^{\mathrm{N}}$. Under this correspondence, the differential of any function $\mathbb{R}^{\mathrm{N}} \longrightarrow \mathbb{R}^{\mathrm{K}}$, when considered as a function between smooth manifolds, is identical to the ordinary differential of the function.

1.6 Definition. Let X be a manifold and let $\mathrm{I} \subseteq \mathbb{R}$ be an open interval. A function $\gamma : \mathrm{I} \longrightarrow \mathrm{X}$ is said to be a *curve* if it is continuous. If X is smooth, we say that γ is a *smooth curve* if it is smooth as a function.

Given any smooth curve $\gamma : \mathrm{I} \longrightarrow \mathrm{X}$ and a point $t_0 \in \mathrm{I}$, we define the *velocity* of the curve at t_0 to be the vector

$$\left.\frac{d\gamma(t)}{dt}\right|_{t_0} := \mathrm{D}\gamma_{t_0}(\partial_t|_{t_0}) \in \mathrm{T}_{\gamma(t)}\mathrm{M},$$

where $\partial_t|_x$ represents $\partial/\partial\mathrm{id}|_x$, the canonical basis vector of $\mathrm{T}_x\mathbb{R}$. Alternatively, the velocity vector at a point t_0 can be denoted as $\gamma'(t_0)$ or $\dot\gamma(t_0)$.

1.7 Proposition (Characterisations of the tangent space)**.** Let X be an N-dimensional smooth manifold with $x \in \mathrm{X}$.

(i) The value of the derivation of a smooth real-valued function from a manifold at a point only depends on the values taken by the function in a neighbourhood of that point. Thus, the tangent space at a point can be equivalently defined as the set of derivations on *germs* of functions: equivalence classes under the relation of local equality around the point.

(ii) There is an isomorphism between $\mathrm{T}_x\mathrm{X}$ and a vector space on the quotient $\mathrm{T}_x^c\mathrm{X}$ of all the curves γ on X such that $\gamma(0) = x$ modulo equality of velocity at 0. Formally,

$$\mathrm{T}_x^c\mathrm{X} := \{\gamma \in C^\infty(]-\varepsilon, \varepsilon[, \mathrm{X}) \mid \varepsilon > 0, \gamma(0) = x\}/\sim,$$

where $\gamma_1 \sim \gamma_2$ if and only if, for any chart φ covering x, $(\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0)$.

Under this isomorphism, the equivalence class of any curve γ is identified with the derivation that maps every $f \in C^\infty(\mathrm{X})$ to $(f \circ \gamma)'(0)$.

(iii) Let C_x be the set of all the charts of the manifold X with x in their domain. There exists an isomorphism between $T_x^c X$ and a vector space defined on the quotient

$$T_x^v X := \left(C_x \times \mathbb{R}^N\right)/\sim,$$

where $(\varphi_1, v_1) \sim (\varphi_2, v_2)$ if and only if $v_2 = D(\varphi_2 \circ \varphi_1^{-1})_{\varphi_1(x)} v_1$.

Under this isomorphism, the equivalence class of any element (φ, v) is identified with the equivalence class of a curve

$$t \longmapsto \varphi^{-1}(\varphi(x) + tv).$$

Proof. (i) Let $f, g \in C^\infty(X)$ agree on an open neighbourhood U of x and let v be a derivation at x. The function $f - g$ must vanish on U. We shall consider a smooth cut-off function $s \in C^\infty(X)$ taking the value 1 at every point outside U and vanishing at x; it is easy to check that such a function can always be constructed. Under these conditions, $(f - g) \cdot s = f - g$ and, consequently,

$$v(f - g) = v((f - g) \cdot s) = v(f - g) \cdot s(x) + (f - g)(x) \cdot vs = 0,$$

since $s(x) = (f - g)(x) = 0$. This yields that, indeed, $v(f) = v(g)$.

(ii) We can equip $T_x^c X$ with a vector space structure by fixing a chart φ covering x and identifying each equivalence class $[\gamma]$ with $(\varphi \circ \gamma)'(0) \in \mathbb{R}^N$. This identification can be easily shown to be injective and well-defined; it is also surjective, because, given any $v \in \mathbb{R}^N$, we can find its pre-image to be the equivalence class of a curve

$$\varphi^{-1} \circ (t \longmapsto \varphi(x) + tv).$$

Hence it follows that we can regard $T_x^c X$ as an N-dimensional vector space with the operations

$$[\gamma_1] + [\gamma_2] = \left[\varphi^{-1} \circ (\varphi \circ \gamma_1 + \varphi \circ \gamma_2)\right], \qquad \lambda[\gamma] = \left[\varphi^{-1} \circ (\lambda \cdot \varphi \circ \gamma)\right]$$

for any $[\gamma], [\gamma_1], [\gamma_2] \in T_x^c X$ and any scalar λ.

Having given a vector space structure to $T_x^c X$, we can define the function

$$\begin{aligned} \Lambda : T_x^c X &\longrightarrow T_x X \\ [\gamma] &\longmapsto (f \in C^\infty(X) \longmapsto (f \circ \gamma)'(0)). \end{aligned}$$

and prove it to be an isomorphism. It is immediate from the definition that this map is well-defined, and its linearity follows from the linearity of differentiation, so it will suffice to show that its kernel is 0 in order to prove that it is a vector space isomorphism.

Let $[\gamma] \in \ker \Lambda$. Taking (i) into account, for every smooth real-valued function f defined on a neighbourhood of x, we will have

$$D(f \circ \gamma)_0 = D(f \circ \varphi^{-1})_{\varphi(x) = \varphi \circ \gamma(0)} \circ D(\varphi \circ \gamma)_0 = 0.$$

This will be true, in particular, if we fix f to be $\pi_i \circ \varphi$ where π_i denotes the i-th cartesian projection in $\mathbb{R}^N$. In this case,

$$D(f \circ \gamma)_0 = D(\pi_i \circ \cancel{\varphi \circ \varphi^{-1}})_{\varphi(x)=\varphi\circ\gamma(0)} \circ D(\varphi \circ c)_0 = D(\pi_i) \circ D(\varphi \circ \gamma)_0.$$

Hence, if $(f \circ \gamma)'(0) = 0$ for every function f, then $D(\pi_i) \circ D(\varphi \circ c)_0 = 0$ for any component $i = 1, \dots, N$. Since $D(\pi_i) = \pi_i$, this implies that $D(\varphi \circ \gamma)_0 = 0$, which shows that $[\gamma] = 0$, just as we wanted to prove.

(iii) We can endow $T_x^v M$ with an N-dimensional vector space by fixing a chart $\varphi \in C_x$ and identifying the equivalence class of any $(\psi, v) \in C_x \times \mathbb{R}^N$ with the vector $D(\varphi \circ \psi^{-1})_{\psi(x)}(v)$. This identification is well-defined for, given any $(\psi_1, v_1) \sim (\psi_2, v_2)$, we will have $v_2 = D(\psi_2 \circ \psi_1^{-1})_{\psi_1(x)}(v_1)$, so

$$D\Big(\varphi \circ \psi_2^{-1}\Big)_{\psi_2(x)}(v_2) = D\Big(\varphi \circ \psi_2^{-1} \circ \psi_2 \circ \psi_1^{-1}\Big)_{\psi_1(x)}(v_1) = D\Big(\varphi \circ \psi_1^{-1}\Big)_{\psi_1(x)}(v_1).$$

We can now define an isomorphism $\Phi : T_x^v M \longrightarrow T_x^c M$ as

$$\Phi(\psi, v) := \big[\psi^{-1} \circ (t \longmapsto \psi(x) + tv)\big].$$

This map is well-defined since, if $(\psi_1, v_1) \sim (\psi_2, v_2)$, then

$$\begin{aligned} D\Big(\psi_2 \circ \Big(\psi_2^{-1} \circ (t \mapsto \psi_2(x) + tv_2)\Big)\Big)_0 &= D\Big(t \mapsto t \cdot D(\psi_2 \circ \psi_1^{-1})_{\psi_1(x)}(v_1)\Big)_0 \\ &= D\Big(\psi_2 \circ \Big(\psi_1^{-1} \circ (t \mapsto \psi_1(x) + tv_1)\Big)\Big)_0. \end{aligned}$$

Moreover, the linearity of Φ can be easily checked. Thus, just as before, we only need to show its kernel to be 0. Indeed, if $\Phi(\psi, v) = [0]$, then

$$0 = (\psi \circ \Phi(\psi, v))' = (t \longmapsto \psi(x) + tv)' = v,$$

which will mean that, for any chart φ covering x, $D(\varphi \circ \psi^{-1})_{\psi(x)}(v) = 0$.

■

1.8 Definition. A smooth function $f : X \longrightarrow Y$ is said to be a *smooth embedding* if it defines a homeomorphism onto its image and if, at every $x \in X$, the differential Df_x is injective.

2 Vector bundles

2.1 Definition. A *vector bundle* of *rank n* over a topological space X is a topological space E together with a continuous surjection $\pi : E \longrightarrow X$ such that:

(B1) For every $x \in X$, $E|_x := E_x := \pi^{-1}(\{x\})$ is an n-dimensional real vector space.

(B2) For every $x \in X$, there exists a neighbourhood U of x and a *local trivialisation*, i.e., a homeomorphism

$$\Phi : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^n$$

such that, for every $u \in U$, the restriction $\Phi|_{\pi^{-1}(u)}$ is a vector space isomorphism onto $\{u\} \times \mathbb{R}^n$.

If X and E have a smooth structure, π is smooth and, around every point, there is a local trivialisation Φ that is a diffeomorphism, then we say that the vector bundle is itself *smooth*.

Given any (smooth) vector bundle (E, π), we define a *section* to be a (smooth) continuous function $\xi : X \longrightarrow E$ such that $\pi \circ \xi = \mathrm{id}_X$. For convenience, we may write $\xi|_x := \xi_x := \xi(x)$. The collection of all the smooth sections of a manifold on a vector bundle (E, π) is denoted as $\Gamma(E)$ and can be given a vector space structure in the obvious way. We can also consider sections locally: a *local section* is a section defined on an open subset of a manifold. In order to emphasise that a given section is not local, we may refer to it as a *global* section.

A frame with respect to a bundle $E \longrightarrow X$ is a collection of sections the images of which at any point x constitute a basis of E_x. If these sections are local, the frame is said to be local as well. If these sections are smooth, the frame is also said to be smooth.

A vector bundle $\pi' : E' \longrightarrow X$ is said to be a *subbundle* of a vector bundle $\pi : E \longrightarrow X$ if (E', π') is itself a vector bundle, E' is a topological subspace of E, $\pi' = \pi|'_E$ and every fiber E'_x is a subspace of E_x. If (E, π) and (E', π') are smooth, then (E', π') is a *smooth subbundle* if, in addition to the previous conditions, E' can be smoothly embedded into E.

2.2 Example. Given a smooth manifold X, the pair $(X \times \mathbb{R}^n, \pi)$ where $\pi(x, v) := x$ defines the n-dimensional *trivial bundle* on X.

2.3 Lemma. Let X be a smooth manifold. Given a natural n, let $\pi : E \longrightarrow X$ be a function such that, for every $x \in X$, $\pi^{-1}(x)$ has the structure of an n-dimensional vector space. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two smooth structures on E such that $(E_1 := (E, \mathcal{S}_1), \pi)$ and $(E_2 := (E, \mathcal{S}_2), \pi)$ are smooth vector bundles over X.

If, around every point $x \in X$, we can find some functions $\xi_1, \ldots, \xi_n$ that are smooth local frames with respect to both (E_1, π) and (E_2, π), then $E_1 = E_2$.

Proof. Let $x \in X$ and let $\{\xi_i\}_i$ be a smooth local frame around x with respect to both E_1 and E_2. It is easy to check that there must exist an open neighbourhood U of X on which we can define a smooth local trivialisation $\Phi : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^n$ with respect to both E_1 and E_2 in such a way that, for every point $u \in U$,

$$\{\Phi(\xi_1(u)), \ldots, \Phi(\xi_n(u))\} = \{(u, e_1), \ldots, (u, e_n)\},$$

where e_k denotes the k-th canonical basis vector of $\mathbb{R}^n$.

If φ is a chart of X around x which, without loss of generality, we will assume to be defined on U, then $(\varphi \times \mathrm{id}_{\mathbb{R}^n}) \circ \Phi$ is a chart of both E_1 and E_2, which shows that E_1 and E_2 have the same smooth structure. ■

2.4 Theorem. Given two natural numbers N and n, let X be an N-dimensional smooth manifold and let $\pi : E \longrightarrow X$ be a function from a set E such that, for every $x \in X$, the pre-image $\pi^{-1}(x)$ is endowed with an n-dimensional vector space structure. We shall assume that we have a countable collection of open sets $\{U_i\}_{i\in I}$ covering X and a family of maps $\{\xi_{i1}, \dots \xi_{in}\}_{i\in I}$ with $\xi_{ir} : U_i \longrightarrow E$ such that $\{\xi_{ir}(x)\}_{r=1}^n$ is a basis of $\pi^{-1}(x)$ for every $x \in U_i$ and $i \in I$. We will further assume that, for every pair of indices i and j such that $U_i \cap U_j \neq \varnothing$, there exist smooth real-valued functions $\alpha_{ijr}^s \in C^\infty(U_i \cap U_j)$ such that

$$\xi_{ir}|_{U_i \cap U_j} = \sum_{s=1}^n \alpha_{ijr}^s \xi_{js}|_{U_i \cap U_j}$$

for $r = 1, \dots, n$, where addition and multiplication by scalars is performed within each vector space $\pi^{-1}(x)$ for every $x \in X$.

Under these conditions, there exists an $(N + n)$-dimensional smooth structure on E that makes (E, π) a smooth vector bundle in which the maps ξ_{ir} are local frames. According to § 2.3, this smooth structure is unique.

Proof. For every $i \in I$, we can construct a function

$$\Phi_i : \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{R}^n$$

defined as follows. For every $e \in \pi^{-1}(U_i)$, there must exist some unique real coefficients λ_k such that $e = \sum_k \lambda_k \cdot \xi_k(\pi(e))$. We set $\Phi(e) := (\pi(e), \lambda)$. This function Φ_i is the inverse of

$$\begin{aligned} U_i \times \mathbb{R}^n &\longrightarrow \pi^{-1}(U_i) \\ (u, \lambda) &\longmapsto \sum_{i=1}^n \lambda_i \xi_i(u). \end{aligned}$$

and is clearly bijective.

Without loss of generality, consider an atlas $\{(U_i, \varphi_i)\}_{i\in I}$ of X. We may now construct an atlas for E by defining, for every $i \in I$, a chart ψ_i as

$$\begin{aligned} \psi_i : \pi^{-1}(U_i) &\longrightarrow \mathbb{R}^{N+n} \\ e &\longmapsto (\psi_i \times \mathrm{id}_{\mathbb{R}^n}) \circ \Phi_i(e). \end{aligned}$$

The compatibility of the resulting atlas follows from the fact that, for any charts ψ_i and ψ_j with non-disjoint domains,

$$\psi_j \circ \psi_i^{-1}|_{U_i \cap U_j} = (\varphi_i \circ \varphi_j^{-1}) \times \left(\lambda \longmapsto \sum_{r=1}^n \lambda_r \cdot (\alpha_{jir}^1, \dots, \alpha_{jir}^n)\right),$$

which is clearly smooth according to our hypotheses.

The atlas that we have just defined endows E with a smooth structure under which π is clearly smooth, for its coordinate representation under the charts ψ_i and φ_i is just the projection onto the first N coordinates. Furthermore, given any point in X and any U_i containing it, it can be readily checked that Φ_i will be a local trivialisation. In particular, we can deduce that Φ_i is a diffeomorphism from the fact that $\psi_i = (\varphi_i \times \mathrm{id}_{\mathbb{R}^n}) \circ \Phi_i$ is a diffeomorphism and so is $\varphi_i \times \mathrm{id}_{\mathbb{R}^n}$. This all shows that (E, π) is a smooth vector bundle. What is more, by construction, it is obvious that the functions ξ_{ir} are smooth local frames of the bundle: their smoothness follows from the fact that their coordinate representation under any chart ψ_i is (with the appropriate domain restriction) $\mathrm{id}_{\mathbb{R}^N} \times (x \mapsto e_r)$, where e_r is the r-th canonical basis vector of $\mathbb{R}^n$. ∎

2.5 Proposition. Let X be a smooth manifold. Let us consider the union

$$\mathrm{TX} := \bigcup_{x \in \mathrm{X}} \mathrm{T}_x\mathrm{X}$$

together with a projection function $\pi : \mathrm{TX} \longrightarrow \mathrm{X}$ mapping any $v \in \mathrm{T}_x\mathrm{X}$ to x. The function π is well-defined and there exists a unique smooth structure on TX that turns (TX, π) into a smooth bundle on X in which, for any chart (U, φ), the functions $x \in \mathrm{U} \longmapsto (x, \partial/\partial\varphi_i|_x) \in \mathrm{TX}$ are smooth local frames.

2.6 Definition. The vector bundle TX introduced in § 2.5 for any smooth manifold X is the *tangent bundle* of X. The sections of a tangent bundle are said to be *vector fields* and the collection of all the vector fields of a manifold X is denoted as $\mathfrak{X}(\mathrm{X}) := \Gamma(\mathrm{TX})$.

In general, given any vector fields χ, η of a smooth manifold, their composition $\eta\chi$ is not a vector field. Nevertheless, it can be shown that their *Lie bracket*

$$[\chi, \eta] := \chi\eta - \eta\chi$$

does indeed define a vector field.

2.7 Proposition. Given any smooth vector bundle $\pi : \mathrm{E} \longrightarrow \mathrm{X}$, let us define

$$\mathrm{E}^* := \bigcup_{x \in \mathrm{X}} \mathrm{E}_x^*$$

and the function $\pi' : \mathrm{E}^* \longrightarrow \mathrm{X}$ taking any $\omega \in \mathrm{E}_x^*$ to x. We are using a star to denote the dual to a vector space.

There exists a unique smooth structure on E^* that turns (E^*, π') into a smooth vector bundle such that, for any smooth local frame $\{\xi_i\}_i$ on an open set $\mathrm{U} \subseteq \mathrm{E}$ and any index i, the function taking any $x \in \mathrm{U}$ to the only $\omega_x \in \mathrm{T}_x^*\mathrm{X}$ with $\omega_x(\xi_k|_x) = \delta_{ik}$ is a smooth local section. The resulting bundle is said to be the *dual bundle* to E.

2.8 Definition. Given any smooth manifold X and any point $x \in \mathrm{X}$, we may consider the dual of its tangent space, $\mathrm{T}_x^*\mathrm{X} := (\mathrm{T}_x\mathrm{X})^*$, which is called the *cotangent space* to x. The elements of this dual are said to be *covectors*, and its sections are called *covector fields*.

The dual bundle to the tangent bundle of X is said to be the *cotangent bundle* of X, and is denoted by $\mathrm{T}^*\mathrm{X} := (\mathrm{TX})^*$.

2.9 Proposition. Let $(\mathrm{E}_1, \pi_1), \dots, (\mathrm{E}_n, \pi_n)$ be some smooth vector bundles defined on a manifold X. We may construct a set

$$\mathrm{E} := \bigcup_{x \in \mathrm{X}} (\mathrm{E}_1|_x \otimes \cdots \otimes \mathrm{E}_n|_x)$$

and define a projection $\pi : \mathrm{E} \longrightarrow \mathrm{X}$ taking any element in $\mathrm{E}_1|_x \otimes \cdots \otimes \mathrm{E}_n|_x$ to x.

There exists a unique smooth structure on E that turns (E, π) into a smooth vector bundle such that, for any smooth local sections ξ_i on E_i (with $i = 1, \dots, n$), the function

$$\begin{aligned} \cap_{i=1}^n \operatorname{dom} \xi_i &\longrightarrow \mathrm{E} \\ x &\longmapsto \xi_1(x) \otimes \cdots \otimes \xi_n(x) \end{aligned}$$

is smooth. The vector bundle (E, π) is said to be a *tensor bundle* of $\mathrm{E}_1, \dots, \mathrm{E}_n$.

2.10 Example. In any finite-dimensional vector space V, the vector space of all k-multilinear forms $\mathrm{T}^k(\mathrm{V}^*)$ is a k-th tensor power of V^*. Hence, given any manifold X and any natural k, we can consider the tensor bundle

$$\mathrm{T}^k\mathrm{T}^*\mathrm{X} := \bigcup_{x \in \mathrm{X}} \mathrm{T}^k(\mathrm{T}^*_{\mathrm{X}}\mathrm{X}).$$

with the smooth structure and projection functions defined in § 2.9. The smooth sections of these bundles are said to be *tensor fields*.

2.11 Definition. Let $\pi : \mathrm{E} \longrightarrow \mathrm{Y}$ be a smooth bundle and let $f : \mathrm{X} \longrightarrow \mathrm{Y}$ be a smooth map. We define the *pullback bundle* $f^*\mathrm{E}$ on X as

$$f^*\mathrm{E} := \{(x, e) \in \mathrm{X} \times \mathrm{E} \mid \pi(e) = f(x)\},$$

with the projection function $\pi^*(x, e) = x$. Given any $(x, e) \in f^*\mathrm{E}$, we may define $\pi^\circ(x, e) := e$. This function π° will make the diagram

$$\begin{tikzcd} f^*\mathrm{E} \arrow[r, "\pi^\circ"] \arrow[d, "\pi^*"] & \mathrm{E} \arrow[d, "\pi"] \\ \mathrm{X} \arrow[r, "f"] & \mathrm{Y} \end{tikzcd}$$

commutative for $\pi \circ \pi^\circ = f \circ \pi^*$.

3 Differential forms and integration

3.1 Let V be a finite-dimensional vector space. The set of all k-multilinear forms on V is a k-th tensor product of V^*; in accordance with § 2.10, we shall denote it by $T^k(V^*)$, and we will define

$$T(V^*) := \bigoplus_{k=0}^{\infty} T^k(V^*).$$

Analogously, we will refer to the set of k-alternating forms on V as $\Lambda^k(V^*)$, and we will write

$$\Lambda(V^*) := \bigoplus_{k=0}^{\dim V^*} \Lambda^k(V^*).$$

Let $v_{(-)}$ and $w_{(-)}$ be some vectors in V. The tensor product in $T(V)$ taking any $\omega \in T^k(V^*)$ and any $\mu \in T^l(V^*)$ to

$$(\omega \otimes \mu)(v_1, \dots, v_k; w_1, \dots, w_l) = \omega(v_1, \dots, v_k) \cdot \mu(w_1, \dots, w_l)$$

defines a tensor algebra of V^*. Similarly, the wedge product in $\Lambda(V^*)$ defined in such a way that, for any $\omega \in \Lambda^k(V)$ and $\mu \in \Lambda^l(V)$,

$$(\omega \wedge \mu)(v_1, \dots, v_{k+l}) := \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma) \cdot (\omega \otimes \mu)\big(v_{\sigma(1)}, \dots, v_{\sigma(k+l)}\big).$$

defines an exterior algebra of V^*.

All the spaces $\Lambda^k(V)$ are subspaces of $T^k(V)$, and so is $\Lambda(V)$ a subspace of $T(V)$. Nevertheless, $\Lambda(V)$ is not itself a subalgebra of $T(V)$ — albeit $\Lambda(V)$ can be embedded, as an algebra, into $T(V)$.

3.2 Proposition. Given a smooth manifold X and a natural number k, let us construct the set

$$\Lambda^k(T^*X) := \bigcup_{x \in X} \Lambda^k(T^*_x X)$$

and define a projection $\pi : \Lambda^k(T^*_x) \longrightarrow X$ taking any element in $\Lambda^k(T^*_x X)$ to x.

There exists a unique smooth structure on $\Lambda^k T^*X$ that turns $(\Lambda^k T^*X, \pi)$ into a smooth vector bundle such that, for any local covector fields $\omega_1, \dots, \omega_k$, the function

$$\begin{aligned} \cap_{i=1}^k \operatorname{dom} \xi_i &\longrightarrow \Lambda^k T^*X \\ x &\longmapsto \omega_1(x) \wedge \cdots \wedge \omega_k(x) \end{aligned}$$

is smooth. Moreover, $(\Lambda^k T^*X, \pi)$ is a smooth subbundle of $T^k T^*X$.

3.3 Definition. The smooth sections of $\Lambda^k T^*X$ are said to be *(differential) k-forms*, or *differential forms* of *degree k*. We denote the set of all differential k-forms over a manifold X by $\Omega^k(X)$.

If $f : \mathrm{X} \longrightarrow \mathrm{Y}$ is a smooth function between manifolds and $\omega \in \Omega^k(\mathrm{Y})$, the *pullback* of ω by f is a differential k-form on X defined, on any $x \in \mathrm{X}$, by

$$(f^*\omega)_x(v_1, \dots, v_k) := \omega_{f(x)}(\mathrm{D}f(v_1), \dots, \mathrm{D}f(v_k)).$$

3.4 Definition. As we mentioned in § 1.5, in the manifold of the real numbers with its canonical smooth structure, we can easily identify the tangent space to any point r with $\mathbb{R}$ itself; this can be achieved with the isomorphism ρ_r that maps the tangent vector $(\partial/\partial \mathrm{id})$ to 1. Given any smooth manifold X with a smooth $\alpha : \mathrm{X} \longrightarrow \mathbb{R}$, the differential of α at a point x is a function $\mathrm{D}\alpha_x : \mathrm{T}_x\mathrm{X} \longrightarrow \mathrm{T}_{\alpha(x)}\mathbb{R}$. If we compose this with $\rho_{\alpha(x)}$, we will have a covector

$$\begin{aligned} d\alpha_x : \mathrm{T}_x\mathrm{X} &\longrightarrow \mathbb{R} \\ v &\longmapsto \rho_{\alpha(x)} \circ \mathrm{D}\alpha_x(v) = v(\alpha), \end{aligned}$$

which will live in the cotangent space $\mathrm{T}_x^*\mathrm{X}$. With this, we get a covector field $d\alpha \in \Omega^1(\mathrm{X})$.

The operator $d : \Omega^0(\mathrm{X}) \longrightarrow \Omega^1(\mathrm{X})$ can be extended to a family of functions $d : \Omega^k(\mathrm{X}) \longrightarrow \Omega^{k+1}(\mathrm{X})$ for every non-negative integer k. The *exterior derivative* of X is the only extension that agrees with our definition of d for $k = 0$, that is linear over $\mathbb{R}$ and such that, for every pair of forms $\omega \in \Omega^k(\mathrm{X})$ and $\mu \in \Omega^l(\mathrm{X})$,

$$d(\omega \wedge \mu) = (d\omega \wedge \mu) + (-1)^k(\omega \wedge d\mu).$$

As a consequence of this properties, the exterior derivative satisfies $d \circ d = 0$.

A differential form ω is said to be *closed* if $d\omega = 0$. Furthermore, it is said to be *exact* if there exists a form μ such that $d\mu = \omega$.

3.5 Scholium. Let $\varphi_1, \dots, \varphi_n$ be the components of a chart at a point x of a smooth manifold X. The family of vectors $\{d\varphi_i\}_i$ of $\mathrm{T}_x^*\mathrm{X}$ is the dual basis for $\{\partial/\partial\varphi_i\}_i$ — in the sense that $d\varphi_i(\partial/\partial\varphi_j) = \delta_{ij}$.

3.6 Proposition. Let X be a smooth N-dimensional manifold. The exterior derivative d over X has the following properties.

(i) Let (U, φ) be a chart of X and let k be a non-negative integer. Given some functions $\omega_{i_1,\dots,i_k} \in C^\infty(\mathrm{U})$, with $i_{(-)} \in \{1, \dots, \mathrm{N}\}$, if

$$\omega = \sum_{i_1,\dots,i_k=1}^{\mathrm{N}} \omega_{i_1,\dots,i_k} \cdot d\varphi_{i_1} \wedge \cdots \wedge d\varphi_{i_k},$$

then we always have

$$d\omega = \sum_{i_1,\dots,i_k=1}^{\mathrm{N}} \sum_{j=1}^{\mathrm{N}} \frac{\partial \omega_{i_1,\dots,i_k}}{\partial \varphi_j} \cdot d\varphi_j \wedge d\varphi_{i_1} \wedge \cdots \wedge d\varphi_{i_k}.$$

(ii) Given, for any integer $k \geqslant 0$, any $\omega \in \Omega^k(\mathrm{X})$, the exterior derivative $d\omega$ is the unique $(k+1)$-form that satisfies, for any vector fields $\chi_0, \ldots, \chi_k$ with $[\chi_i, \chi_j] = 0$ for $i, j = 0, \ldots, k$,

$$d\omega(\chi_0, \ldots, \chi_k) = \sum_{i=0}^{k} (-1)^i \chi_i \omega(\chi_0, \ldots, \hat{\chi}_i, \ldots, \chi_k),$$

where the hat denotes omission.

(iii) If $f : \mathrm{X} \longrightarrow \mathrm{Y}$ is a smooth function and ω is a differential k-form over Y, then $f^*(d\omega) = d(f^*\omega)$.

Proof. Results 14.24, 14.26 and 14.32 in *Introduction to smooth manifolds* [**10**]. ■

3.7 We will soon define how differential forms can be integrated over manifolds, but we first need to introduce two important notions: those of an oriented manifold and a partition of unity.

An *oriented manifold* is a smooth (N-dimensional) manifold X together with a nowhere-vanishing top-degree form o. A chart φ is said to be positively oriented with respect to o if $o(\partial/\partial\varphi_1, \ldots, \partial/\partial\varphi_{\mathrm{N}}) > 0$; otherwise, it is said to be negatively oriented.

Given an open cover $\{\mathrm{U}_i\}_{i\in\mathrm{I}}$ of a smooth manifold X, a collection of smooth functions $\{\alpha_i : \mathrm{X} \longrightarrow [0,1]\}_{i\in\mathrm{I}}$ is a *smooth partition of unity* subordinate to $\{\mathrm{U}_i\}_i$ if α_i is supported in U_i for every $i \in \mathrm{I}$, and if, for every $x \in \mathrm{X}$, there exists a neighbourhood of x in which only a finite number of functions α_i are non-zero and we have $\sum_{i\in\mathrm{I}} \alpha_i(x) = 1$. Partitions of unity can always be constructed on any open cover of a smooth manifold [**10**, Th. 2.23].

3.8 Definition. Let X be an oriented N-dimensional smooth manifold and let ω be a compactly-supported N-form defined on it. We shall consider a finite collection of charts $\{(\mathrm{U}_i, \varphi_i)\}_i$ covering the support of ω together with a partition of unity $\{\alpha_i\}_i$ subordinate to $\{\mathrm{U}_i\}_i$.

For any chart $(\mathrm{U}_i, \varphi_i)$, we will write $\tilde{\omega}_i$ to denote the only smooth real-valued function on U_i such that $\omega = \tilde{\omega}_i \cdot d\varphi_1 \wedge \cdots \wedge d\varphi_{\mathrm{N}}$. Then, the integral of ω over X is

$$\int_{\mathrm{X}} \omega := \sum_i \int_{\varphi(\mathrm{U}_i)} (o_i \alpha_i \cdot \tilde{\omega}_i) \circ \varphi^{-1},$$

where $o_i := +1$ if the chart (U_i, Φ_i) is positively oriented and $o_i := -1$ otherwise.

3.9 Theorem (Stokes)**.** The integral of an exact top-degree differential form with compact support over an oriented smooth manifold is zero.

Proof. Theorem 16.11 in *Introduction to smooth manifolds* [**10**]. Our result is a particular case of a more general statement concerning 'manifolds with boundary', which we are not considering in this work. ■

4 Connections

4.1 Definition. Let X be a smooth manifold and let $\pi : E \longrightarrow X$ be a smooth vector bundle defined on it. A *connection* in this bundle is a map

$$\begin{aligned} \nabla : \mathfrak{X}(E) \times \Gamma(E) &\longrightarrow \Gamma(E) \\ (\chi, \xi) &\longmapsto \nabla_\chi \xi \end{aligned}$$

that verifies the following conditions:

(C1) The function $\nabla_\chi \xi$ is $C^\infty(X)$-linear in its argument χ.
(C2) The function $\nabla_\chi \xi$ is $\mathbb{R}$-linear in its argument ξ.
(C3) For any $\alpha \in C^\infty(X)$, $\chi \in \mathfrak{X}(E)$ and $\xi \in \Gamma(E)$, the following *product rule* is satisfied:

$$\nabla_\chi \alpha\xi = \alpha\nabla_\chi \xi + \chi(\alpha)\cdot\xi.$$

We read $\nabla_\chi \xi$ as the *covariant derivative* of ξ along χ.

Given a connection ∇ on the tangent bundle and a smooth local frame $\{E_i\}_i$, the *connection coefficients* with respect to that frame are the only real-valued functions Γ_{ij}^k such that

$$\nabla_{E_i} E_j|_x = \sum_k \Gamma_{ij}^k E_k.$$

4.2 Proposition. Let ∇ be a connection on the tangent bundle of a smooth manifold X, let $\{E_i\}_i$ be a smooth frame defined on an open $U \subseteq X$, and let $\chi, \xi \in \mathfrak{X}(U)$. If we write $\chi = \sum_i \chi_i E_i$ and $\xi = \sum_j \xi_j E_j$, then

$$\nabla_\chi \xi = \sum_k \Big(\chi(\xi_k) + \chi_i \xi_j \Gamma_{ij}^k\Big) E_k,$$

where Γ_{ij}^k are the connection coefficients that we defined above.

4.3 Lemma. Let X be a manifold, $\pi : E \longrightarrow X$ a vector bundle, $\chi \in \mathfrak{X}(X)$ and $\xi \in \Gamma(E)$. Let ∇ is any connection on E. Under these conditions, the following statements are true.

(i) If, at any given $x \in X$, $\chi(x) = 0$, then $\nabla_\chi \xi|_x = 0$.
(ii) The value of $\nabla_\chi \xi$ at any point $x \in X$ only depends on $\chi(x)$ and on the value of ξ along any curve γ such that $\gamma(0) = x$ and $\gamma'(0) = \xi(x)$.

Proof. (i) Picking any chart around x, we can always write $\chi = \sum_i \chi_i \partial_i$, hence it follows that

$$\nabla_\chi \xi = \nabla_{\sum_i \chi_i \partial_i} \xi = \sum_i \nabla_{\chi_i \partial_i} \xi = \sum_i \chi_i \nabla_{\partial_i} \xi,$$

which, evaluated at the point x, reduces to

$$\nabla_\chi \xi|_x = \sum_i 0 \nabla_{\partial_i} \xi = 0.$$

(ii) Let $\chi, \eta \in \Gamma(\mathrm{TX})$ such that $\chi(x) = \eta(x)$. Then,

$$(\nabla_\chi \xi - \nabla_\eta \xi)_x = \nabla_{\chi-\eta} \xi|_x = 0,$$

since $(\xi - \eta)(x) = 0$ and according to (i). This will mean, in turn, that $\nabla_\chi \xi = \nabla_\eta \xi$.

In addition, fixing a smooth frame E_i around x, we will be able to write $\xi = \sum_i \xi_i \mathrm{E}_i$. Thus,

$$\nabla_\chi \xi = \nabla_\chi \left(\sum_i \xi_i \mathrm{E}_i \right) = \sum_i \chi(\xi_i)\mathrm{E}_i + \xi_i \nabla_\chi \mathrm{E}_i.$$

When this expression is evaluated at any point x, the last term in the expression only depends on ξ_x. The first one, on the other hand, depends on the derivatives $\chi_x(\xi_i)$ of the components of ξ in the direction χ_x; thus, it only depends on the values taken by χ on any curve crossing x and having χ_x as tangent vector at that point.

∎

4.4 Proposition. Let ∇ be any connection on some smooth bundles on a manifold X. We can define ∇ on the trivial bundle $\mathrm{X} \times \mathbb{R}$ as follows:

(C4) If $\chi \in \mathfrak{X}(\mathrm{X})$ and $\alpha \in C^\infty(\mathrm{X}) = \Gamma(\mathrm{X} \times \mathbb{R})$, we set

$$\nabla_\chi \alpha|_x = \chi_x(\alpha).$$

Moreover, we can uniquely extend ∇ to the dual to any bundle on which it is defined and to any tensor products of bundles where it is defined if we impose the following conditions:

(C5) If $\chi \in \mathfrak{X}(\mathrm{X})$ and if, for some bundle E on which ∇ is defined, $\omega \in \Gamma\mathrm{E}^*$ and $\xi \in \Gamma\mathrm{E}$, then

$$\nabla_\chi(\omega \circ \xi) = (\nabla_\chi \omega) \circ \xi + \omega \circ (\nabla_\chi \xi).$$

(C6) If $\chi \in \mathfrak{X}(\mathrm{X})$ and if ξ and ζ are sections of bundles on which ∇ has already been defined, then

$$\nabla_\chi(\xi \otimes \zeta) = (\nabla_\chi \xi) \otimes \zeta + \xi \otimes (\nabla_\chi \zeta.$$

Proof. Axiom (C4) together with (C5) implies that, for any $\omega \in \Gamma\mathrm{E}^*$ and $\xi \in \Gamma\mathrm{E}$,

$$(\nabla_\chi \omega) \circ \xi = \nabla_\chi(\omega \circ \xi) - \omega \circ (\nabla_\chi \xi) = \chi(\omega \circ \xi) - \omega \circ (\nabla_\chi \xi),$$

which fully characterises the extension of ∇ to sections of E^*. It can be readily checked that this indeed determines a connection in E^*.

Lastly, (C6), together with (C2) and (C3), fully specifies how ∇ can be extended to tensor products of bundles where it has already been defined. It is straightforward to verify that the resulting function does define a connection and that it is 'associative', in the sense that

$$\nabla_\chi((\xi_1 \otimes \xi_2) \otimes \xi_3) = \nabla_\chi(\xi_1 \otimes (\xi_2 \otimes \xi_3))$$

for any suitable sections ξ_1, ξ_2 and ξ_3. ∎

4.5 Example. Given some bundles E and F on a smooth manifold X, if we fix any $x \in X$, the space $\mathrm{Hom}(E_x, F_x)$ is a tensor product $E^* \otimes F$. Thus, we can consider a tensor bundle $\mathrm{Hom}(E, F)$ where, for every $x \in X$, the fibre $\mathrm{Hom}(E, F)_x$ is taken to be $\mathrm{Hom}(E_x, F_x)$.

Applying § 4.4, it then follows that, if a connection ∇ has been defined on E and F, then it can be extended to $\mathrm{Hom}(E, F)$ as

$$(\nabla_\chi A)(\xi) = \nabla_\chi (A\xi) - A(\nabla_\chi \xi)$$

for every $A \in \Gamma\,\mathrm{Hom}(E, F)$, $\chi \in \mathfrak{X}(X)$ and $\xi \in E$.

4.6 Proposition. Let $f : X \longrightarrow Y$ be a smooth function between two manifolds and let $\pi : E \longrightarrow Y$ be a smooth vector bundle. Given any local section $\eta : U \longrightarrow E$, we can pull it back to $f^{-1}(U)$ as

$$\tilde{\eta} : x \in f^{-1}(U) \longmapsto (x, \eta \circ f(x)) \in f^*E.$$

In a minor abuse of notation, we will sometimes drop the tilde in $\tilde{\eta}$.

If ∇ is a connection in E, then there exists a unique connection ∇^* in f^*E that satisfies, for any local section η,

$$\pi^\circ\left(\nabla^*_\chi \tilde{\eta}\right) = \left(\nabla_{\mathrm{D}f(\chi)} \eta\right) \circ f,$$

where π° is defined as in § 2.11.

Proof. Let $x \in X$ be an arbitrary point and let $\{E_k\}_k$ be a local frame defined on a neighbourhood U of $f(x)$. Clearly, $\{\tilde{E}_k\}_k$ will be a local frame with respect to f^*E on $f^{-1}(U)$. Therefore, in $f^{-1}(U)$, we can write any smooth section $\xi \in \Gamma f^*E$ as $\xi = \sum_i \xi_i \tilde{E}_i$ for some $\xi_i \in C^\infty(f^{-1}(U))$. Thus, our induced covariant derivative will satisfy, over $f^{-1}(U)$,

$$\begin{aligned}
\nabla^*_\chi \xi &= \sum_i (\chi \xi_i)\tilde{E}_i + \xi_i \nabla^*_\chi \tilde{E}_i \\
&= \sum_i (\chi \xi_i)\tilde{E}_i + \xi_i \Big(x \mapsto \big(x, \left(\nabla_{\mathrm{D}f(\chi)} E_i\right) \circ f(x)\big)\Big).
\end{aligned}$$

Note that $\mathrm{D}f \circ \chi$ does not define a vector field on $f(\chi)$ unless f is injective. Nonetheless, we can still make sense out of the expression above thanks to § 4.3(ii).

The expression that we have derived proves the uniqueness of ∇^*. Its existence, i.e., the fact that ∇^* is indeed a well-defined connection, is easy to check from this point. ∎

5 Riemannian manifolds

5.1 Definition. A *Riemannian manifold* is a smooth manifold X together with a tensor field $g \in \Gamma(T^2T^*X)$ that defines, at every point $x \in X$, a scalar product on T_xX.

Such a tensor field is said to be a *Riemannian metric*. We may often write $g(-,-)$ as $\langle - \mid - \rangle_g$ or, if there is no risk of ambiguity, as simply $\langle - \mid - \rangle$.

5.2 Proposition. Let X be an oriented Riemannian manifold. The manifold admits a unique *volume form*: a top-degree form ω such that, for any (local) positively-oriented orthonormal frame $\{E_i\}_i$, $\omega(E_i)_i = 1$.

Proof. Proposition 15.29 in *Introduction to smooth manifolds* [**10**]. ■

5.3 We will denote the volume form of an oriented Riemannian manifold X as $*1_X$ or just $*1$ if there's no risk of ambiguity. The symbol $*$ represents Hodge's star operator.

The volume form of an oriented Riemannian manifold X automatically induces a measure μ on it by $\mu(E) := \int_X (\chi_E)(*1_X)$, where χ_E is the characteristic function of E. In particular, we define the volume of a manifold X to be

$$\operatorname{vol} X := \int_X (*1_X).$$

This measure can also be defined on Riemannian manifolds without an orientation by means of a Riemannian 'density function'. [**9**, Prop. 2.44]

When given a smooth real-valued function f defined on an oriented Riemannian manifold X, we will sometimes resort to the usual notation

$$\int_X f(x)\,dx := \int_X f \cdot (*1_X).$$

5.4 Example. (i) The Euclidean space $\mathbb{R}^N$ together with the metric induced by its ordinary scalar product on each tangent space [§ 1.5] is a Riemannian manifold. Its volume form can be trivially defined.

(ii) If an N-dimensional smooth manifold X can be smoothly embedded into Euclidean space $\mathbb{R}^K$ (with $N \leqslant K$) through a map $\iota : X \longrightarrow \mathbb{R}^K$, then we may consider the metric g induced by the Euclidean metric on X as the one defined, at every $x \in X$ and for every $v, w \in T_x X$, by

$$g_x(v,w) := \langle D\iota_x(v) \mid D\iota_x(w) \rangle,$$

where $\langle - \mid - \rangle$ denotes the usual scalar product in $T_{\iota(x)}\mathbb{R}^N$, as in § (i). This is the Riemannian metric that we will use on spheres.

In this setting, given any $v \in TX$, we will say that $D\iota(v)$ is the *Euclidean representation* of v.

5.5 Definition. Let ∇ be a connection on the tangent bundle of a Riemannian manifold X. We say that ∇ is *compatible with the metric* if, for any $\chi, \eta_1, \eta_2 \in \mathfrak{X}(X)$,

$$\chi\langle \eta_1 \mid \eta_2 \rangle = \left\langle \nabla_\chi \eta_1 \,\middle|\, \eta_2 \right\rangle + \left\langle \eta_1 \,\middle|\, \nabla_\chi \eta_2 \right\rangle.$$

In addition, we say that ∇ is *torsion-free* or *symmetric* if, for any vector fields χ, η,

$$[\chi, \eta] = \nabla_\chi \eta - \nabla_\eta \chi.$$

5.6 Theorem. On any Riemannian manifold, there exists a unique connection that is both compatible with the metric and torsion-free. We call it the *Levi-Civita* connection.

Proof. Theorem 5.10 in *Introduction to Riemannian manifolds* [**9**]. ■

5.7 Definition. Let X be a Riemannian manifold and let ∇ denote its Levi-Civita connection. The *divergence* of a vector field $\eta \in \mathfrak{X}(\mathrm{X})$ is the trace of $\nabla_{(-)}\eta$. Thus, if $\{\mathrm{E}_i\}_i$ is an orthonormal frame around a point, we may define

$$\operatorname{div} \eta := \operatorname{tr} \nabla \eta = \sum_{i=1}^{n} \left\langle \nabla_{\mathrm{E}_i} \eta \;\middle|\; \mathrm{E}_i \right\rangle.$$

5.8 Theorem (Divergence)**.** Given a vector field η on a compact oriented Riemannian manifold X, the integral of $\operatorname{div}(\eta) \cdot (*1_{\mathrm{X}})$ over X is zero.

Proof. Theorem 2.1 in Chapter 2 of *Partial differential equations I: Basic theory* [**16**]. Our result is a particular case of a more general statement concerning 'manifolds with boundary', and it can be deduced from Stokes' Theorem [§ 3.9] and the fact that $\operatorname{div}(\eta) \cdot (*1_{\mathrm{X}})$ is exact. ■

5.9 Definition. Given a Riemannian manifold (X, g) and a curve $\gamma : [a, b] \longrightarrow \mathrm{X}$, the *length* of γ is defined to be

$$\mathrm{L}(\gamma) := \int_a^b \|\gamma'(t)\|_g \, dt,$$

where $\| - \|$ denotes the norm induced by the Riemannian metric.

5.10 In a Riemannian manifold X, the distance between two points x and y is defined as the infimum of the lengths of all the curves that join x and y, this is,

$$d(x, y) := \inf\{\mathrm{L}(\gamma) \mid \gamma \text{ joins } x \text{ and } y\}.$$

This definition turns every Riemannian manifold into a metric space and, what is more, the topology induced by this distance coincides with the topology of X.

5.11 Proposition. Let X be a Riemannian manifold. There exists an open set $\mathcal{E} \subseteq \mathrm{TX}$ on which an *exponential* function $\exp : \mathcal{E} \longrightarrow \mathrm{X}$ can be defined in such a way that, for any $x \in \mathrm{X}$ and any $v \in \mathrm{T}_x\mathrm{X}$, the curve $\gamma : t \mapsto \exp(tv)$ goes through x at $t = 0$ with tangent vector v.

Moreover, the set $\mathcal{E}$ is open and, for every $x \in \mathrm{X}$, the tangent vector 0_x is contained in E.

Proof. Proposition 5.19 in *Introduction to Riemannian manifolds* [**9**]. ■

5.12 Proposition (Normal coordinates)**.** Let X be a Riemannian manifold. For any point $x \in \mathrm{X}$, there exists a *normal chart* (U, φ) with $x \in \mathrm{U}$ such that $\varphi(x) = 0$, such that the coordinate tangent vectors $\partial/\partial\varphi_i$ are orthonormal at x, and such that, for any $u \in \mathrm{U}$, $d(x, u) = \|\varphi(u)\|$. We call the coordinates induced by such a chart *normal coordinates*.

Proof. Proposition 5.24 in *Introduction to Riemannian manifolds* [**9**]. ■

Chapter II

Other fundamental notions

1 Algebraic topology

1.1 Given two topological spaces X and Y, we will write $X \approx Y$ to denote that they are homeomorphic. If X is a topological space and $x_0 \in X$ is a point in it, the structure (X, x_0) is called a *pointed space* (with *base-point* x_0). A function f between two pointed spaces (X, x_0) and (Y, y_0) will be said to be *pointed* if $f(x_0) = y_0$. What is more, two pointed spaces are *homeomorphic* as pointed spaces if there exists a pointed homeomorphism between them.

If $A_1, \dots, A_n$ are some subsets of a topological space X, we may consider the quotient space $X/A_1, \dots, A_n$ induced by the equivalence relation satisfied by a pair $(x, y) \in X \times X$ if and only if $x = y$ or $x, y \in A_i$ for an index $i = 1, \dots, n$.

1.2 **Definition.** Let (X, x_0) and (Y, y_0) be a pair of pointed spaces. Their *wedge sum* is the topological subspace $X \vee Y := (X \times \{y_0\}) \cup (\{x_0\} \times Y) \subseteq X \times Y$. Their *smash product* is the pointed space $X \wedge Y := (X \times Y)/(X \vee Y)$ with base-point $[(x_0, y_0)]$.

1.3 **Definition.** The *suspension* of a topological space X is the space

$$SX := X \times [0,1] \Big/ (X \times \{0\}), (X \times \{1\}) \cdot$$

There exists an analogous notion for pointed spaces that preserves their base-points: the *reduced suspension* of a pointed space (X, x_0) is the pointed space

$$\Sigma X := X \times [0,1] \Big/ (X \times \{0\}) \cup (X \times \{1\}) \cup (\{x_0\} \times [0,1])$$

with base-point $[(x_0, 0)]$.

1.4 If $f : X \longrightarrow Y$ is a continuous function between topological spaces, we define its *suspension* to be the function

$$\begin{aligned} Sf : SX &\longrightarrow SY \\ [(x,t)] &\longmapsto [(f(x), t)]. \end{aligned}$$

It can be easily checked that Sf is well-defined and continuous.

1.5 Proposition. The reduced suspension of a pointed topological space (X, x_0) is homeomorphic to its smash product with a pointed circle (S^1, s_0) [§ I-1.2], i.e.,

$$\Sigma(\mathrm{X}, x_0) \approx (\mathrm{X}, x_0) \wedge (\mathrm{S}^1, s_0).$$

Proof. Taking into account that the circle S^1 can be identified with the quotient space $[0,1]/\{0,1\}$ through a homeomorphism taking s_0 to $[0]$, it follows that

$$\begin{aligned}
(\mathrm{X}, x_0) \wedge (\mathrm{S}^1, s_0) &= \Big(\mathrm{X} \times \mathrm{S}^1 \Big/ (\mathrm{X} \times \{s_0\}) \cup (\{x_0\} \times \mathrm{S}^1), [(x_0, s_0)]\Big) \\
&\approx \Big(\mathrm{X} \times ([0,1]/\{0,1\}) \Big/ (\mathrm{X} \times \{0\}) \cup (\{x_0\} \times [0,1]), [(x_0, [0])]\Big) \\
&\approx \Big(\mathrm{X} \times [0,1] \Big/ (\mathrm{X} \times \{0\}) \cup (\{x_0\} \times [0,1]) \cup (\mathrm{X} \times \{0,1\}), [(x_0, 0)]\Big) \\
&= \Big(\mathrm{X} \times [0,1] \Big/ (\mathrm{X} \times \{0\}) \cup (\mathrm{X} \times \{1\}) \cup (\{x_0\} \times [0,1]), [(x_0, 0)]\Big),
\end{aligned}$$

which is, precisely, the pointed space $\Sigma(\mathrm{X}, x_0)$, just as we wanted to prove. ■

1.6 Proposition. Given any natural number N, the sphere S^N is homeomorphic to the suspension $\mathrm{SS}^{\mathrm{N}-1}$ and, as a pointed space, to the reduced suspension $\Sigma\mathrm{S}^{\mathrm{N}-1}$ obtained by fixing any base-point.

1.7 Definition. Two continuous maps $f, g : \mathrm{X} \longrightarrow \mathrm{Y}$ are said to be *homotopic* if there exists a *homotopy* between them: a continuous $h : \mathrm{X} \times [0,1] \longrightarrow \mathrm{Y}$ such that $h(-,0) = f$ and $h(-,1) = g$. We say that the homotopy is *relative* to a subset $\mathrm{A} \subseteq \mathrm{X}$ if, in addition, for every $a \in \mathrm{A}$, we have $h(a,-) = f(a) = g(a)$. The relations 'being homotopic to' and 'being homotopic to (...) relative to a subset A' are equivalence relations on the set of continuous functions from X to Y. We will use the symbols $\simeq$ and $\simeq_\mathrm{A}$ to denote these relations, although, in an abuse of notation we may drop the subindex A if doing so leads to no ambiguity. Moreover, if X is a pointed space with base-point x_0, we may write $\simeq_*$ in lieu of $\simeq_{\{x_0\}}$. We will refer to the equivalence classes induced by all these relations as *homotopy classes*.

Any map that is homotopic to a constant map is said to be *null-homotopic*.

If X and Y are topological spaces, we will denote the collection of all their homotopy classes as $[\mathrm{X}; \mathrm{Y}]$. Moreover, if (X, x_0) and $(\mathrm{Y}, \mathrm{Y}_0)$ are pointed spaces, we will write $[(\mathrm{X}, x_0); (\mathrm{Y}, y_0)]_*$ to denote all the homotopy classes of maps with homotopies h such that $h(x_0, t) = y_0$ for any $t \in [0,1]$.

1.8 Definition. Let X and Y be topological spaces. Given any pair of homotopies $\mathrm{H}_1, \mathrm{H}_2 : \mathrm{X} \times [0,1] \longrightarrow \mathrm{Y}$, if $\mathrm{H}_1(x,1) = \mathrm{H}_2(x,0)$ for every $x \in \mathrm{X}$, then the *concatenation* of H_1 with H_2 is the homotopy $\mathrm{H}_1 * \mathrm{H}_2$ defined by

$$(\mathrm{H}_1 * \mathrm{H}_2)(x,t) := \begin{cases} \mathrm{H}_1(x, 2t), & t \leqslant 1/2 \\ \mathrm{H}_2(x, 2(t-1/2)), & t > 1/2. \end{cases}$$

1.9 Let X and Y be pointed spaces. Given any pair of pointed continuous functions, $f, g : \Sigma X \longrightarrow Y$, we may compose them with the projection function $\pi : X \times [0,1] \longrightarrow \Sigma X$ in order to obtain two functions: $f \circ \pi$ and $g \circ \pi$, from $X \times [0,1]$ to Y. These functions are homotopies and can be concatenated, giving rise to the homotopy $(f \circ \pi) * (g \circ \pi)$. Using the universal property that defines quotient spaces, we know that there must exist a function $f * g$ for which

$$\begin{tikzcd} X \times [0,1] \arrow[r, "(f\circ\pi)*(g\circ\pi)"] \arrow[d, "\pi"] & Y \\ \Sigma X \arrow[ur, dashed, "f*g"'] & \end{tikzcd}$$

will be a commutative diagram.

Given some pointed continuous functions $f, f', g, g' : \Sigma X \longrightarrow Y$, if $f \simeq_* f'$ and $g \simeq_* g'$, then we must necessarily have, $f * g \simeq_* f' * g'$. Thus, we can take our newly-defined $*$ as a law of composition in $[\Sigma X, Y]_*$ that takes any homotopy classes $[f]$ and $[g]$ to $[f] * [g] := [f * g]$. This law of composition can be proved to satisfy the group axioms.

1.10 Definition. For some natural N, let us consider the pointed N-dimensional sphere S^N constructed as the N-th reduced suspension of $(S^0, +1)$ [§ 1.6]. The N-th *homotopy group* of a pointed space (X, x_0) is the group $\pi_N(X, x_0)$ defined on the set $[S^N, X]_*$ with the law of composition $*$ introduced in § 1.9.

1.11 (i) If a space X is path connected, then, given any natural number N, all the groups $\pi_N(X, x_0)$ are isomorphic for any base-point $x_0 \in X$. That is the reason why we will often drop the base-point when working with path-connected spaces.

(ii) The first homotopy group of a space is said to be its *fundamental group*. The fundamental group of a space may not be abelian, but the other homotopy groups always are.

1.12 Proposition. Let N and K be natural numbers. The following statements about the homotopy groups of spheres are true; their proofs can be found in the references [**5**].

(i) All the homotopy groups of spheres are abelian.
(ii) For any N, the group $\pi_N(S^N)$ is isomorphic to $\mathbb{Z}$.
(iii) If $N < K$, the group $\pi_N(S^K)$ is trivial.

The classification of some homotopy groups of spheres can be found in table 1.1. Of particular importance is the fact that $\pi_3(S^2)$ is isomorphic to $\mathbb{Z}$.

1.13 Theorem (Degree)**.** Let N be a natural number. We can define a function $\deg : \pi_N(S^N) \longrightarrow \mathbb{Z}$ by considering, for each homotopy class in $\pi_N(S^N)$, a smooth

$\pi_1(S^1) = \mathbb{Z}$	$\pi_1(S^2) = 0$	$\pi_1(S^3) = 0$	$\pi_1(S^4) = 0$	$\pi_1(S^5) = 0$
$\pi_2(S^1) = 0$	$\pi_2(S^2) = \mathbb{Z}$	$\pi_2(S^3) = 0$	$\pi_2(S^4) = 0$	$\pi_2(S^5) = 0$
$\pi_3(S^1) = 0$	$\pi_3(S^2) = \mathbb{Z}$	$\pi_3(S^3) = \mathbb{Z}$	$\pi_3(S^4) = 0$	$\pi_3(S^5) = 0$
$\pi_4(S^1) = 0$	$\pi_4(S^2) = \mathbb{Z}_2$	$\pi_4(S^3) = \mathbb{Z}_2$	$\pi_4(S^4) = \mathbb{Z}$	$\pi_4(S^5) = 0$
$\pi_5(S^1) = 0$	$\pi_5(S^2) = \mathbb{Z}_2$	$\pi_5(S^3) = \mathbb{Z}_2$	$\pi_5(S^4) = \mathbb{Z}_2$	$\pi_5(S^5) = \mathbb{Z}$

Table 1.1: First five homotopy groups for the spheres in dimensions one to five.

representative f and setting

$$\deg(f) := \frac{\int_{S^N} f^*(*1_{S^N})}{\int_{S^N}(*1_{S^N})}.$$

We say that $\deg(f)$ is the *degree* of f. The function deg is well-defined and is a group isomorphism.

Proof. Results III-2.4, III-2.5 and V-2.1 of *Mapping degree theory* [**12**]. ■

1.14 Theorem (Hopf invariant)**.** Let K be a natural number. We can construct a function $h : \pi_{2K+1}(S^K) \longrightarrow \mathbb{Z}$ by considering, for each homotopy class in $\pi_{2K+1}(S^K)$, a smooth representative f and setting

$$h(f) := \frac{1}{(\operatorname{vol} S^K)^2} \int_{S^N} f^*(*1_{S^K}) \wedge d^{-1} f^*(*1_{S^K}),$$

where we write $d^{-1}\omega$ to denote a form such that $d(d^{-1}\omega) = \omega$. We say that $h(f)$ is the *Hopf invariant* of f. The function h is well-defined and is a group homomorphism.

The expression that we have used to define $h(f)$ is known as *Whitehead's integral formula*.

Proof. Proposition 17.22 of *Differential forms in algebraic topology* [**3**]. ■

2 Calculus of variations

2.1 When we are given a function f between two finite-dimensional normed spaces, we say that a point x_0 in its domain is critical if the differential of f vanishes at x_0, i.e., $\mathrm{D}f_{x_0} = 0$. The differential of a function between two normed spaces with no additional smooth structure is defined in terms of the norms of the spaces; nevertheless, all norms in finite-dimensional vector spaces are equivalent, so this notion of criticality is not only well-defined — it is also independent of choice of norms when we work with spaces of finite dimension.

With the following chapters in mind, we need to extend this notion of criticality to a different domain, namely to spaces of smooth functions between manifolds. Thus, we need to ask ourselves the following question: if X and Y are smooth manifolds and we are given some function $F : \mathcal{C}^\infty(X, Y) \longrightarrow \mathbb{R}$, how can we define what it means for a 'point' $f \in \mathcal{C}^\infty(X, Y)$ to be critical with respect to F? A reasonable possibility could be to endow $\mathcal{C}^\infty(X, Y)$ with a smooth structure and define a function $f \in \mathcal{C}^\infty(X, Y)$ to be critical when $DF_f = 0$. Nevertheless, as valid as this approach could be, constructing a smooth structure for $\mathcal{C}^\infty(X, Y)$ and working with it would not necessarily be an easy task. Thus, in pursuit of simplicity, we will instead consider a *variational approach* to handle optimisation in function spaces.

2.2 Definition. Let X and Y be smooth manifolds and let $C \subseteq \mathcal{C}^\infty(X, Y)$. Given any smooth function $f : X \longrightarrow Y$, a *variation* of f in C is a smooth function

$$\begin{aligned} f_{(-)} : \,]-\varepsilon, \varepsilon[\, \times X &\longrightarrow Y \\ (t, x) &\longmapsto f_t(x) \end{aligned}$$

defined for some $\varepsilon > 0$ and extending f: satisfying $f_0 = f$. We also require that, for every $t \in \,]-\varepsilon, \varepsilon[$, $f_t \in C$.

2.3 Definition. Given a compact smooth manifold X and a Riemannian manifold Y, let us consider a functional $F : \mathcal{C}^\infty(X, Y) \longrightarrow \mathbb{R}$. We will require that, for any variation $f_{(-)}$ of any $f \in \mathcal{C}^\infty(X, Y)$, the function $t \mapsto F(f_t)$ be differentiable at $t = 0$.

A function $f \in \mathcal{C}^\infty(X, Y)$ is *critical* with respect to F in a subset $C \subseteq \mathcal{C}^\infty(X, Y)$ if, for every variation $f_{(-)}$ of f in C,

$$\left.\frac{d}{dt} F(f_t)\right|_{t=0} = 0.$$

Unless otherwise specified, the set C will be taken to be the collection of all smooth functions $\mathcal{C}^\infty(X, Y)$.

2.4 If $f : X \longrightarrow \mathbb{R}$ is a continuous real-valued function defined from some Hausdorff space X, we say that a point x_0 is a local minimiser or maximiser of f if there exists an open neighbourhood U of x_0 in which, for every $x \in U$, $f(x_0) \leqslant f(x)$ or $f(x_0) \geqslant f(x)$ respectively. A point that is a local minimiser or a local maximiser is said to be a *local extremum point*. In particular, the points at which f reaches its maximum or minimum values are, obviously, local maximisers or minimisers of f, respectively.

If the notion of criticality that we have introduced is adequate, being a critical point should be a necessary condition for being a local extremum with respect to some topology. We will now prove that this is indeed the case for the uniform convergence topology, but, before we can do that, we will have to consider a somewhat elementary lemma.

2.5 Lemma. Let X and C be topological spaces. If C is compact and if a function $\mathrm{F} : \mathrm{X} \times \mathrm{C} \longrightarrow \mathbb{R}$ is continuous, then the functions

$$f_{\max} : \mathrm{X} \longrightarrow \mathbb{R} \qquad\qquad f_{\min} : \mathrm{X} \longrightarrow \mathbb{R}$$
$$x \longmapsto \max_{c \in \mathrm{C}} \mathrm{F}(x, c), \qquad\qquad x \longmapsto \min_{c \in \mathrm{C}} \mathrm{F}(x, c)$$

are also continuous.

Proof. This proof will only rely on elementary topological notions. Any sequence-oriented readers are referred to § A.5 for an alternative proof that they might find more entertaining.

We will prove the result for $f := f_{\max}$. This will suffice as

$$f_{\min}(x) = -\max_{c \in \mathrm{C}} -\mathrm{F}(x, c).$$

Given an arbitrary real number r, let $\mathrm{I}_r :=]-\infty, r[$. We aim to prove that $f^{-1}(\mathrm{I}_r)$ is open. The empty set is trivially open, so suppose $f^{-1}(\mathrm{I}_r) \neq \varnothing$ and consider an arbitrary element $x \in f^{-1}(\mathrm{I}_r)$. By the definition of f, we must have $\mathrm{F}(x, c) \in \mathrm{I}_r$ for every $c \in \mathrm{C}$. Thus, by the continuity of F, for any $c \in \mathrm{C}$, there has to exist an open neighbourhood $\mathrm{W}_c \subseteq \mathrm{X} \times \mathrm{C}$ of (x, c) such that $\mathrm{F}(\mathrm{W}_c) \subseteq \mathrm{I}_r$. Without loss of generality, assume that, for each $c \in \mathrm{C}$, $\mathrm{W}_c = \mathrm{U}_c \times \mathrm{V}_c$ for some open $\mathrm{U}_c \subseteq \mathrm{X}$ and $\mathrm{V}_c \subseteq \mathrm{C}$. Clearly, the family $\{\mathrm{W}_c\}_{c \in \mathrm{C}}$ covers $\{x\} \times \mathrm{C}$; hence, by virtue of the compactness of C, we can be sure of the existence a finite subfamily $\{\mathrm{W}_i\}_{i=1}^n$ of $\{\mathrm{W}_c\}_{c \in \mathrm{C}}$ covering $\{x\} \times \mathrm{C}$.

The set $\mathrm{W} := \mathrm{W}_1 \cup \cdots \cup \mathrm{W}_n$ is an open set in $\mathrm{X} \times \mathrm{C}$ such that $\{x\} \times \mathrm{C} \subseteq \mathrm{W}$, so $\mathrm{U} := \mathrm{U}_1 \cap \cdots \cap \mathrm{U}_n$ must be an open neighbourhood of x such that $\mathrm{U} \times \mathrm{C} \subseteq \mathrm{W}$. Moreover, $\mathrm{F}(\mathrm{W}) \subseteq \mathrm{I}_r$, which, in turn, means that $\mathrm{U} \subseteq f^{-1}(\mathrm{I}_r)$.

Lastly, regarding the sets $]r, \infty[$ for $r \in \mathbb{R}$, we know that

$$f^{-1}(]r, \infty[) = \pi_{\mathrm{X}}\Big(\mathrm{F}^{-1}(]r, \infty[)\Big),$$

where π_{X} is the projection function $\mathrm{X} \times \mathrm{C} \longrightarrow \mathrm{X}$. This shows that $f^{-1}(]r, \infty[)$ is open because of the continuity of F and the openness of π_{X}. ■

2.6 Proposition. If a function $f \in \mathcal{C}^\infty(\mathrm{X}, \mathrm{Y})$ between a compact smooth manifold X and a Riemannian manifold Y is a local extremum of a functional $\mathrm{F} : \mathcal{C}^\infty(\mathrm{X}, \mathrm{Y}) \longrightarrow \mathbb{R}$ with respect to the uniform convergence topology on $\mathcal{C}^\infty(\mathrm{X}, \mathrm{Y})$, then it is also a critical function of F.

Proof. Since Y is Riemannian, we can consider its natural distance function d [§ I-5.10]. We know that, for any variation $f_{(-)}$ of f, $(t, x) \mapsto d(f_t(x), f(x))$ is a continuous function, so — since X is assumed to be compact — the function δ taking

$$\delta(t) := d_\infty(f_t, f_0) = \max_{x \in \mathrm{X}} d(f_t(x), f(x))$$

must be continuous as well by virtue of Lemma 2.5.

Let us assume that there exists a variation $f_{(-)}$ for which $(d/dt)\mathrm{F}(f_t)|_{t=0} \neq 0$. Given any radius $r > 0$, we may consider an open ball B of radius r around f in which f be a global maximum or minimum. We can also take an open neighbourhood $\mathrm{U} \subseteq \mathbb{R}$ of 0 such that $\delta(\mathrm{U}) \subseteq [0, r[$, which must exist because of the continuity of δ. According to our assumption of a non-zero derivative of $\mathrm{F}(f_t)$ at $t = 0$, there must exist some elements $t_1, t_2 \in \mathrm{U}$ for which $\mathrm{F}(f_{t_1}) < \mathrm{F}(f_t) < \mathrm{F}(f_{t_2})$. Since $\delta(\mathrm{U}) \subseteq [0, r[$, this means that $f_1, f_2 \in \mathrm{B}$, hence f can be neither a local minimiser nor a local maximiser as r can be arbitrarily small. ∎

2.7 Theorem (Euler-Lagrange)**.** For a fixed natural number N, an open set $\mathrm{U} \subseteq \mathbb{R}^{\mathrm{N}}$ and a pair of points $x_0, x_1 \in \mathrm{U}$, let us consider the set C of smooth curves in $C([0,1], \mathrm{U})$ joining x_0 and x_1. Given any differentiable *Lagrangian* $\mathrm{L} : \mathbb{R}^{\mathrm{N}} \times \mathbb{R}^{\mathrm{N}} \longrightarrow \mathbb{R}$ for some natural N, we may define the *action* functional induced by L as

$$\begin{aligned} \mathcal{A}_{\mathrm{L}} : C &\longrightarrow \mathbb{R} \\ \gamma &\longmapsto \int_0^1 \mathrm{L}(\gamma(t), \gamma'(t))\, dt \end{aligned}$$

We will refer to the first N arguments of L with the variables $q_1, \ldots, q_{\mathrm{N}}$ and to the last N arguments with $\dot{q}_1, \ldots, \dot{q}_{\mathrm{N}}$.

A curve γ is a critical point of $\mathcal{A}_{\mathrm{L}}$ if and only if at every point $t \in]0,1[$

$$\left.\frac{\partial \mathrm{L}}{\partial q_i}\right|_{(\gamma(t),\gamma'(t))} - \frac{d}{dt}\left.\left(\frac{\partial \mathrm{L}}{\partial \dot{q}_i}\right)\right|_{(\gamma(t),\gamma'(t)))} = 0$$

for $i = 1, \ldots, \mathrm{N}$.

Proof. Let us consider an arbitrary variation γ_s in C of some curve γ whose criticality we seek to characterise. The function $s \mapsto \mathcal{A}_{\mathrm{L}}(\gamma_s)$ will clearly be differentiable at $s = 0$ and, what is more,

$$\left.\frac{d}{ds}\mathcal{A}_{\mathrm{L}}(\gamma_s)\right|_{s=0} = \int_0^1 \left.\frac{d}{ds}\mathrm{L}(\gamma_s(t), \gamma_s'(t))\right|_{s=0} dt.$$

In order to expand the integrand in the expression above, let us consider an arbitrary value $t \in]0,1[$. We have that

$$\begin{aligned} \frac{d}{ds}\mathrm{L}(\gamma_s(t)) &= \sum_{i=1}^{\mathrm{N}} \frac{\partial \mathrm{L}}{\partial q_i} \cdot \frac{\partial(\gamma_s(t))_i}{\partial s} + \frac{\partial \mathrm{L}}{\partial \dot{q}_i} \cdot \frac{\partial^2(\gamma_s(t))_i}{\partial s \partial t} \\ &= \sum_{i=1}^{\mathrm{N}} \frac{\partial \mathrm{L}}{\partial q_i} \cdot \frac{\partial(\gamma_s(t))_i}{\partial s} + \frac{\partial \mathrm{L}}{\partial \dot{q}_i} \cdot \frac{\partial}{\partial t}\frac{\partial(\gamma_s(t))_i}{\partial s} \\ &= \sum_{i=1}^{\mathrm{N}} \frac{\partial \mathrm{L}}{\partial q_i} \cdot \frac{\partial(\gamma_s(t))_i}{\partial s} + \frac{\partial}{\partial t}\left(\frac{\partial \mathrm{L}}{\partial \dot{q}_i} \cdot \frac{\partial(\gamma_s(t))_i}{\partial s}\right) - \frac{\partial}{\partial t}\frac{\partial \mathrm{L}}{\partial \dot{q}_i}\frac{\partial(\gamma_s(t))_i}{\partial s}, \end{aligned}$$

where all the derivatives of the Lagrangian are evaluated at $(\gamma_s(t), \gamma_s'(t))$. If we plug this back into our original expression, we obtain that $(d/ds)\mathcal{A}_{\mathrm{L}}(\gamma_s)|_{s=0}$ must equal

$$\sum_{i=1}^{\mathrm{N}} \int_0^1 \frac{\partial \mathrm{L}}{\partial q_i} \cdot \frac{\partial(\gamma_s(t))_i}{\partial s} - \frac{\partial}{\partial t}\frac{\partial \mathrm{L}}{\partial \dot{q}_i}\frac{\partial(\gamma_s(t))_i}{\partial s}\bigg|_{s=0} dt + \cancel{\left[\frac{\partial \mathrm{L}}{\partial \dot{q}_i} \cdot \frac{\partial(\gamma_s(t))_i}{\partial s}\bigg|_{s=0}\right]_{t=0}^{t=1}},$$

where the last term vanishes since, by the construction of C, $\gamma_s(0) = x_0$ and $\gamma_s(1) = x_1$ for any value of s.

Simplifying and rearranging, we get that

$$\frac{d}{ds}\mathcal{A}_{\mathrm{L}}(\gamma_s)\bigg|_{s=0} = \sum_{i=1}^{\mathrm{N}} \int_0^1 \left(\frac{\partial \mathrm{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathrm{L}}{\partial \dot{q}_i}\right)_{(\gamma_0(t),\gamma_0'(t)} \cdot \frac{\partial(\gamma_s(t))_i}{\partial s}\bigg|_{s=0} dt,$$

which — taking into account that U is open by hypothesis — is zero for any variation $\gamma_{(-)}$ if and only if

$$\frac{\partial \mathrm{L}}{\partial q_i}\bigg|_{(\gamma(t),\gamma'(t))} - \frac{d}{dt}\frac{\partial \mathrm{L}}{\partial \dot{q}_i}\bigg|_{(\gamma(t),\gamma'(t))} = 0$$

for $i = 1, \ldots, \mathrm{N}$ at almost every point t. ∎

2.8 The Euler-Lagrange equation can be trivially generalised to any Riemannian manifold that be diffeomorphic to an open set of $\mathbb{R}^{\mathrm{N}}$. In this case, the domain of the Lagrangian would be the tangent bundle of the manifold.

2.9 Example. To better illustrate how the Euler-Lagrange equation can be used in practice, we will now discuss a simple example. Let us consider, for some arbitrary $k \in \mathbb{Z}$, the collection of all the smooth functions in the homotopy class of $\pi_1(\mathrm{S}^1)$ of maps with degree k. Under the identification of the pointed sphere (S^1, s_0) with $([0,1]/\{0,1\}, [0])$ and with $(\mathbb{R}/2\pi\mathbb{Z}, [0])$, we can think of this collection of functions as the set C_k of all smooth functions $\gamma : [0,1] \longrightarrow \mathbb{R}$ with fixed endpoints $\gamma(0) = 0$ and $\gamma(1) = 2\pi k$.

We will now get ahead of ourselves for a moment and define, for a real number $p \geqslant 2$, the action E_p associated to the Lagrangian $\mathrm{L}_k(q, \dot{q}) := |\dot{q}|^p$. We seek to find all the critical curves in C_k with respect to E_p and, to this end, we will rely on the Euler-Lagrange equation.

Let $\gamma \in C_k$ and $t \in [0,1]$. We should first notice that

$$\frac{\partial \mathrm{L}}{\partial \dot{q}}\bigg|_{(\gamma(t),\gamma'(t))} = p\gamma'(t)|\gamma'(t)|^{p-2}.$$

Hence, at these points, the Euler-Lagrange equation reduces to $\gamma''(t) = 0$, meaning that, according to the definition of C_k, the only possible critical curve for E_p is given by the function $\gamma(t) := (2\pi k)t$.

In Chapter III, we will see how this E_p functional is a particular case of what we shall call 'generalised Dirichlet energies'. Moreover, in Section III-2, we will derive a general Euler-Lagrange equation for these Dirichlet energies, which can be defined not just for curves, bur for functions from any compact Riemannian manifold.

3 Functional analysis

3.1 Definition. Given any (real) topological vector space V, its *topological dual* is the vector space V^* of all the continuous real-valued functions from V. The *weak topology* of a topological vector space V is the coarsest topology in which all the elements in the topological dual of V are continuous. We may sometimes refer to the 'original' topology of a topological vector space V as its *strong* topology, in order to distinguish it from its weak topology.

3.2 Theorem (Eberlein–Šmulian)**.** A subset of a Banach space is precompact in the weak topology if and only if it is sequentially precompact in the weak topology.

Proof. An elementary proof of the Eberlein-Šmulian Theorem [**17**]. ■

3.3 Definition. A normed space V is said to be *reflexive* if the function taking any $v \in V$ to the element $Jv \in V^{**}$ defined by $Jv : w \in V^* \longmapsto w(v)$ is an isometric linear isomorphism.

3.4 Theorem (Banach–Alaoglu)**.** Let V be a reflexive normed space. Any closed ball in V is weakly compact.

Proof. Theorems 3.16 and 3.17 of *Functional analysis, Sobolev spaces and partial differential equations* [**6**]. ■

3.5 Proposition. Every bounded set in a reflexive normed space is sequentially precompact in the weak topology.

Proof. Let B be any closed ball including such a bounded set. We know that B is compact by § 3.4, and § 3.2 then yields the sequential precompactness of B, which is automatically inherited by any subset of it. ■

3.6 Definition. Let X be a compact Riemannian manifold and $K \in \mathbb{N}$. We define the *Sobolev space* $H^p(X, \mathbb{R}^K)$ as the completion of the space of smooth functions from X to $\mathbb{R}^K$ with respect to the norm

$$\|f\|_{1p} := \left(\int_X \|f(x)\|^p + \|Df_x\|^p \, dx\right)^{1/p},$$

where $\|f(x)\|$ is the Euclidean norm of $f(x)$ in $\mathbb{R}^K$ and $\|Df_x\|$ is the operator norm of Df. By construction, these Sobolev spaces are Banach spaces.

3.7 Theorem (Rellich–Kondrachov)**.** Let X be a compact N-dimensional Riemannian manifold and let K be a natural number. Given any real number $p \geqslant 1$, the space $H^p(X, \mathbb{R}^K)$ is compactly embedded into $L^p(X, \mathbb{R}^K)$ [§ A.1].

Proof. Result 2.34 in *Nonlinear analysis on manifolds. Monge-Ampère equations* [**1**].

■

3.8 Definition. Let p be any real number. If X is a compact Riemannian manifold and Y is a compact subset of some $\mathbb{R}^K$, we define $H^p(X, Y)$ to be the set of functions $f \in H^p(X, \mathbb{R}^K)$ such that, for almost every $x \in X$, $f(x) \in Y$.

3.9 Lemma. In a normed space, every weakly convergent sequence is bounded.

Proof. Proposition 3.5(ii) in *Functional analysis, Sobolev spaces and partial differential equations* [**6**]. ■

3.10 Proposition. Let X be any compact N-dimensional Riemannian manifold and let Y be a compact subset of some $\mathbb{R}^K$. Given any real $p \geqslant 1$, the space $H^p(X, Y)$ is a closed subset of $H^p(X, \mathbb{R}^K)$ in the weak topology.

Proof. Consider an arbitrary sequence $\{f_n\}_{n\in\mathbb{N}} \subseteq H^p(X, Y)$ converging weakly to some function f in $H^p(X, \mathbb{R}^K)$. By virtue of Lemma 3.9, the sequence is bounded in $H^p(X, \mathbb{R}^K)$. Taking into account Theorem 3.7 together with the fact that $L^p(X, \mathbb{R}^K)$ is complete, this implies that $\{f_n\}_n$ must have a convergent subsequence $\{f_{n_i}\}_{i\in\mathbb{N}}$ in $L^p(X, \mathbb{R}^K)$.

Since convergence in the strong topology implies convergence in the weak topology, the limit of $\{f_{n_i}\}_i$ in the strong topology has to be f. Therefore, $\{f_{n_i}\}_i$ converges point-wise to $f(x)$ for almost every $x \in X$. Since Y is a closed subset of $\mathbb{R}^K$, this implies that $f(x) \in Y$ for almost every $x \in X$, thus proving that $f \in H^p(X, Y)$. ■

3.11 Proposition. Given any compact Riemannian manifold X, any real $1 < p < \infty$ and any natural K, the Sobolev space $H^p(X, \mathbb{R}^K)$ is reflexive.

Proof. Proposition 9.1 in *Functional analysis, Sobolev spaces and partial differential equations* [**6**]. ■

Chapter III

Harmonic maps

1 The generalised Dirichlet energy

1.1 Given any natural number N, we will consider the collection T_N of all the symmetric and positive semi-definite real $N \times N$ matrices. This set is a closed subset of the $(N(N+1)/2)$-dimensional vector spaces of symmetric $N \times N$ real matrices. We will furthermore consider the quotient set $[T_N]$ of matrices in T_N modulo conjugation by orthogonal matrices.

In accordance with this notation, for every endomorphism L in an N-dimensional inner product space, we will write $[L]$ to denote the set of all the matrices of f with respect to orthonormal bases. Clearly, if $[L]$ is symmetric and positive semi-definite, $[L] \in [T_N]$.

Given any function $\Phi : T_N \longrightarrow \mathbb{R}$ invariant under conjugation by orthogonal matrices, there exists a unique function $[\Phi] : [T_N] \longrightarrow \mathbb{R}$ satisfying $[\Phi]([A]) = \Phi(A)$ for every $A \in T_N$.

1.2 Lemma. Let X be a Riemannian N-dimensional manifold and let $F : X \longrightarrow \mathrm{Hom}(TX, TX)$ be any smooth section. If $\Phi : T_N \longrightarrow \mathbb{R}$ is smooth and invariant under conjugation by orthogonal matrices, then the function $x \in X \longmapsto [\Phi]([F(x)])$ is smooth.

Proof. Let us consider a point $x_0 \in X$ and a local orthonormal frame $\{E_i\}_i$ for TX defined on an open neighbourhood U of x_0. The function that maps every $x \in U$ to the matrix A_x of $F(x)$ with respect to $\{E_i\}_i$ is smooth, hence so is the function

$$x \in U \mapsto \Phi(A_x) = [\Phi]([F(x)]).$$

Since x_0 is arbitrary, the result follows. ■

1.3 Definition. Let X be a compact N-dimensional Riemannian manifold and let $\Phi : T_N \longrightarrow \mathbb{R}$ be smooth and invariant under conjugation by orthogonal matrices.

Given any Riemannian manifold Y, the *(generalised) Dirichlet energy* induced by Φ for functions between X and Y is the functional

$$\begin{aligned} E_\Phi : \mathcal{C}^\infty(X, Y) &\longrightarrow \mathbb{R} \\ f &\longmapsto \int_X [\Phi]([Df^* Df]) \cdot (*1_X), \end{aligned}$$

which is well-defined by virtue of § 1.2. We will usually refer to E_Φ as the Φ-*energy*.

A function $f : X \longrightarrow Y$ is said to be Φ-*harmonic* if it is a critical point of the Φ-energy.

1.4 Scholium. As in the definition, let X be an arbitrary compact Riemannian manifold of dimension N. When we consider a function Φ in order to define the Φ-energy of a function from X, what we would really want to have is a smooth function from $\mathrm{Hom}(TX, TX)$ to the set of real numbers. However, if we defined the energy in terms of such a function, we would need to specify Φ for each and every possible manifold.

Taking the admittedly cumbersome approach of assuming Φ to be a function from T_N, however, enables us to define a notion of Φ-energy on any X — for, as we have shown, this Φ, if invariant under conjugation by orthonormal matrices, does induce a unique smooth function from the bundle $\mathrm{Hom}(TX, TX)$ of X. We should remark that not every smooth function from $\mathrm{Hom}(TX, TX)$ to the real numbers is induced by some function Φ from T_N.

The perspicacious reader may have noticed that, since all symmetric matrices are orthonormally diagonalisable, we could have considered Φ as a function of the eigenvalues of a linear operator. Nevertheless, this would mean that we would need Φ to be invariant under permutations of arguments — which would not really lead to a simpler definition of the Dirichlet energies. Furthermore, such an approach would complicate the proof of one of the key results in this chapter: the first variational form of Dirichlet energies [§ 2.3].

1.5 (i) The *classical energy* is the Dirichlet energy induced by $(1/2)\,\mathrm{tr}(-)$. It is common across the literature to refer to this functional as the 'Dirichlet energy' of a function, and this is why we have appended the adjective 'generalised' to our notion of the Dirichlet energy.

We shall refer to the critical points of the classical energy as *classical harmonic maps*. It should be remarked that the term 'harmonic map' is often used in the literature to refer to what we would call a classical harmonic map.

(ii) For any real $p \geqslant 2$, the *p-energy* is the Dirichlet energy induced by the function $\mathrm{tr}(-)^{p/2}$. Critical functions of this energy are said to be *p-harmonic* maps.

Recalling that the Hilbert-Schmidt norm of a linear map L between inner-product spaces is defined as $\|L\|_{HS} := \sqrt{\mathrm{tr}\, L^* L}$, we can observe how the p-energy of any $f \in C^\infty(X, Y)$, for some Riemannian manifolds X and Y, is

$$E_p(f) = \int_X \|Df\|_{HS}^p \cdot (*1_X).$$

The classical energy of a map is half its 2-energy, so, clearly, a map is 2-harmonic if and only if it is classically harmonic.

(iii) The *exponential energy* is the Dirichlet energy induced by $\exp(\mathrm{tr}(-))$.

1.6 Example. Computing a Dirichlet energy of a function analytically is usually a difficult task, but it is perfectly feasible in some cases.

Using a stereographic projection [§ I-1.2], let us identify S^2 with the compactification of the complex numbers. For any natural number n, let $f_n : S^2 \longrightarrow S^2$ be a function which, under this identification, takes $z \mapsto z^n$. Its 2-energy is

$$E_2(f_n) = 8\pi n.$$

Proof. Combining the stereographic projection considered in the statement with the use of polar coordinates, we can define a chart $(U, \varphi = (r, \theta))$ for S^2 defined on a subset U equal to S^2 up to a measure-zero subset, and with

$$\begin{aligned} \varphi^{-1} :]0,\infty[\times]0, 2\pi[&\longrightarrow S^2 \subseteq \mathbb{R}^3 \\ (r, \theta) &\longmapsto \left(\frac{2r\cos\theta}{r^2+1}, \frac{2r\sin\theta}{r^2+1}, \frac{r^2-1}{r^2+1} \right). \end{aligned}$$

Under the coordinate representation in (U, φ), f_n takes any point (r, θ) to $(r^n, \theta n)$. For the sake of simplicity, in the sequel, we will drop the subindex n in f_n.

The 2-energy of f will be given by the integral

$$E_2(f) = \int_{S^2} \|Df\|_{HS} \cdot (*1_{S^2}).$$

Let us compute the two factors in the integrand separately.

Going back to the definition of the metric that we are considering on the sphere [§ I-5.4(ii)], we can easily check that ∂_r and ∂_θ are orthogonal at any point. Under these conditions, given any point $x = \varphi^{-1}(r, \theta)$ for $r > 0$ and $0 < \Theta < 2\pi$, we can compute the norm of Df_x as

$$\begin{aligned} \|Df_x\|_{HS} = \operatorname{tr}(Df_x^* Df_x) &= \frac{\left\langle Df_x^* Df_x(\partial_r|_x) \mid \partial_r|_x \right\rangle}{\|\partial_r|_x\|^2} + \frac{\left\langle Df_x^* Df_x(\partial_\theta|_x) \mid \partial_\theta|_x \right\rangle}{\|\partial_\theta|_x\|^2} \\ &= \frac{\|Df_x(\partial_r|_x)\|^2}{\|\partial_r|_x\|^2} + \frac{\|Df_x(\partial_\theta|_x)\|^2}{\|\partial_\theta|_x\|^2} = \left(nr^{n-1}\right)^2 \frac{\|\partial_r|_{f(x)}\|^2}{\|\partial_r|_x\|^2} + n^2 \frac{\|\partial_\theta|_{f(x)}\|^2}{\|\partial_\theta|_x\|^2}, \end{aligned}$$

where the scalar products and the norms are the ones provided by the Riemannian metric. Using this metric, we can find that

$$\|\partial_r|_{\varphi(r,\theta)}\| = \left\| \frac{\partial \varphi^{-1}}{\partial r} \right\| = \frac{2}{1+r^2}, \qquad \|\partial_\theta|_{\varphi(r,\theta)}\| = \left\| \frac{\partial \varphi^{-1}}{\partial \theta} \right\| = \frac{2r}{1+r^2}.$$

Plugging this result into the previous expression leads us to conclude that

$$\|Df_x\|_{HS} = \left(nr^{n-1}\right)^2 \left(\frac{1+r^2}{1+r^{2n}} \right)^2 + n^2 \left(\frac{r^{n-1}(1+r^2)}{(1+r^{2n})} \right)^2.$$

In regard to the volume form, using the orthogonality of the vectors ∂_r and ∂_θ, we have that

$$(*1_{S^2}) = (*1_{S^2})(\partial_r, \partial_\theta)(dr \wedge d\theta)$$

$$= \left\| \mathrm{D}\varphi^{-1}(\partial_1) \right\| \cdot \left\| \mathrm{D}\varphi^{-1}(\partial_2) \right\|_{\mathbb{R}^3} (dr \wedge d\theta)$$
$$= \frac{4r}{\left(1 + r^2\right)^2} (dr \wedge d\theta).$$

Thus, if we put everything together, we may conclude that

$$\mathrm{E}_2(f) = \int_0^{2\pi} \int_{0^\infty} \left(\left(nr^{n-1} \right)^2 \left(\frac{1 + r^2}{1 + r^{2n}} \right)^2 + n^2 \left(\frac{r^{n-1}(1 + r^2)}{(1 + r^{2n})} \right)^2 \right) \frac{4r}{\left(1 + r^2\right)^2} \, dr \, d\theta$$
$$= \int_0^{2\pi} \int_0^\infty \frac{8n^2 r^{2n-1}}{\left(r^{2n} + 1\right)^2} \, dr \, d\theta = 8\pi n,$$

which completes the computation.

■

2 The first variational form

2.1 So far, we haven't said much about harmonic maps. In this section, we will fully characterise when functions are critical points of some Dirichlet energy in terms of its *first variational form*. The proof of this result shares some analogies with that of § II-2.7.

Before we can introduce this result, however, we should fix some notation. Given any natural N, since $\mathrm{T_N}$ is the collection of all $\mathrm{N} \times \mathrm{N}$ symmetric matrices, we can think of Φ as a function with N^2 arguments — one for each matrix entry. For any $1 \leqslant i, j \leqslant \mathrm{N}$, we will then write $\partial_{ij}\Phi(\mathrm{A})$ to denote the partial derivative of Φ at $\mathrm{A} \in \mathrm{T_N}$ with respect to the (i, j)-th entry of the input matrix.

As we mentioned before, $\mathrm{T_N}$ has dimension $\mathrm{N}(\mathrm{N}+1)/2 < \mathrm{N}^2$, so thinking of Φ as a function with N^2 variables introduces some redundancy. It could thus make sense to take Φ to be a function on the $\mathrm{N}(\mathrm{N} + 1)/2$ entries of the upper triangular portion of matrices in $\mathrm{T_N}$, but this would be slightly more impractical from a computational perspective.

Even if a function Φ is invariant under conjugation by orthogonal matrices, its partial derivatives ∂_{ij} may not be. For this reason, these partial derivatives will not induce functions from $[\mathrm{T_N}]$. Thus, when given an endomorphism L, it will be sometimes necessary to refer to its matrix with respect to a fixed basis $\{e_i\}_i$. We will denote this matrix as $(\mathrm{L})_e$.

2.2 **Lemma.** For any natural N and any smooth $\Phi : \mathrm{T_N} \longrightarrow \mathbb{R}$ invariant under conjugation by orthogonal matrices, let us consider an endomorphism L in an N-dimensional vector space V and a pair of orthonormal basis $\{e_i\}_{i=1}^{\mathrm{N}}$ and $\{\tilde{e}_i\}_{i=1}^{\mathrm{N}}$ of V. If v is an arbitrary vector in V, then

$$\sum_{i,j} (\partial_{ij} + \partial_{ji})\Phi((\mathrm{L})_e) \cdot \langle v \mid e_j \rangle e_i = \sum_{i,j} (\partial_{ij} + \partial_{ji})\Phi((\mathrm{L})_{\tilde{e}}) \cdot \langle v \mid \tilde{e}_j \rangle \tilde{e}_i,$$

which is to say that this expression is invariant under changes of orthonormal basis.

Proof. Let $A \in T_N$ be the matrix of L with respect to $\{e_i\}_i$ and let P be the transition matrix from $\{e_i\}_i$ to $\{\tilde{e}_i\}_i$, in such a way that the matrix of L with respect to $\{\tilde{e}_i\}_i$ be $P^{-1}AP$. For the sake of clarity, we will refer to the entries of P as p_{ij} and to those of P^{-1} as q_{ij}, but, since P must be orthogonal, we will have $q_{ij} = p_{ji}$.

We should first notice how, if we define a function Ξ taking any $M \in T_N$ to $P^{-1}MP \in T_N$, we need to have

$$(\partial_{ij} + \partial_{ji})\Xi_{\alpha\beta}(M) = (\partial_{ij} + \partial_{ji}) \sum_{r,s} q_{\alpha r} m_{rs} p_{s\beta} = p_{\alpha i} p_{\beta j} + p_{\alpha j} p_{\beta i}.$$

Since Φ is invariant under conjugation by orthogonal matrices, we know that $\Phi = \Phi \circ \Xi$, hence

$$\begin{aligned}(\partial_{ij} + \partial_{ji})\Phi(A) = (\partial_{ij} + \partial_{ji})\Phi \circ \Xi(A) &= \sum_{\alpha,\beta} \partial_{\alpha\beta}(PAP^{-1}) \cdot (\partial_{ij} + \partial_{ji})\Xi_{\alpha\beta}(A) \\ &= \sum_{\alpha,\beta} \partial_{\alpha\beta}(PAP^{-1}) \cdot (p_{\alpha i} p_{\beta j} + p_{\alpha j} p_{\beta i}) \\ &= \sum_{\alpha,\beta} (\partial_{\alpha\beta} + \partial_{\beta\alpha})(PAP^{-1}) \cdot (p_{\alpha i} p_{\beta j}).\end{aligned}$$

Moreover, by construction, we know that $e_i = \sum_\mu q_{i\mu} \tilde{e}_\mu = \sum_\mu p_{\mu i} \tilde{e}_\mu$. Together with the fact that

$$\sum_{i,j} p_{\alpha i} p_{\beta j} \cdot p_{\mu i} p_{\sigma j} = \sum_j \left(\sum_i q_{i\alpha} p_{i\mu} \right) q_{j\beta} p_{j\sigma} = \delta_{\alpha\mu} \delta_{\beta j},$$

this all leads us to the desired result through a direct computation. ∎

2.3 Theorem (First variational form)**.** Consider two Riemannian manifolds X and Y, with X compact, and a function $\Phi : T_N \longrightarrow \mathbb{R}$ which we will assume to be invariant under conjugation by orthogonal matrices.

Let $f : X \longrightarrow Y$ be a smooth function. Let $\{E_i\}_i$ be an orthonormal frame around an arbitrary point $x \in X$ and let us define the Φ-*tension* of f at that point as

$$\begin{aligned}\tau := \sum_{i,k} \nabla^*_{E_i} \Big((\partial_{ik} + \partial_{ki}) \Phi((Df^* Df)_E)(Df E_k) \Big) \\ + \sum_j \Big((\partial_{jk} + \partial_{kj}) \Phi((Df^* Df)_E) \Gamma^i_{ij} \Big)(Df E_k),\end{aligned}$$

where ∇^* represents the induced connection [§ I-4.6] in f^*TY and Γ is used to denote the Christoffel symbols at x with respect to the local frame $\{E_i\}_i$.

Under these conditions, f is Φ-harmonic, i.e., a critical function of the Φ-energy, if and only if $\tau = 0$ everywhere. For this reason, τ is said to be the first variational form of E_Φ and the equation $\tau = 0$ is often referred to as the 'Euler-Lagrange equation' for the Φ-energy.

Proof. Let us take an arbitrary variation $f_{(-)} :]-\varepsilon, \varepsilon[\times \mathrm{X} \longrightarrow \mathrm{Y}$ of f in $C^\infty(\mathrm{X}, \mathrm{Y})$ and let $\xi \in \Gamma(f^*\mathrm{TY})$ be the smooth section defined by $\xi_x := \mathrm{D}f_t(\partial_t|_{(0,x)})$.

Consider an arbitrary point $x \in \mathrm{X}$ and an orthonormal frame $\{\mathrm{E}_i\}_i$ defined in a neighbourhood U around it. We will extend it to $\mathrm{U} \times]-\varepsilon, \varepsilon[$ in such a way that $[\partial_t, \mathrm{E}_i] = 0$ (for $i = 1, \ldots, \mathrm{N}$). In an attempt to keep our notation simpler, we will construct, for any $i, k \in \{1, \ldots, \mathrm{N}\}$, the smooth function

$$\Phi'_{ik} : x \in \mathrm{U} \longmapsto (\partial_{ik} + \partial_{ki})\Phi((\mathrm{D}f_x^*\mathrm{D}f_x)_\mathrm{E}).$$

With all of this, we know that, at the point x,

$$\begin{aligned} \frac{d}{dt}[\Phi]\big([\mathrm{D}f_t^*\mathrm{D}f_t]\big)\bigg|_{t=0} &= \sum_{i,k} \partial_{ik}\Phi\big((\mathrm{D}f_0^*\mathrm{D}f_0)_\mathrm{E}\big) \cdot \partial_t|_0\langle \mathrm{D}f_t(\mathrm{E}_i) \mid \mathrm{D}f_t\mathrm{E}_k\rangle \\ &= \sum_{i,k} \partial_{ik}\Phi((\mathrm{D}f^*\mathrm{D}f)_\mathrm{E}) \cdot \Big(\big\langle \tilde\nabla_{\partial_t}\mathrm{D}f_t(\mathrm{E}_i) \,\big|\, \mathrm{D}f(\mathrm{E}_k)\big\rangle + \big\langle \mathrm{D}f(\mathrm{E}_i) \,\big|\, \tilde\nabla_{\partial_t}\mathrm{D}f_t\mathrm{E}_k\big\rangle\Big) \\ &= \sum_{i,k} \Phi'_{ik} \cdot \big\langle \tilde\nabla_{\partial_t}\mathrm{D}\tilde f_t(\mathrm{E}_i) \,\big|\, \mathrm{D}f(\mathrm{E}_k)\big\rangle, \end{aligned}$$

where $\tilde\nabla$ denotes the induced connection in $f_{(-)}^*\mathrm{TY}$.

Since the (Levi-Civita) connection in Y is symmetric and, by construction, $[\partial_t, \mathrm{E}_i] = 0$, we know that $\tilde\nabla_{\partial_t}\mathrm{D}f_t(\mathrm{E}_i)|_{(0,x)} = \nabla^*_{\mathrm{E}_i}\mathrm{D}f_t(\partial_t|_{(0,x)})|_x$, hence

$$\frac{d}{dt}[\Phi]\big([\mathrm{D}\tilde f_t^*\mathrm{D}\tilde f_t]\big)\bigg|_{t=0} = \sum_{i,k} \Phi'_{ik} \cdot \big\langle \nabla^*_{\mathrm{E}_i}\xi \,\big|\, \mathrm{D}f\mathrm{E}_k\big\rangle.$$

Given any orthonormal frame $\{\mathrm{E}_i\}_i$, we know [§ I-5.7] that the divergence of a vector field $\eta = \sum_i \eta_i\mathrm{E}_i$ is equal to

$$\begin{aligned} \operatorname{div}\eta = \operatorname{tr}\nabla\eta = \sum_i \big\langle \nabla_{\mathrm{E}_i}\eta \mid \mathrm{E}_i\big\rangle &= \sum_i \left\langle \sum_k \left(\mathrm{E}_i(\eta_k) + \sum_j \eta_j\Gamma^k_{ij}\right)\mathrm{E}_k \,\middle|\, \mathrm{E}_i\right\rangle \\ &= \sum_i \mathrm{E}_i(\eta_i) + \sum_j \eta_j\Gamma^i_{ij}, \end{aligned}$$

where we have used the fact that $\langle \mathrm{E}_k \mid \mathrm{E}_i\rangle = \delta_{ik}$ and where Γ denotes the Christoffel symbols. This can be particularised to deduce that

$$\begin{aligned} &\operatorname{div}\left(\sum_{i,k}(\Phi'_{ik}\langle\xi \mid \mathrm{D}f\mathrm{E}_k\rangle)\mathrm{E}_i\right) = \\ &= \sum_i \mathrm{E}_i\underbrace{\left(\sum_k \Phi'_{ik}\langle\xi \mid \mathrm{D}f\mathrm{E}_k\rangle\right)}_{\eta_i} + \sum_j \underbrace{\left(\sum_k \Phi'_{jk}\langle\xi \mid \mathrm{D}f\mathrm{E}_k\rangle\right)}_{\eta_j}\Gamma^i_{ij} \\ &= \sum_{i,k} \mathrm{E}_i\big(\Phi'_{ik}\langle\xi \mid \mathrm{D}f\mathrm{E}_k\rangle\big) + \sum_j \Big(\Phi'_{jk}\langle\xi \mid \mathrm{D}f\mathrm{E}_k\rangle\Big)\Gamma^i_{ij} \end{aligned}$$

$$= \sum_{i,k} \mathrm{E}_i \langle \xi \mid \Phi'_{ik} \mathrm{D}f\mathrm{E}_k \rangle + \left\langle \xi \;\middle|\; \sum_j \Phi'_{jk} \Gamma^i_{ij} \mathrm{D}f\mathrm{E}_k \right\rangle$$

$$= \sum_{i,k} \Phi'_{ik} \big\langle \nabla^*_{\mathrm{E}_i} \xi \;\big|\; \mathrm{D}f\mathrm{E}_k \big\rangle + \big\langle \xi \;\big|\; \nabla^*_{\mathrm{E}_i} \Phi'_{ik} \mathrm{D}f\mathrm{E}_k \big\rangle + \left\langle \xi \;\middle|\; \sum_j \Phi'_{jk} \Gamma^i_{ij} \mathrm{D}f\mathrm{E}_k \right\rangle$$

$$= \sum_{i,k} \Phi'_{ik} \big\langle \nabla^*_{\mathrm{E}_i} \xi \;\big|\; \mathrm{D}f\mathrm{E}_k \big\rangle + \left\langle \xi \;\middle|\; \nabla^*_{\mathrm{E}_i} \Phi'_{ik} \mathrm{D}f\mathrm{E}_k + \sum_j \Phi'_{jk} \Gamma^i_{ij} \mathrm{D}f\mathrm{E}_k \right\rangle,$$

which, by virtue of Lemma 2.2, is the divergence of a global vector field — invariant under change of orthonormal frame of reference. From this result, we can deduce that

$$\begin{aligned} \frac{d}{dt}[\Phi]\big([\mathrm{D}f_t^*\mathrm{D}f_t]\big)\Big|_{t=0} &= \sum_{i,k} \Phi'_{ik} \big\langle \nabla^*_{\mathrm{E}_i} \xi \;\big|\; \mathrm{D}f\mathrm{E}_k \big\rangle \\ &= \operatorname{div}(\cdots) - \sum_{i,k} \left\langle \xi \;\middle|\; \nabla^*_{\mathrm{E}_i} \Phi'_{ik} \mathrm{D}f\mathrm{E}_k + \sum_j \Phi'_{jk} \Gamma^i_{ij} \mathrm{D}f\mathrm{E}_k \right\rangle. \end{aligned}$$

By the Divergence Theorem [§ I-5.8], for the arbitrary variation that we have considered,

$$\frac{d}{dt}\mathrm{E}_\Phi(f_t)\Big|_{t=0} = \int_{\mathrm{X}} \frac{d}{dt}[\Phi]\big([\mathrm{D}f_t^*\mathrm{D}f_t]\big)\Big|_{t=0} \cdot (*1_{\mathrm{X}}) = \int_{\mathrm{X}} \langle \xi \mid \tau \rangle \cdot (*1_{\mathrm{X}}),$$

which is zero everywhere for an arbitrary variation — and, therefore, for an arbitrary ξ — if and only if $\tau = 0$ everywhere. This follows from the fact that we can always find a variation with $\xi = \tau$; such a variation can be defined, thanks to the compactness of X, as

$$f_t(x) := \exp(t \cdot \tau_x),$$

where exp denotes the exponential of X [§ I-5.11]. ■

2.4 Corollary. Given any N-dimensional compact Riemannian manifold X and any smooth $\Phi : \mathrm{T}_{\mathrm{N}} \longrightarrow \mathbb{R}$ invariant under conjugation by orthogonal matrices, the identity function id_{X} is Φ-harmonic.

Proof. Let us fix an arbitrary point $x \in \mathrm{X}$ and an orthonormal frame $\{\mathrm{E}_i\}_i$ around it such that $\nabla_{\mathrm{E}_i}\mathrm{E}_j|_x = 0$; this will mean, in particular, that all the Christoffel symbols will vanish at x. The matrix of the identity map with respect to any basis is always the identity matrix, hence the partial derivatives $\partial_{ij}\Phi((\mathrm{Did}^*_{\mathrm{X}}\mathrm{Did}_{\mathrm{X}})_{\mathrm{E}})$ must be equal to some constants c_{ik} independent of x and of the choice of frame $\{\mathrm{E}_i\}_i$. The Φ-tension field of f can thus be simplified to

$$\tau_\Phi(f)_x = \sum_{ik} c_{ik} \nabla_{\mathrm{E}_i}\mathrm{E}_k + 0 = 0.$$

Since our choice of x is arbitrary, this shows that the tension field vanishes everywhere and, therefore, that the identity is Φ-harmonic. ■

2.5 Bibliographical remarks. The first variational formula for generalised Dirichlet energies can be found in the paper *Tension field and index form of energy-type functionals* [**2**].

The variational formula that we have discussed is a slight generalisation of the one in the original paper, for, in our definition of the tension field at x, we have not assumed the orthonormal frame $\{E_i\}_i$ to be parallel at x, i.e., we have not assumed the Christoffel symbols to vanish at x.

3 Sequences of functions with decreasing energy

3.1 A natural question that may arise from the characterisation of harmonic maps that the first variational form gives us is whether it is actually possible to find harmonic maps — at least beyond the identity map. We know, by virtue of § II-2.6, that any local energy minimiser in the uniform convergence topology must be a harmonic map. Thus, when given any sequence of harmonic maps with decreasing energy, we may reasonably wonder whether we can always find its limit with respect to some topology, and whether this limit will be a function with minimal energy with respect to the sequence. In this section we will address the first question, but we will leave the second one open.

To this end, we will take two complementary approaches: a more abstract approach, relying on the results from functional analysis that we introduced in § II-3, and a more constructive approach, which will be built on the Arzelà-Ascoli Theorem.

3.2 Theorem. Let X be an N-dimensional compact Riemannian manifold and let Y be a compact Riemannian manifold, which we will assume to be a subset of $\mathbb{R}^K$. Given a real number $p \geqslant 2$, if A is a subset of $H^p(X, Y)$ on which the p-energy is bounded, then A is sequentially precompact in the weak topology of $H^p(X, Y)$.

Proof. The subset A must be bounded in $H^p(X, \mathbb{R}^K)$ because of the compactness of Y. Since Sobolev spaces are reflexive [§ II-3.11], we can apply § II-3.5 to deduce that A must be sequentially precompact with respect to the weak topology of $H^p(X, \mathbb{R}^K)$. Sequential precompactness within $H^p(X, Y)$ then follows from § II-3.10. ∎

3.3 Scholium. The previous theorem shows that, in a subset of functions with bounded energy, if we consider any sequence of functions (in particular one with decreasing p-energy), that sequence must have a converging subsequence in the weak topology. However, the theorem offers us no guarantee of whether such a function will be smooth or just regular enough for its p-energy to even be defined.

3.4 Bibliographical remarks. We have adapted our proof of Theorem 3.2 from the book *Geometry of harmonic maps* by Xin [**18**]. The book only considers the 2-energy, and we have adapted its results to any p-energy.

3.5 Theorem (Arzelà-Ascoli)**.** Let X and Y be compact metric spaces, and let $F \subseteq C(X, Y)$ be a set of continuous functions from X to Y. If F is equicontinuous [§ A.3], then it is precompact in the uniform convergence topology.

Proof. Theorem 47.1 of *Topology* [**11**]. ■

3.6 Lemma. Let X be a Riemannian manifold and let (U, φ) be a chart of X. The function $y \in \varphi(U) \mapsto \|D\varphi_y^{-1}\|_{op}$ is continuous, where $\| - \|_{op}$ denotes the operator norm with respect to the Euclidean metric in $\varphi(U)$ and the Riemannian metric in TX.

Proof. Let N be the dimension of X and let g be its Riemannian metric.

Since the tangent bundle is a vector bundle and can therefore be locally trivialised, we can consider a local trivialisation $\Psi : TU \longrightarrow U \times \mathbb{R}^N$. This local trivialisation is a diffeomorphism and it induces a smooth function $\bar{g}_{(-)}$, from U to the set of metrics in $\mathbb{R}^N$, defined, at every $u \in U$ and for every $v_1, v_2 \in \mathbb{R}^N$, as

$$\bar{g}_u(v_1, v_1) := g_u\Big(\Psi^{-1}(u, v_1), \Psi^{-1}(u, v_2)\Big).$$

Given any metric m in $\mathbb{R}^N$, we can always define a linear function Ξ_m taking any orthonormal basis with respect to m to an orthonormal basis with respect to the Euclidean metric. It is also clear that, if $\| - \|$ denotes the Euclidean norm, then $\| - \|_m = \|\Xi_m(-)\|$, and, what is more, Ξ can be constructed in such a way that Ξ_m be continuously dependent on the metric m — just following the Gram-Schmidt algorithm. In addition to all of this, we know that, if we consider the standard metric in $\mathbb{R}^N$, then the operator norm, as a function $\| - \|_* : \mathrm{Hom}(\mathbb{R}^N, \mathbb{R}^N) \longrightarrow \mathbb{R}$, is continuous.

Therefore, putting everything together, it follows that the function

$$y \in \varphi(U) \mapsto \left\| \Xi_{\bar{g}_{\varphi^{-1}(y)}}\Big(D\varphi_y^{-1}\Big) \right\|_* = \|D_x \varphi^{-1}\|_{op}$$

must be continuous. ■

3.7 Theorem. Let X and Y be Riemannian manifolds with X compact. Let N be the dimension of X. If F is a collection of smooth functions from X to Y with bounded p-energy for $p > N$ (and $p \geqslant 2$), then F is equicontinuous.

Proof. Pick an arbitrary $f \in F$ and an $x \in X$. In order to prove the equicontinuity of F, it will suffice to show that there exists an open neighbourhood V of x in which, for every $z \in V$, we can bound $d(f(x), f(z))$ in terms of $E_p(f)$ and $d(x, z)$.

Let us take any $\Delta > 0$. Using Lemma 3.6, consider a normal chart (U, φ) around x such that $\|D\varphi_y^{-1}\|_{op} \leqslant 1 + \Delta$ for every $y \in \varphi(U)$. By construction, we will have $\varphi(x) = 0$ and $\|D\varphi_{\varphi(x)}^{-1}\|_{op} = 1$. [§ I-5.12]

Let $\delta > 0$ be small enough as to have $B^\circ(0, \delta) \subseteq \varphi(U)$ [§ A.2]. We will take

$$V := \varphi^{-1}(B^\circ(0, \delta/2))$$

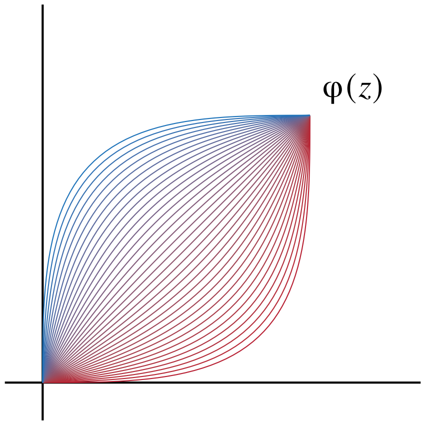

Figure 3.1: Taking $N = 2$ and given a z, the image of the function $\varphi \circ \Lambda(-, v)$ is depicted in different colours for different values of v.

and consider an arbitrary $z \in V$. If we let $w := \varphi(z)/\|\varphi(z)\|$, we can define the set

$$D_w^{N-1} := \{v \in \mathbb{R}^N \mid \|v\| \leqslant 1,\ \langle v \mid w\rangle = 0\},$$

which is nothing more than the $(N-1)$-dimensional unit disc D^{N-1} in the orthogonal complement of w. We would like to warn the reader not to mistake the set D_w^{N-1} with the differential operator.

Setting $l := d(x, z)/\pi$, let us construct the function

$$\begin{aligned} \Lambda : [0, \pi] \times D_w^{N-1} &\longrightarrow X \\ (t, v) &\longmapsto \varphi^{-1}(l \cdot (tw + v \sin t)). \end{aligned}$$

We should remark that this function is well-defined, for, given any $w \in S^{N-1}$, any $t \in [0, \pi]$ and any $v \in D_w^{N-1}$, we have

$$\|l(tw + v \sin t)\| \leqslant l(t\|w\| + \sin t\|v\|) \leqslant l(\pi + 1) \leqslant 2l\pi = 2d(x, z) < \delta,$$

and we know that $B^\circ(0, \delta) \subseteq \varphi(U)$. Notice how we have used the fact that $d(x, z) = \|\varphi(z)\| < \delta/2$, as φ is a normal chart around x.

If we consider an arbitrary $v \in D_w^{N-1}$, we will have $x = \Lambda(0, v)$ and $z = \Lambda(\pi, v)$, which means that $\Lambda(-, v)$ will parametrise a curve joining x and z; this is depicted in Figure 3.1. Consequently, for any $v \in D_w^{N-1}$,

$$d(f(x), f(z)) \leqslant \int_0^\pi \left\| \frac{d}{dt}(f \circ \Lambda)(t, v) \right\|_X dt \leqslant \int_0^\pi \left\| Df_{\Lambda(t,v)} \right\|_{op} \left\| D\varphi^{-1}_{\varphi(\Lambda(t,v))} \right\|_{op} 2l\, dt,$$

where we have used the fact that the derivative of $l(tw + v \sin t)$ with respect to t is bounded by $2l$, and where we have used $\| - \|_X$ to denote the Riemannian norm in TX.

Since the estimate that we have just obtained holds for every possible value of v, we must have

$$d(f(x), f(z)) \leqslant \frac{2l}{\operatorname{vol} D^{N-1}} \int_{D^{N-1}} \int_0^\pi \left\| Df_{\Lambda(t,v)} \right\|_{op} \cdot \left\| D\varphi^{-1}_{\varphi(\Lambda(t,v))} \right\|_{op} dt\, dv$$

$$\leqslant \frac{2l(1+\Delta)}{\operatorname{vol} D^{N-1}} \int_{D^{N-1}} \int_0^{\pi} \left\| D f_{\Lambda(t,v)} \right\|_{\mathrm{op}} dt\, dv.$$

When we have a linear function $L_y : \mathbb{R}^N \longrightarrow T_y X$, we can compute its determinant from one of its matrices provided that the volume of the bases with respect to which the matrix is considered is the same (as computed with the corresponding volume forms). We know that the determinant is a continuous function of the entries of a matrix. Moreover, since X is a Riemannian manifold, its metric is continuous. Therefore, using the Gram-Schmidt method, the change of basis matrix towards an orthonormal matrix can be continuous. Thus, we may conclude that $\det L_y$ must be a continuous function on y if L_y is continuous. This applies, in particular, to $D\varphi^{-1}$, so we may assume — without any loss of generality — that $\det D\varphi^{-1} \in \,]1-\varepsilon, 1+\varepsilon[$ throughout $\varphi(U)$ for an arbitrary yet fixed $0 < \varepsilon < 1$. Here we have used the fact that $\det D\varphi_x = 1$.

Taking into consideration the preceding discussion and the fact that

$$\det D(\varphi \circ \Lambda)_{(t,v)} = \begin{vmatrix} l & 0 & \cdots & 0 \\ \vdots & & & \\ lv\cos t & & l \sin t \times \mathrm{id}_{N-1} & \\ \vdots & & & \end{vmatrix} = l^N \sin(t)^{N-1},$$

we may continue with our estimate as

$$\begin{aligned} d(f(x), f(z)) &\leqslant \frac{2l(1+\Delta)}{\operatorname{vol} D^{N-1}} \int_{D^{N-1}} \int_0^{\pi} \| D f_{\Lambda(t,v)} \| \frac{1}{\det D\Lambda} \det D\Lambda \, dt\, dv \\ &\leqslant \frac{2l(1+\Delta)}{\operatorname{vol} D^{N-1}} \int_{\Omega := \operatorname{im} \Lambda} \| D f_y \| \frac{1}{(1-\varepsilon) l^N \sin(t(y))^{N-1}} \, dy, \end{aligned}$$

where we have implicitly defined

$$t(y) := \frac{1}{l} \left\langle \varphi(y), \frac{\varphi(z)}{\|\varphi(z)\|} \right\rangle.$$

Applying Hölder's inequality [§ B.1], we get that

$$d(f(x), f(z)) \leqslant \frac{2(1+\Delta)}{(1-\varepsilon) l^{N-1} \operatorname{vol} D^{N-1}} \| D f \|_{p,\Omega} \left\| \frac{1}{\sin(t(y))^{N-1}} \right\|_{p^*,\Omega}.$$

Since, by hypothesis, $p > N$, we need have $-1 < (1-p^*)(N-1)$. Thus, we can obtain the bound

$$\begin{aligned} \left\| \frac{1}{\sin(t(y))^{N-1}} \right\|_{p^*,\Omega}^{p^*} &= \int_{\Omega} \frac{1}{\sin(t(y))^{(N-1)p^*}} (\det D\Lambda)(\det D\Lambda^{-1}) \, dy \\ &\leqslant \int_{D^{N-1}} \int_0^{\pi} l^N \sin(t)^{(1-p^*)(N-1)} (1+\varepsilon) \, dt\, dv \\ &= l^N C, \end{aligned}$$

for some C depending on no other variables than p, N and ε.

Putting everything together, we have shown that

$$d(f(x), f(z)) \leqslant \frac{2(1+\Delta)\mathrm{C}^{1/p^*}}{(1-\varepsilon)\operatorname{vol}\mathrm{D}^{\mathrm{N}-1}}\mathrm{E}_p(f)^{1/p} l^{(\mathrm{N}/p^*)-(\mathrm{N}-1)},$$

where, since $p > \mathrm{N}$, the exponent of l is positive.

In conclusion, we have proven the existence of an open set V such that, for every $z \in \mathrm{V}$, we have

$$d(f(x), f(z)) \leqslant \kappa \cdot \mathrm{E}_p(f)^{\alpha} l^{\beta}$$

where κ, α and β are greater than zero and only depend on N, p and the fixed yet arbitrary constants Δ and ε. This concludes the proof. ■

3.8 Corollary. Let X and Y be compact Riemannian manifolds and let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions such that $\mathrm{E}_p(f_n) \to \mathrm{E}_0$. If X is N-dimensional and $p > \mathrm{N}$, then there exists a subsequence of f_n that converges uniformly to a function f.

Proof. Direct consequence of Theorems 3.5 and 3.7. ■

3.9 Scholium. Just like Theorem 3.2, this theorem can guarantee the existence of a limit for any sequence of functions with decreasing p-energy, but it cannot give us any guarantees regarding the regularity of such a limit.

Nevertheless, the conclusion that we obtain in this case is more powerful than the one we had in Theorem 3.2. In our previous approach to this problem, we only proved the existence of a limit with respect to the weak topology, whereas, now, we have a limit with respect to the uniform convergence topology. This will imply, in particular, that if all the functions in the sequence belong to a homotopy class, so will the limit function.

Chapter IV

Maps between spheres

1 Non-existence of energy minimisers

1.1 **Lemma.** For any natural number $\mathrm{N} > 1$, consider the smooth embedding

$$\begin{aligned}\iota : \mathrm{S}^{\mathrm{N}-1} \times \,]-\pi/2, \pi/2[\, &\longrightarrow \mathrm{S}^{\mathrm{N}} \\ (y, \theta) &\longmapsto (y \cos\theta, \sin\theta),\end{aligned}$$

whose image is the whole N-dimensional sphere up to a measure-zero subset. We can define, at every point $x \in \operatorname{im}\iota$, a tangent vector $\partial_\theta|_x$ mapping any $\alpha \in C^\infty(\mathrm{S}^{\mathrm{N}})$ to the real number $\partial/\partial\theta(\alpha \circ \iota)|_x$.

Let us fix an arbitrary point $x = \iota(y, \theta) \in \operatorname{im}\iota$ and take an orthonormal basis $\{v_k\}_k$ of the tangent space to $\mathrm{S}^{\mathrm{N}-1}$ at y.

(i) The family $\{\partial_\theta, \mathrm{D}\iota(v_1), \dots, \mathrm{D}\iota(v_{\mathrm{N}-1})\}$ is an orthogonal basis of $\mathrm{T}_x\mathrm{S}^{\mathrm{N}-1}$.
(ii) In the Riemannian metric of $\mathrm{T}_x\mathrm{S}^{\mathrm{N}}$, $\|\partial_\theta\| = 1$ and $\|\mathrm{D}\iota(v_k)\| = \cos\theta$ for any index $1 \leqslant k \leqslant \mathrm{N}-1$.
(iii) We have $(*1)_{\mathrm{S}^{\mathrm{N}}} = (\cos\theta)^{\mathrm{N}-1}(d\theta \wedge (\iota^{-1})^*(*1)_{\mathrm{S}^{\mathrm{N}-1}})$.

When referring to the function ι, we will sometimes write $\iota_{(\mathrm{N})}$ in order to specify the dimension of the spheres on which it is defined.

Proof. We may first notice how $\|\partial_\theta\| = 1$ at $x = \iota(y, \theta)$ since

$$\begin{aligned}\|\partial_\theta\|^2 = \sum_{k=1}^{\mathrm{N}} \left(\frac{\partial\iota_k}{\partial\theta}\right)^2 &= \left((\sin\theta)^2 \sum_{k=1}^{\mathrm{N}-1} y_k^2\right) + (\cos\theta)^2 \\ &= (\cos\theta)^2 + (\sin\theta)^2 = 1,\end{aligned}$$

where we have used the fact that, since $y \in \mathrm{S}^{\mathrm{N}-1}$, it must have unit norm.

It is straightforward to check that $\langle \partial_\theta \mid \mathrm{D}\iota v_k\rangle = 0$, for, at $x = \iota(y, \theta)$,

$$\begin{aligned}\langle \partial_\theta \mid \mathrm{D}\iota v_k\rangle = \left\langle \frac{\partial\iota}{\partial\theta} \,\middle|\, \mathrm{D}\iota v_k\right\rangle_{\mathbb{R}^{\mathrm{N}+1}} &= \sum_{r=1}^{\mathrm{N}-1}(\sin\theta \cdot y_r) \cdot (\cos\theta \cdot (v_k)_r) \\ &= \sin\theta\cos\theta \cdot \sum_{r=1}^{\mathrm{N}-1} y_r \cdot (v_k)_r = 0,\end{aligned}$$

where we have implicitly identified the vectors v_k and $\mathrm{D}\iota v_k$ with their Euclidean representations: their image under the differential of the identity from the spheres (as manifolds) into Euclidean space. Notice how we have relied on the fact that — under this same identification — the vector $y \in \mathrm{S}^{\mathrm{N}-1} \subseteq \mathbb{R}^{\mathrm{N}}$ needs to be orthogonal to the tangent vector $v_k \in \mathbb{R}^{\mathrm{N}}$.

The fact that $\|\mathrm{D}\iota v_k\| = \cos\theta$ at x is trivial, which means that the family

$$\left\{\partial_\theta, \frac{\mathrm{D}\iota v_1}{\cos\theta}, \dots, \frac{\mathrm{D}\iota v_{\mathrm{N}-1}}{\cos\theta}\right\}$$

must be an orthonormal basis of the tangent space to S^{N} at x. Statement (iii) follows directly by evaluating the form on this orthonormal basis. ■

1.2 Given any smooth $f : \mathrm{S}^{\mathrm{N}-1} \longrightarrow \mathrm{S}^{\mathrm{K}-1}$ and any continuous $\alpha : [-\pi/2, \pi/2] \longrightarrow [-\pi/2, \pi/2]$ with $\alpha(\pm\pi/2) = \pm\pi/2$, we can extend f to its *suspension (with respect to* α*)* $\mathrm{S}_\alpha f$. This suspension takes any point $(s\cos t, \sin t) \in \mathrm{S}^{\mathrm{N}}$, with $s \in \mathrm{S}^{\mathrm{N}-1}$ and $t \in [-\pi/2, \pi/2]$, and maps it as

$$\mathrm{S}_\alpha f : (s\cos t, \sin t) \in \mathrm{S}^{\mathrm{N}} \longmapsto (f(s)\cos\alpha(t), \sin\alpha(t)) \in \mathrm{S}^{\mathrm{K}}.$$

This suspension $\mathrm{S}_\alpha f$ for any choice of a suitable α is homotopic to the topological suspension $\mathrm{S}f$ that we considered in § II-1.4.

Considering the functions $\iota_{(\mathrm{N})}$ and $\iota_{(\mathrm{K})}$ that we introduced in § 1.1, the suspension $\mathrm{S}_\alpha f$ would map

$$\iota_{(\mathrm{N})}(y, \theta) \longmapsto \iota_{(\mathrm{K})}(f(y), \alpha(\theta)).$$

It is easy to see that the Hilbert-Schmidt norm of the differential of $\mathrm{S}_\alpha f$ at a point $\iota_{(\mathrm{N})}(y, \theta)$ will be

$$\|\mathrm{D}(\mathrm{S}f)_{\iota_{(\mathrm{N})}(y,\theta)}\|^2 = \|\mathrm{D}f_y\|^2 \cdot \frac{\cos^2\alpha(\theta)}{\cos^2\theta} + \alpha'(\theta)^2,$$

so the p-energy of $\mathrm{S}_\alpha f$ is

$$\begin{aligned}
\mathrm{E}_p(\mathrm{S}_\alpha f) &= \int_{-\pi/2}^{\pi/2}\int_{\mathrm{S}^{\mathrm{N}-1}} \left[\|\mathrm{D}f_y\|^2 \cdot \frac{\cos(\alpha(\theta))^2}{\cos(\theta)^2} + \alpha'(\theta)^2\right]^{p/2} \cos(\theta)^{\mathrm{N}-1}\, dy\, d\theta \\
&\leqslant 2^{p/2}\mathrm{E}_p(f) \int_{-\pi/2}^{\pi/2} \cos(\alpha(\theta))^p \cdot \cos(\theta)^{\mathrm{N}-p-1}\, d\theta \\
&\quad + 2^{p/2}\,\mathrm{vol}(\mathrm{S}^{\mathrm{N}-1}) \int_{-\pi/2}^{\pi/2} \alpha'(\theta)^p \cos(\theta)^{\mathrm{N}-1}\, d\theta,
\end{aligned}$$

where we have used the fact that, in general, given any $a, b \geqslant 0$ and any real $p \geqslant 2$,

$$(a+b)^{p/2} \leqslant 2^{p/2}\left(a^{p/2} + b^{p/2}\right).$$

1.3 Lemma. Let $r, \delta, a > 0$. There exists a real-valued function $\beta_{r\delta a} = \beta$ such that $\beta(0) = 0$ and $\beta(x) = a$ for $x \geqslant r + \delta$, and such that $\beta'(x) \leqslant a/r$.

Proof. Letting $f(t) = \exp(-1/t) \cdot \chi_{(t>0)}$, where $\chi_{(-)}$ is the characteristic function of a subset, we know that

$$g(t) := \frac{f(t)}{f(t) + f(\delta/2 - t)}$$

is smooth and non-negative, and takes the value 0 at 0 and the value 1 at any $x \geqslant \delta/2$. Thus, if we let

$$h(t) := \frac{a}{r}\big(g(t)\chi_{(t\leqslant\delta/2)} + \chi_{(\delta/2<t<r+\delta/2)} + g(\delta/2 - (t-r))\chi_{(t\geqslant r+\delta/2)}\big),$$

we will have a smooth non-negative function with $h(0) = 0$, $h(t) = a/r$ whenever $\delta/2 \leqslant t \leqslant r + \delta/2$ and $h(t) = 0$ when $t \geqslant r + \delta$.

It then follows that the function

$$t \mapsto \int_0^t h(t)\, dt$$

will be non-decreasing, 0 at $t = 0$, and will take a constant value $\mathrm{C} > a$ whenever $t \geqslant r + \delta$. Therefore,

$$\beta_{r\delta a}(t) = \frac{a}{\mathrm{C}} \int_0^t h(t)\, dt$$

will satisfy the conditions in the statement of this lemma. ∎

1.4 Theorem (Morrey). Let $f : \mathrm{S}^{\mathrm{N}-1} \longrightarrow \mathrm{S}^{\mathrm{K}-1}$ and let $p < \mathrm{N}$. The infimum p-energy in the homotopy class of the suspension of f is zero.

Proof. We know that all the suspensions of f by any α are homotopic, so we may as well consider the suspension given by $\alpha_r(\theta) = \beta_{rr\pi}(\theta+\pi/2)-\pi/2$, where β is defined as in Lemma 1.3, but fixing $a = \pi$ and making $r = \delta$; of course, we need $r < \pi/2$. In order to prove our result, we will just show that $\lim_{r\to 0} \mathrm{E}_p(\mathrm{S}_{\alpha_r} f) = 0$.

According to § 1.2, the p-energy of $\mathrm{S}_\alpha f$ is bounded above by

$$\begin{aligned} 2^{p/2}\mathrm{E}_p(f) &\int_{-\pi/2}^{\pi/2} \cos(\alpha_r(\theta))^p \cos(\theta)^{\mathrm{N}-p-1}\, d\theta \\ &+ 2^{p/2}\operatorname{vol}(\mathrm{S}^n) \int_{-\pi/2}^{\pi/2} \alpha_r'(\theta)^p \cos(\theta)^{\mathrm{N}-1}\, d\theta. \end{aligned}$$

The first integrand is bounded by 1, and equal to 0 when $\alpha(\theta) = \pi/2$, so the integral will converge to 0 as $r \to 0$. In the second integral, the integrand will be non-zero only when $\alpha'(\theta) \neq 0$, so its support will be $]-\pi/2, -\pi/2 + r + r[$. In addition, we know that $(\alpha')^p$ is bounded by r^{-p}, so, taking into account that $\cos(t - \pi/2) \leqslant t$,

$$0 \leqslant \lim_{r\to 0} \int_{-\pi/2}^{\pi/2} (\alpha_r'(\theta))^p (\cos\theta)^{\mathrm{N}-1}\, d\theta \leqslant \lim_{r\to 0} (2r) r^{(\mathrm{N}-1)-p} = 0$$

if $\mathrm{N} - 1 - p > -1$, which is equivalent to $p < \mathrm{N}$. ∎

1.5 Corollary. The infimum p-energy in every homotopy class of functions from a sphere S^N to a Riemannian manifold X is zero if $p < N$. Therefore, there are no minimisers of the p-energy within any homotopy class of functions $S^N \longrightarrow X$ other than in the class of null-homotopic functions.

Proof. The identity function in S^N is homotopic to any suspension of the identity in S^{N-1}. Thus, by Theorem 1.4, we can consider a sequence of smooth functions $\{h_n\}_{n\in\mathbb{N}}$ which are homotopic to the identity and such that $E_p(h_n) \to 0$.

Let $f : S^N \longrightarrow X$ be any smooth function. All the functions in the sequence $\{f \circ h_n\}_{n\in\mathbb{N}}$ will be homotopic to f. Moreover, since spheres are compact,

$$E_p(f \circ h_n) \leqslant \left(\max_{x\in S^N} \|Df_x\|_{HS}^p\right) \int_{S^N} \|Dh_n\|_{HS}^p \cdot (*1_{S^N}) \xrightarrow{n\to\infty} 0,$$

which completes the proof. ■

1.6 Bibliographical remarks. The main result in this section, Theorem 1.4, was introduced for the 2-energy in *Harmonic mappings of Riemannian manifolds* [**8**]. The proof that we have presented is adapted from there.

2 Energy estimation in terms of the degree

2.1 We now know that, in a fixed homotopy class of maps from a sphere S^N to another sphere, the infimum p-energy is always zero if $p < N$.

In this section, we will formulate a lower bound for the p-energy of functions $S^N \longrightarrow S^N$ when $p \geqslant N$, and we will do it in terms of an invariant that fully characterises the homotopy class of functions in $\pi_N(S^N)$: the degree.

2.2 Proposition. Let $f : X \longrightarrow Y$ be any smooth function between compact Riemannian manifolds. If $q < p$ are a pair of positive real numbers, then

$$E_p(f) \geqslant \frac{E_q^{p/q}}{(\operatorname{vol} X)^{p/q-1}}.$$

Proof. Since $p > q$, the function $(-)^{p/q}$ is convex, which implies that

$$\begin{aligned}\left(\frac{E_q}{\operatorname{vol} X}\right)^{p/q} &= \left(\frac{\int_X \|Df\|_{HS}^q (*1_{S^N})}{\operatorname{vol} X}\right)^{p/q} \\ &\leqslant \frac{\int_X \|Df\|_{HS}^{qp/q} (*1_{S^N})}{\operatorname{vol} X} = \frac{E_p(f)}{\operatorname{vol} X},\end{aligned}$$

from where the result follows. ■

2.3 Lemma. For any natural N, let $f : \mathrm{S}^{\mathrm{N}} \longrightarrow \mathrm{S}^{\mathrm{N}}$ be a smooth function. The N-energy of f is bounded below by

$$\mathrm{E}_{\mathrm{N}}(f) \geqslant \mathrm{N}^{\mathrm{N}/2} \cdot \deg f \cdot \mathrm{vol}(\mathrm{S}^{\mathrm{N}}),$$

where $\deg f$ represents the degree of f [§ II-1.13].

Proof. We know that the degree of any such function f is well-defined and equal to

$$\deg f = \frac{\int_{\mathrm{S}^{\mathrm{N}}} f^*(*1_{\mathrm{S}^{\mathrm{N}}})}{\int_{\mathrm{S}^{\mathrm{N}}}(*1_{\mathrm{S}^{\mathrm{N}}})} = \frac{\int_{\mathrm{S}^{\mathrm{N}}}(\det \mathrm{D}f)(*1_{\mathrm{S}^{\mathrm{N}}})}{\mathrm{vol}\, \mathrm{S}^{\mathrm{N}}}.$$

Taking into account that $|\det \mathrm{D}f| = \sqrt{\det \mathrm{D}f^*\mathrm{D}f}$ and the inequality of arithmetic and geometry means, we have that

$$|\det \mathrm{D}f| = \sqrt{\det \mathrm{D}f^*\mathrm{D}f} = \left(\sqrt[\mathrm{N}]{\det \mathrm{D}f^*\mathrm{D}f}\right)^{\mathrm{N}/2} \leqslant \left(\frac{\mathrm{tr}\, \mathrm{D}f^*\mathrm{D}f}{\mathrm{N}}\right)^{\mathrm{N}/2}.$$

We may then conclude that

$$\deg f \cdot \mathrm{vol}\, \mathrm{S}^{\mathrm{N}} \leqslant \frac{1}{\mathrm{N}^{\mathrm{N}/2}} \int_{\mathrm{S}^{\mathrm{N}}} (\mathrm{tr}\, \mathrm{D}f^*\mathrm{D}f)^{\mathrm{N}/2}(*1_{\mathrm{S}^{\mathrm{N}}}) = \frac{\mathrm{E}_{\mathrm{N}}(f)}{\mathrm{N}^{\mathrm{N}/2}},$$

and the result follows by a simple rearrangement of the terms in this expression. ■

2.4 Theorem. Given any natural N and a smooth function $f : \mathrm{S}^{\mathrm{N}} \longrightarrow \mathrm{S}^{\mathrm{N}}$, if $p \geqslant \mathrm{N}$ is real, then

$$\mathrm{E}_p(f) \geqslant \mathrm{N}^{p/2} \cdot (\deg f)^{p/\mathrm{N}} \cdot \mathrm{vol}\, \mathrm{S}^{\mathrm{N}}.$$

Proof. The result is a direct consequence of § 2.2 and § 2.3. ■

2.5 Corollary. Let n be a natural number. We know that the 2-dimensional sphere can be easily identified with the completion of the complex plane. The function $f_n : \mathrm{S}^2 \longrightarrow \mathrm{S}^2$ that, under that identification, takes any $z \in \bar{\mathbb{C}}$ to z^n is a minimiser of the 2-energy within its homotopy class.

Proof. According to Theorem 2.4, the smallest possible 2-energy of a function from S^2 to itself with degree n must be

$$2^{2/2} \cdot n^{2/2} \cdot (4\pi) = 8\pi n.$$

The result then follows from the fact that $\deg f_n = n$ and from § III-1.6. ■

3 A general lower bound for the energy

3.1 In the previous section, we were able to estimate the energy of a function from a sphere onto itself in terms of the degree. The estimate was simple and elegant, but its scope of application was very limited, for it did not work for functions between spheres of different dimension.

In this section, we will find a general lower bound for the p-energy: a lower bound for the p-energy of any surjective function $f : \mathrm{S}^{\mathrm{N}} \longrightarrow \mathrm{S}^{\mathrm{K}}$ between spheres of any dimension, only subject — in accordance with our results from section 1 — to the condition that $p > \mathrm{N}$.

As promising as this may seem, there are a few caveats. For starters, this general lower bound will be significantly more difficult to both prove and compute. Moreover, it will only provide a global estimate for surjective functions, but, beyond that, it will not allow us to obtain different estimates within different homotopy classes.

The estimate will be introduced in Theorem 3.5, and its proof will rely on some technical lemmas that precede it. The proofs of these lemmas could very well be omitted on a first reading.

3.2 Lemma. Let $\lambda \geqslant 0$ be a real number and let us consider the function

$$\begin{aligned} \Xi_\lambda : \mathbb{R}_{>0} \times \mathrm{S}^{\mathrm{N}-1} &\longrightarrow \mathbb{R}^{\mathrm{N}} \\ (t, \Theta) &\longmapsto (\lambda, 0, \ldots, 0) + t \cdot \Theta, \end{aligned}$$

together with the inverse chart of a stereographic projection of the N-dimensional sphere:

$$\begin{aligned} \varphi^{-1} : \mathbb{R}^{\mathrm{N}} &\longrightarrow \mathrm{S}^{\mathrm{N}} \\ y &\longmapsto \frac{1}{1 + \|y\|^2}\Big(2y_1, \ldots, 2y_{\mathrm{N}}, 1 - \|y\|^2\Big). \end{aligned}$$

If we define the function $\Lambda_\lambda := \varphi^{-1} \circ \Xi_\lambda$, then, at an arbitrary yet fixed point (t, Θ) in $\mathbb{R}_{>0} \times \mathrm{S}^{\mathrm{N}-1}$,

$$\|\mathrm{D}\Lambda_\lambda(\partial_t)\| = \frac{2}{1 + \|t\Theta + \lambda e_1\|^2}, \qquad \det \mathrm{D}\Lambda_\lambda = t^{\mathrm{N}-1}\left(\frac{2}{1 + \|t\Theta + \lambda e_1\|^2}\right)^{\mathrm{N}},$$

where $e_1 = (1, 0, \ldots, 0)$ is the canonical basis vector of $\mathbb{R}^{\mathrm{N}}$.

Proof. Let $v_1, \ldots, v_{\mathrm{N}-1}$ be an orthonormal basis of the tangent space to $\mathrm{S}^{\mathrm{N}-1}$ at Θ. It is easy to check that, at (t, Θ), for any $i = 1, \ldots, \mathrm{N} - 1$,

$$\|\mathrm{D}\Xi_\lambda(v_i)\| = t, \qquad \|\mathrm{D}\Xi_\lambda(\partial_t)\| = 1.$$

What is more, the images of these basis vectors (v_i and ∂_t) under $\mathrm{D}\Xi_\lambda$ can be readily shown to be orthogonal, so we must have

$$\det \mathrm{D}\Xi_\lambda = t^{\mathrm{N}-1}.$$

In regard to the mapping φ^{-1}, consider an arbitrary point $y \in \mathbb{R}^N$. Given any $i, k = 1, \dots, N$, we have

$$\frac{\partial}{\partial y_i}\varphi_k^{-1} = \frac{\partial}{\partial y_i}\frac{2y_k}{1 + y_1^2 + \cdots + y_N^2} = \frac{2\big(1 + y_1^2 + \cdots + y_N^2\big)\delta_{ik} - 4y_i y_k}{\big(1 + y_1^2 + \cdots + y_N^2\big)^2}.$$

$$\frac{\partial}{\partial y_i}\varphi_{N+1}^{-1} = \frac{\partial}{\partial y_i}\frac{1 - \big(y_1^2 + \cdots + y_N^2\big)}{1 + y_1^2 + \cdots + y_N^2} = -\frac{4y_i}{\big(1 + y_1^2 + \cdots + y_N^2\big)^2},$$

which can be simplified and rewritten as

$$\frac{\partial}{\partial y_i}\varphi_k^{-1} = \frac{2\big(1 + \|y\|^2\big)\delta_{ik} - 4y_i y_k}{\big(1 + \|y\|^2\big)^2}, \qquad \frac{\partial}{\partial y_i}\varphi_{N+1}^{-1} = -\frac{4y_i}{\big(1 + \|y\|^2\big)^2}.$$

With this information, we can compute the norm of $D\varphi^{-1}(\partial_{y_i})$ at y as

$$\begin{aligned} &\frac{1}{\big(1 + \|y\|^2\big)^2}\sqrt{4\big(1 + \|y\|^2\big)^2 - 16\big(1 + \|y\|^2\big)y_i^2 + 16y_i^2 + 16\sum_{k=1}^{N}(y_i y_k)^2} \\ &= \frac{1}{\big(1 + \|y\|^2\big)^2}\sqrt{4\big(1 + \|y\|^2\big)^2 - \cancel{16\Big(1 + \|y\|^2\Big)y_i^2 + 16y_i^2 + 16y_i^2\|y\|^2}} \\ &= \frac{2}{1 + \|y\|^2} = \|D\varphi^{-1}(\partial_{y_i})\|. \end{aligned}$$

With our expressions for $\partial\varphi^{-1}/\partial y_i$, it is straightforward to check that the vectors $\{D\varphi^{-1}(\partial_{y_i})\}$ are orthogonal. Taking this into account together with our result for $\|D\varphi^{-1}(\partial_{y_i})\|$ and the fact that $\{\partial_{y_i}\}_i$ is orthonormal yields that, for any $y \in \mathbb{R}^N$ and any $v \in T_y\mathbb{R}^N$,

$$\|D\varphi^{-1}(v)\| = \frac{2}{1 + \|y\|^2}\|v\|.$$

Thus, we will obviously have

$$\det D\varphi_y^{-1} = \left(\frac{2}{1 + \|y\|^2}\right)^N.$$

We can now compute the norm of the derivative with respect to t of Λ_λ at a point (t, Θ) as

$$\|D\Lambda_\lambda(\partial_t|_{(t,\theta)})\| = \big\|D\varphi^{-1}\big(D\Xi_l(\partial_t|_{(t,\theta)})\big)\big\| = \frac{2}{1 + \|le_1 + t\Theta\|^2}.$$

Lastly, regarding the determinant of $D\Lambda_\lambda$, we have that

$$\det D\Lambda_\lambda|_{(t,\Theta)} = \det D\varphi_{\lambda e_1 + t\Theta}^{-1} \cdot \det D\Xi_\lambda|_{(t,\Theta)} = t^{N-1}\left(\frac{2}{1 + \|\lambda e_1 + t\Theta\|^2}\right)^N.$$

This concludes the proof. ∎

3.3 Scholium. Under the hypotheses of 3.2, the function Ξ_λ is a family of curves in the sense that, for every $\Theta \in S^{N-1}$, the function $\Xi(-,\Theta)$ defines a ray from λe_1 to infinity in the direction Θ. Consequently, if we consider a stereographic projection such as the φ taken in 3.2, $\Lambda_\lambda = \varphi^{-1} \circ \Xi_\lambda$ will be a family of curves joining the points taken to λe_1 and infinity by the stereographic projection.

As a final important remark, the function Λ_λ defines a diffeomorphism between $\mathbb{R}_{>0} \times S^{N-1}$ and the N-dimensional sphere up to a measure-zero subset.

3.4 Lemma. Let N be a natural number and let $p > N$ be a real number. If we define the function

$$\begin{aligned} I : \mathbb{R}_{\geqslant 0} \times S^{N-1} \times \mathbb{R} &\longrightarrow \mathbb{R} \\ (\lambda, \Theta, t) &\longmapsto t^{-\frac{N-1}{p-1}} \left[\frac{2}{1 + \|\lambda e_1 + t\Theta\|^2} \right]^{\frac{p-N}{p-1}}, \end{aligned}$$

where $e_1 \in \mathbb{R}^N$ is the canonical basis vector, then we can bound

$$\int_{S^{N-1}} \int_0^\infty I(\lambda, \Theta, t)\, dt\, d\Theta \leqslant \operatorname{vol} S^{N-1} \cdot \int_0^\infty t^{-\frac{N-1}{p-1}} \left(\frac{2}{1+t^2} \right)^{\frac{p-N}{p-1}} dt.$$

Proof. Doing some basic algebra, we can expand

$$\begin{aligned} I(\lambda, \Theta, t) &= t^{-\frac{N-1}{p-1}} \left[\frac{2}{1 + \|\lambda e_1 + t\Theta\|^2} \right]^{\frac{p-N}{p-1}} \\ &= t^{-\frac{N-1}{p-1}} \left[\frac{2}{1 + \lambda^2 + t^2 + 2\langle \Theta \mid e_1 \rangle \lambda t} \right]^{\frac{p-N}{p-1}} \end{aligned} \tag{1}$$

Fixing some arbitrary values for λ and Θ, let us prove that $\int_0^\infty I(\lambda, \Theta, t)\, dt$ converges. On the one hand, the integral does not diverge at the limit $t \longrightarrow 0$ because, since $p > N$, we have $-1 < -(N-1)/(p-1) < 0$. On the other hand, it does not diverge at the limit $t \longrightarrow \infty$ because

$$-\frac{N-1}{p-1} - 2\frac{p-N}{p-1} = -\frac{N-1+2p-2N}{p-1} = -\frac{p-1+(p-N)}{p-1} < -\frac{p-1}{p-1} = -1.$$

Now that we know that the integral $\int_0^\infty I(\lambda, \Theta, t)\, dt$ converges for any values of Θ and λ, we aim to prove that

$$\max_{\lambda \geqslant 0} \int_{S^{N-1}} \int_0^\infty I(\lambda, \Theta, t)\, dt\, d\Theta = \int_{S^{N-1}} \int_0^\infty I(0, \Theta, t)\, dt\, d\Theta.$$

If we show this, the result will then follow trivially just by having a look at (1). To that end, we should first notice how, for any $\lambda \geqslant 0$,

$$\int_{S^{N-1}} \int_0^\infty I(\lambda, \Theta, t)\, dt\, d\Theta = \int_{S^{N-1}} \int_0^\infty t^{-\frac{N-1}{p-1}} \left[\frac{2}{1 + \lambda^2 + t^2 + 2\langle \Theta \mid e_1 \rangle \lambda t} \right]^{\frac{p-N}{p-1}} dt\, d\Theta$$

$$= \int_{\mathrm{S}^{\mathrm{N}-1}} \int_0^\infty \underbrace{|t|^{-\frac{\mathrm{N}-1}{p-1}} \left[\frac{2}{1 + \lambda^2 + t^2 + 2\langle \Theta \mid e_1 \rangle \lambda t} \right]^{\frac{p-\mathrm{N}}{p-1}}}_{\mathrm{J}(\lambda,\Theta,t)} \, dt \, d\Theta$$

$$= \frac{1}{2} \int_{\mathrm{S}^{\mathrm{N}-1}} \int_{-\infty}^\infty \mathrm{J}(\lambda, \Theta, t) \, dt \, d\Theta$$

because, for any real t and any $\Theta \in \mathrm{S}^{\mathrm{N}-1}$, we have $\mathrm{J}(\lambda, \Theta, -t) = \mathrm{J}(\lambda, -\Theta, t)$. Thus, if we manage to prove that, for almost every $\Theta \in \mathrm{S}^{\mathrm{N}-1}$,

$$\max_{\lambda \geqslant 0} \int_{-\infty}^\infty \mathrm{J}(\lambda, \Theta, t) \, dt = \int_{-\infty}^\infty \mathrm{J}(0, \Theta, t) \, dt, \tag{2}$$

we will have already proved our result. With all of this in mind, let us first show that

$$\lim_{\lambda \to \infty} \int_{-\infty}^\infty \mathrm{J}(\lambda, \Theta, t) \, dt = 0$$

for almost every $\Theta \in \mathrm{S}^{\mathrm{N}-1}$.

If we take any $\Theta \in \mathrm{S}^{\mathrm{N}-1}$ and any real t, it is obvious that the limit of $\mathrm{J}(\lambda, \Theta, t)$ as $\lambda \to \infty$ has to be zero. Thus, according to the Dominated Convergence Theorem [§ B.2], if, for any fixed $\Theta \in \mathrm{S}^{\mathrm{N}-1}$, we can find an integrable function g_Θ bounding $\mathrm{J}(\lambda, \Theta, -)$ for every $\lambda > 0$, then

$$\lim_{\lambda \to \infty} \int_{-\infty}^\infty \mathrm{J}(\lambda, \Theta, t) \, dt = \int_{-\infty}^\infty \lim_{\lambda \to \infty} \mathrm{J}(\lambda, \Theta, t) \, dt = 0.$$

Let us denote $\Theta_1 := \langle \Theta \mid e_1 \rangle$ for our fixed value of Θ. For any real $t, \lambda \geqslant 0$, we have

$$\lambda^2 + 2\Theta_1 \lambda t \geqslant \lambda^2 - 2|\Theta_1|\lambda t \geqslant -(|\Theta_1| t)^2 \tag{3}$$

since the function $\lambda \mapsto \lambda^2 - 2|\Theta_1|\lambda t$ reaches its minimum at $\lambda = |\Theta_1| t$. Therefore, we can just consider

$$g_\Theta(t) := t^{-\frac{\mathrm{N}-1}{p-1}} \left[\frac{2}{1 + \left(1 - |\Theta_1|^2\right) t^2} \right]^{\frac{p-\mathrm{N}}{p-1}} \geqslant t^{-\frac{\mathrm{N}-1}{p-1}} \left[\frac{2}{1 + t^2 + \lambda^2 + 2\Theta_1 \lambda t} \right]^{\frac{p-\mathrm{N}}{p-1}}.$$

as dominating function under the assumption that $|\Theta_1| \neq 1$. This is not a problem, for the set of points $\Theta \in \mathrm{S}^{\mathrm{N}-1}$ with $|\Theta_1| = 1$ has measure zero.

We thus know that, for almost every $\Theta \in \mathrm{S}^{\mathrm{N}-1}$, the limit of $\int_{-\infty}^\infty \mathrm{J}(\lambda, \Theta, t) \, dt$ as $\lambda \to \infty$ is zero. Let us now try to compute, for any Θ, the derivative

$$\frac{\partial}{\partial \lambda} \int_0^\infty \mathrm{J}(\lambda, \Theta, t) \, dt.$$

If we prove that — for almost every Θ — this derivative is zero only when $\lambda = 0$ and different from 0 elsewhere, we will know that the maximum value of the integral $\int_0^\infty \mathrm{J}(\lambda, \Theta, t) \, dt$ is attained when $\lambda = 0$ (for almost every Θ), and thus we will have proved (2), and hence our result. It is worth mentioning that, if a positive differentiable function that approaches zero at infinity has a single critical point, it must be its global maximum.

In what follows, just as before, fix any $\Theta \in S^{N-1}$ with $|\Theta_1| \neq 1$. Using (3), we can bound J by an integrable function with no dependence on λ as

$$J(\lambda, \Theta, t) \leqslant |t|^{-\frac{N-1}{p-1}} \left[\frac{2}{1 + \left(1 - |\Theta_1|^2\right) t^2} \right]^{\frac{p-N}{p-1}}.$$

Moreover, regarding the derivative of J with respect to λ, we have

$$\frac{\partial J}{\partial \lambda} = -\frac{p-N}{p-1} |t|^{-\frac{N-1}{p-1}} \left[\frac{2}{1 + t^2 + \lambda^2 + 2\Theta_1 \lambda t} \right]^{\frac{p-N}{p-1}+1} (\lambda + t\Theta_1).$$

Let us say that we want to find the derivative $\partial_\lambda \int_0^\infty J(\lambda, \Theta, t)\, dt$ at some point $\lambda = \lambda_0$. We may restrict λ to an interval $[\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$ and, on this interval, we can always bound $\partial_\lambda J$ by a function with no dependence on λ as

$$\left| \frac{\partial J}{\partial \lambda} \right| \leqslant \frac{p-N}{p-1} |t|^{-\frac{N-1}{p-1}} \left[\frac{2}{1 + (1 - |\Theta_1|^2) t^2} \right]^{\frac{p-N}{p-1}+1} ((\lambda_0 + \varepsilon) + t|\Theta_1|),$$

which is integrable over $t \in \mathbb{R}$.

We can hence apply the Leibniz integral rule [§ B.3] to deduce that

$$\frac{\partial}{\partial \lambda} \int_{-\infty}^{\infty} J(\lambda, \Theta, t)\, dt = -\frac{p-N}{p-1} \int_{-\infty}^{\infty} \underbrace{|t|^{-\frac{N-1}{p-1}}}_{\rho(t)} \underbrace{\left[\frac{2}{1 + t^2 + \lambda^2 + 2\Theta_1 \lambda t} \right]^{\frac{p-N}{p-1}+1}}_{R(\lambda,\Theta,t)} (\lambda + t\Theta_1)\, dt,$$

which clearly vanishes at $\lambda = 0$. If we now show that this derivative does not vanish at any point other than $\lambda = 0$, we will have concluded our proof. In order to do this, we just have to prove that, if $\lambda \neq 0$, we necessarily have

$$\int_{-\infty}^{-\lambda/\Theta_1} \rho(t) \cdot R(\lambda, \Theta, t) \cdot (\lambda + t\Theta_1)\, dt + \int_{-\lambda/\Theta_1}^{\infty} \rho(t) \cdot R(\lambda, \Theta, t) \cdot (\lambda + t\Theta_1) \neq 0. \quad (4)$$

The set of points with $\Theta_1 = 0$ has measure zero, so we can safely assume that $\Theta_1 \neq 0$. Under this assumption, the inequality is equivalent to

$$\int_{-\infty}^{-\lambda/\Theta_1} \rho(t) \cdot R(\lambda, \Theta, t) \cdot \left| t + \frac{\lambda}{\Theta_1} \right| dt \neq \int_{-\lambda/\Theta_1}^{\infty} \rho(t) \cdot R(\lambda, \Theta, t) \cdot \left| t + \frac{\lambda}{\Theta_1} \right| dt. \quad (4)$$

We shall now prove that this holds for any $\lambda \neq 0$ (and any Θ with $\Theta_1 \neq 0$).

We know that $t \mapsto \rho(t)$ is symmetric and has a unique maximum at $t = 0$, and it is easy to verify that $t \mapsto R(\lambda, \Theta, t)$ is also symmetric around a unique maximum at $t = -\Theta_1 \lambda$. What is more, both of these functions are positive and strictly monotonous except at their maxima, and their limits as $t \longrightarrow \pm\infty$ are zero.

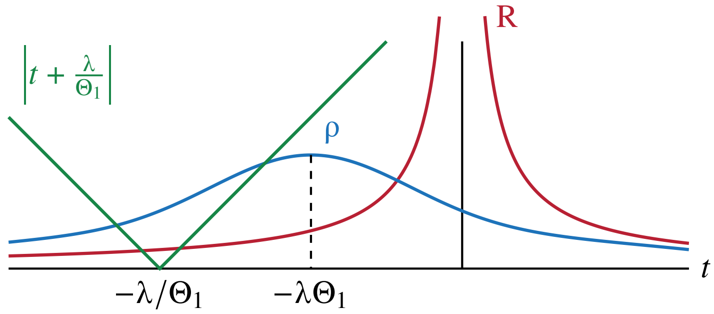

Figure 3.1: Schematic representation (not to scale) of the functions ρ, R and $|t+\lambda/\Theta_1|$ when $\lambda \neq 0$ and $\Theta_1 > 0$.

Without loss of generality, let us assume that $\Theta_1 > 0$; the case $\Theta_1 < 0$ would be analogous. Since $0 < \Theta_1 < 1$, it is clear how

$$-\frac{\lambda}{\Theta_1} < -\lambda\Theta_1 < 0,$$

so $-\lambda/\Theta_1$ is smaller than the points at which R and ρ reach its maxima. We can then easily deduce that the integral on the right hand side of (4) will necessarily be greater than then one of the left hand side, thus making equality impossible under the assumption that $\lambda \neq 0$. This is illustrated in Figure 3.1. ∎

3.5 Theorem. Let N and K be a pair of natural numbers and let $p > \mathrm{N}$ be a real number. If $p \geqslant 2$ and $f : \mathrm{S}^\mathrm{N} \longrightarrow \mathrm{S}^\mathrm{K}$ is surjective, then we can bound its p-energy as

$$\mathrm{E}_p(f) \geqslant \frac{\pi^p \cdot \operatorname{vol}\mathrm{S}^{\mathrm{N}-1}}{\mathrm{N}^{p/2}}\left[\int_0^\infty t^{-\frac{\mathrm{N}-1}{p-1}}\left(\frac{2}{1+t^2}\right)^{\frac{p-\mathrm{N}}{p-1}} dt\right]^{1-p}.$$

Proof. Consider two arbitrary points a and b on the sphere S^N such that $f(a) = -f(b)$. Since f is assumed to be surjective, we can always find such a pair of points. We may now define a stereographic projection φ taking b to infinity and a to a point of the form λe_1 with $\lambda \geqslant 0$, where $e_1 \in \mathbb{R}^\mathrm{N}$ is the canonical basis vector. We can then define a function Λ_λ like the one introduced in § 3.2, but using our stereographic projection φ and our value for λ.

As we discussed in § 3.3, for every $\Theta \in \mathrm{S}^{\mathrm{N}-1}$, the curve $t \in \mathbb{R}_{\geqslant 0} \mapsto \Lambda(t,\Theta)$ will join a with b, which means that

$$\pi = d(f(a), f(b)) \leqslant \int_0^\infty \|\mathrm{D}f_{\Lambda_\lambda(t,\Theta)}\|_{\mathrm{op}} \cdot \|\mathrm{D}\Lambda_\lambda(\partial_t|_{(t,\Theta)})\|\, dt.$$

Taking into account that Λ_λ is a diffeomorphism between $\mathbb{R}_{>0} \times \mathrm{S}^{\mathrm{N}-1}$ and S^N up to a measure-zero subset, we can integrate over all the possible Θ in $\mathrm{S}^{\mathrm{N}-1}$ as follows:

$$\pi \leqslant \frac{1}{\operatorname{vol}\mathrm{S}^{\mathrm{N}-1}} \int_{\mathrm{S}^{\mathrm{N}-1}} \int_0^\infty \|\mathrm{D}f_{\Lambda_\lambda(t,\Theta)}\|_{\mathrm{op}} \|\mathrm{D}\Lambda_\lambda(\partial_t|_{(t,\Theta)})\|\, dt\, d\Theta$$

$$
\begin{aligned}
&= \frac{1}{\operatorname{vol} S^{N-1}} \int_{S^N} \|Df_{\Lambda_\lambda(x)}\|_{op} \|D\Lambda_\lambda(\partial_t|_x)\| \frac{1}{\det D\Lambda_\lambda}\, dx \\
&= \frac{\sqrt{N}}{\operatorname{vol} S^{N-1}} \left(\int_{S^N} \|Df\|_{HS}^p (*1_{S^N}) \right)^{1/p} \cdot \left(\int_{S^N} \left(\frac{\|D\Lambda_\lambda(\partial_t)\|}{\det D\Lambda_\lambda} \right)^{p^*} (*1_{S^N}) \right)^{1/p^*} \\
&= \frac{\sqrt{N}}{\operatorname{vol} S^{N-1}} E_p(f)^{1/p} \left[\int_{S^{N-1}} \int_0^\infty \|D\Lambda_\lambda(\partial_t|_{(t,\Theta)})\|^{p^*} (\det D\Lambda_\lambda)^{1-p^*}\, dt\, d\Theta \right]^{1/p^*},
\end{aligned}
$$

where we have used Hölder's inequality [§ B.1] and the fact that we can always bound $\|Df\|_{HS} \leqslant \sqrt{N}\|Df\|_{op}$. Regarding the exponents in the integral, it is easy to check that

$$
p^* = \frac{p}{p-1}, \qquad 1 - p^* = -\frac{1}{p-1},
$$

so we know that

$$
\begin{aligned}
&\int_{S^{N-1}} \int_0^\infty \|D\Lambda_\lambda(\partial_t|_{(t,\Theta)})\|^{p^*} (\det D\Lambda_\lambda)^{1-p^*}\, dt\, d\Theta = \\
&\quad = \int_{S^N} \int_0^\infty \left(\frac{2}{1 + \|\lambda e_1 + \Theta t\|^2} \right)^{p^*} \left[t^{N-1} \left(\frac{2}{1 + \|\lambda e_1 + \Theta t\|^2} \right)^N \right]^{1-p^*} dt\, d\Theta \\
&\quad = \int_{S^N} \int_0^\infty t^{(N-1)(1-p^*)} \left[\frac{2}{1 + \|\lambda e_1 + \Theta t\|^2} \right]^{p^* + N(1-p^*)} dt\, d\Theta \\
&\quad = \int_{S^N} \int_0^\infty t^{-\frac{N-1}{p-1}} \left[\frac{2}{1 + \|\lambda e_1 + \Theta t\|^2} \right]^{\frac{p-N}{p-1}} dt\, d\Theta.
\end{aligned}
$$

We can hence invoke § 3.4 to conclude that

$$
\pi \leqslant \frac{\sqrt{N}}{\operatorname{vol} S^{N-1}} E_p(f)^{1/p} \left[\operatorname{vol} S^{N-1} \int_0^\infty t^{-\frac{N-1}{p-1}} \left(\frac{2}{1+t^2} \right)^{\frac{p-N}{p-1}} dt \right]^{1/p^*}.
$$

This expression can be rearranged as

$$
E_p(f)^{1/p} \geqslant \frac{\pi (\operatorname{vol} S^{N-1})^{1/p}}{\sqrt{N}} \left[\int_0^\infty t^{-\frac{N-1}{p-1}} \left(\frac{2}{1+t^2} \right)^{\frac{p-N}{p-1}} dt \right]^{-1/p^*},
$$

from where the result follows trivially by noticing that $p/p^* = p - 1$. ∎

4 Energy estimation in terms of the Hopf invariant

4.1 In Section 2, we introduced an estimate for the p-energy of a smooth function from a sphere to itself in terms the degree of the function.

In this section, we will introduce an estimate for the p-energy of a function in terms of its Hopf invariant. Just as the degree is only defined for functions from a sphere to itself, the Hopf invariant can only be computed for functions from a sphere of dimension $2K + 1$ to one of dimension K, for a fixed K. This will mean, in turn, that our estimate will only work for functions between spheres of these dimensions.

We will construct our estimate of the energy in terms of the Hopf invariant using Whitehead's integral formula [§ II-1.14]. This formula contains an instance of the 'inverse exterior derivative' of a form, which is denoted as d^{-1}. Thus, before we can introduce our estimate for the energy, which we will do in Theorem 4.7, we will have to spend some time discussing how such an inverse can be computed.

4.2 Notation and conventions. Given any vector $v \in \mathbb{R}^N$ and any variable x, we will write v_x to denote differentiation on the variable x in the direction v. Moreover, if w is also a vector in $\mathbb{R}^N$, we will denote the orthogonal component of v with respect to w as $(v)^{\perp w}$. For any vector v, we will write $\hat{v}$ to denote its normalisation: $\hat{v} := v/\|v\|$.

We will identify the tangent vectors to S^{N-1} or $\mathbb{R}^N$ (as a smooth manifold) with their Euclidean representation [§ I-5.4(ii)] in $\mathbb{R}^N$ (as a Euclidean vector space).

4.3 Proposition. Let ω be a smooth $(k+1)$-form defined over the Euclidean space $\mathbb{R}^N$ except for a measure-zero subset, and let $v_1, \ldots, v_k$ be vectors in that space. We define

$$\mu_x(v_1, \ldots, v_k) := \frac{1}{\operatorname{vol} S^{N-1}} \int_{\mathbb{R}^N} \frac{\omega_y(x-y, v_1, \ldots, v_k)}{\|x-y\|^N}\, dy.$$

Let $U \subseteq \mathbb{R}^N$ be an open set over which ω is smooth. If the integral defining μ converges absolutely for every $x \in U$ and if, for any vector $v \in \mathbb{R}^N$, so does the integral

$$\int_{S^{N-1}} \int_0^\infty v_x \omega_{x+r\Theta}(\Theta, v_1, \ldots, v_k)\, dr\, d\Theta$$

for $x \in U$, then μ is a k-form in U such that $d\mu = \omega$.

Proof. Let us take an arbitrary $x \in U$. We begin by defining a change of coordinates through the function

$$\begin{aligned} y_x : S^{N-1} \times \mathbb{R}_{>0} &\longrightarrow \mathbb{R}^N \\ (\Theta, r) &\longmapsto x + r\Theta, \end{aligned}$$

which amounts to considering spherical coordinates around x.

Clearly, $\det \mathrm{D}y_x = r^{N-1}$, so we can transform μ as

$$\begin{aligned} \mu_x(v_1, \ldots, v_k) &:= \frac{1}{\operatorname{vol} S^{N-1}} \int_{S^{N-1}} \int_0^\infty \frac{\omega_{x+r\Theta}(-r\Theta, v_1, \ldots, v_k)}{r^N} r^{N-1}\, dr\, d\Theta \\ &= \frac{1}{\operatorname{vol} S^{N-1}} \int_{S^{N-1}} \int_\infty^0 \omega_{x+r\Theta}(\Theta, v_1, \ldots, v_k)\, dr\, d\Theta. \end{aligned}$$

Since the integrand of the expression above and its partial derivatives are both integrable by hypothesis, we can use result I-3.6(ii) in conjunction with Leibniz's integral rule [§ B.3] to deduce that

$$(d\mu)_x(v_0, \ldots, v_k) =$$

$$= \frac{1}{\operatorname{vol} S^{N-1}} \sum_i (-1)^i (v_i)_x \int_{S^{N-1}} \int_\infty^0 \omega_{x+r\Theta}(\Theta, v_0, \dots, \hat{v}_i, \dots, v_k)\, dr\, d\Theta$$
$$= \frac{1}{\operatorname{vol} S^{N-1}} \int_{S^{N-1}} \int_\infty^0 \sum_i (-1)^i (v_i)_x \omega_{x+r\Theta}(\Theta, v_0, \dots, \hat{v}_i, \dots, v_k)\, dr\, d\Theta.$$

Since we are assuming ω to be closed, and hence $d\omega_{x+r\Theta}(\Theta, v_0, \dots, v_k) = 0$, then we must have

$$\Theta_x \omega_{x+r\Theta}(v_0, \dots, v_k) - \sum_{i=0}^{k} (-1)^i (v_i)_x \omega_{x+r\Theta}(\Theta, v_0, \dots, \hat{v}_i, \dots, v_k) = 0,$$

so our expression for $d\mu$ can be further reduced to

$$\begin{aligned}(d\mu)_x(v_0, \dots, v_k) &= \frac{1}{\operatorname{vol} S^{N-1}} \int_{S^{N-1}} \int_\infty^0 -0 + \Theta_x \omega_{x+r\Theta}(v_0, \dots, v_k)\, dr\, d\Theta \\ &= \frac{1}{\operatorname{vol} S^{N-1}} \int_{S^{N-1}} \omega_x(v_0, \dots, v_k)\, d\Theta = \omega_x(v_0, \dots, v_k).\end{aligned}$$

In this last deduction, we have relied on the fact that differentiating $\omega_{x+r\Theta}(-)$ with respect to x in the direction Θ yields the same result as differentiating it with respect to the variable r.

We have then shown that, indeed, $(d\mu)_x = \omega_x$ for every $x \in U$. ■

4.4 Scholium. The hypotheses concerning the absolute integrability of functions in the previous result are automatically satisfied if ω is a differential form with compact support.

4.5 Lemma. Let x_0 be a point in $\mathbb{R}^N$ and let U be an open ball around it. Given any real $k < N$, if f is a bounded function over U, then

$$\int_U \left| \frac{f(x)}{\|x - x_0\|^k} \right| dx < \infty.$$

Proof. Since the function f is bounded over U,

$$\int_U \left| \frac{f(x)}{\|x - x_0\|^k} \right| dx \leqslant \|f\|_{L^\infty(U)} \int_U \frac{1}{\|x - x_0\|^k}\, dx,$$

so we only need to be concerned with the integral of $\|x - x_0\|^{-k}$. For this, we can switch to spherical coordinates around x_0 through $x = x_0 + r\Theta$ for $r > 0$ and $\Theta \in S^{N-1}$. Thus, if $R > 0$ is the radius of U,

$$\int_U \frac{1}{\|x - x_0\|^k}\, dx = \int_{S^{N-1}} \int_0^R \frac{r^{N-1}}{r^k}\, dr\, d\Theta.$$

As $k < N$, this integral must be absolutely convergent. ■

4.6 Theorem. Let ω be a $(k+1)$-form on the $(\mathrm{N}-1)$-dimensional sphere. If ω is closed and $1 \leqslant k+1 < \mathrm{N}-1$, then the k-form

$$\mu_x(v_1, \ldots, v_k) = \frac{1}{\operatorname{vol} \mathrm{S}^{\mathrm{N}-1}} \int_{\mathrm{S}^{\mathrm{N}-1}} \omega_\Theta\left(\int_0^\infty \frac{r^{\mathrm{N}-k-2}}{\|x - r\Theta\|^{\mathrm{N}}}\, dr\, x^{\perp\Theta}, v_1^{\perp\Theta}, \ldots, v_k^{\perp\Theta}\right) d\Theta,$$

is well-defined on the sphere and verifies $d\mu = \omega$. In the sequel, we will write $d^{-1}\omega$ to represent this form μ.

Proof. Let us fist extend $\omega \in \Omega^{k+1}(\mathrm{S}^{\mathrm{N}-1})$ to a form $\bar{\omega} \in \Omega^{k+1}(\mathbb{R}^{\mathrm{N}} \setminus \{0\})$ which we define to be the pullback of ω through

$$\begin{aligned} \mathbb{R}^{\mathrm{N}} \setminus \{0\} &\longrightarrow \mathrm{S}^{\mathrm{N}-1} \\ x &\longmapsto \frac{x}{\|x\|}. \end{aligned}$$

If $v_0, \ldots, v_k$ are tangent vectors to $\mathrm{S}^{\mathrm{N}-1}$ at a point x, we have

$$\bar{\omega}_x(v_0, \ldots, v_k) = \frac{1}{\|x\|^{k+1}} \omega_{x/\|x\|}(v_0^\perp, \ldots, v_k^\perp),$$

where — as we will do throughout the proof — we have used $v_i^\perp$ to denote the perpendicular component of v_i with respect to the base-point of ω (in our case, x). This shows how $\bar{\omega}$ is, indeed, an extension of ω. Of course, in the expression above, we are identifying the tangent vectors to a point x in $\mathbb{R}^{\mathrm{N}}$ with the tangent vectors to $x/\|x\|$ that have the same Euclidean representation.

The pullback of any closed differential form is closed [§ I-3.6(iii)], so — according to our hypotheses — the extension $\bar{\omega}$ must be closed itself.

Let $v_1, \ldots, v_k$ be some vectors in $\mathbb{R}^{\mathrm{N}}$. We will compute the inverse exterior derivative of forms ω in spheres as the inverse of their corresponding extensions $\bar{\omega}$. Thus, with an eye on § 4.3, we may consider

$$\begin{aligned} \bar{\mu}_x(\tilde{v}_1, \ldots, \tilde{v}_k) &:= \frac{1}{\operatorname{vol} \mathrm{S}^{\mathrm{N}-1}} \int_{\mathbb{R}^{\mathrm{N}}} \frac{\bar{\omega}_y(x-y, v_1^\perp, \ldots, v_k^\perp)}{\|x-y\|^{\mathrm{N}}}\, dy \\ &= \frac{1}{\operatorname{vol} \mathrm{S}^{\mathrm{N}-1}} \int_{\mathbb{R}^{\mathrm{N}}} \frac{\omega_{y/\|y\|}\big((x-y)^\perp, v_1^\perp, \ldots, v_k^\perp\big)}{\|y\|^{k+1}\|x-y\|^{\mathrm{N}}}\, dy = (*). \end{aligned}$$

We should keep in mind that, as before, we have used the ⊥ superscript to refer to the orthogonal component of a vector with respect to the base-point of the differential form that it is being fed to.

Now we need to ask ourselves whether the integral above converges absolutely or not, but, before doing so, we should refine our expression for its integrand. Let us first deal with the term $(x-y)^\perp$, which represents the orthogonal component of $x-y$ with respect to y. Since

$$(x-y)^{\perp y} = (x-y) - \frac{\langle x-y \mid y\rangle}{\|y\|^2}(y) = \|x-y\|(\widehat{x-y} - \langle \widehat{x-y} \mid \hat{y}\rangle \hat{y}),$$

we may rewrite our integral as

$$(*) = \frac{1}{\operatorname{vol} S^{N-1}} \int_{\mathbb{R}^N} \frac{\omega_{\hat{y}}\big(\widehat{x-y} - \langle \widehat{x-y} \mid \hat{y}\rangle \hat{y}, v_1^{\perp}, \ldots, v_k^{\perp}\big)}{\|y\|^{k+1}\|x-y\|^{N-1}}\, dy$$

Let us fix any $x \neq 0$. We will partition $\mathbb{R}^N$ into three subsets: an open ball U_1 around 0, an open ball U_2 around x, and a subset U_0 which shall be the complement of $U_1 \cup U_2$. Of course, if our integrand is absolutely integrable over these three domains, it will be absolutely integrable over the whole Euclidean space. To prove integrability over U_1 and U_2, we only need to invoke Lemma 4.5. To prove integrability over U_0, we just have to take care of integrability when $y \to \infty$.

If we look at the first argument of ω in our integrand, we can rewrite it as

$$\begin{aligned}
\frac{x-y}{\|x-y\|} - \frac{\langle x-y \mid y\rangle}{\|x-y\|\|y\|^2} y &= \frac{1}{\|y\|}\left(\frac{\|y\|}{\|x-y\|}x - \left(\frac{\|y\|}{\|x-y\|} + \frac{\langle x-y \mid y\rangle}{\|x-y\|\|y\|}\right) y\right)\\
&= \frac{1}{\|y\|}\left(\frac{\|y\|}{\|x-y\|}x - \left(\frac{\|y\|^2}{\|x-y\|} + \frac{\langle x-y \mid y\rangle}{\|x-y\|}\right) \hat{y}\right)\\
&= \frac{1}{\|y\|}\left(\frac{\|y\|}{\|x-y\|}x - \left(\frac{\langle y \mid y\rangle + \langle x-y \mid y\rangle}{\|x-y\|}\right) \hat{y}\right)\\
&= \frac{1}{\|y\|}\underbrace{\left(\frac{\|y\|}{\|x-y\|}x - \left(\frac{\langle x \mid y\rangle}{\|x-y\|}\right) \hat{y}\right)}_{\text{bounded as } y \to \infty}.
\end{aligned}$$

It is then clear how, in the limit $y \to \infty$, the norm of this vector behaves as $\|y\|^{-1}$, hence the absolute value of the whole integrand will behave as

$$\frac{1}{\|y\|^{k+2} \cdot \|y\|^{N-1}},$$

which, since $k \geqslant 0$ by hypothesis, is absolutely integrable in the limit $y \to \infty$.

This proves that our integral converges absolutely, but — since $\bar{\omega}$ does not have compact support — we still have to show whether, for any vector $v \in \mathbb{R}^N$,

$$\int_{S^{N-1}} \int_0^{\infty} v_x \left(\frac{\omega_{\widehat{x+r\Theta}}\big(\Theta^{\perp}, v_1^{\perp}, \ldots, v_k^{\perp}\big)}{\|x+r\Theta\|^{k+1}}\right) dr\, d\Theta$$

converges absolutely.

For this, we should first notice how the integrand above can be expanded as the sum of two terms, namely.

$$\frac{v_x \omega_{\widehat{x+r\Theta}}\big(\Theta^{\perp}, v_1^{\perp}, \ldots, v_k^{\perp}\big)}{\|x+r\Theta\|^{k+1}} + \omega_{\widehat{x+r\Theta}}\big(\Theta^{\perp}, v_1^{\perp}, \ldots, v_k^{\perp}\big) v_x \frac{1}{\|x+r\Theta\|^{k+1}}.$$

We will analyse the two terms separately.

Let us work with the first term in the sum. We will begin by trying to find an upper bound for the absolute value of its numerator. Since spheres are compact, we know that the operator norm of the differential of

$$s \in \mathrm{S}^{\mathrm{N}-1} \longmapsto \omega_s(\Theta^{\perp}, v_1^{\perp}, \ldots, v_k^{\perp})$$

must be bounded, for any $\Theta \in \mathrm{S}^{\mathrm{N}-1}$, in its entire domain. On the other hand,

$$\begin{aligned}\left\| \mathrm{D}\left(x \mapsto \frac{x + r\Theta}{\|x + r\Theta\|} \right)_x \right\|_{\mathrm{op}} &= \|\mathrm{D}(x \mapsto x + r\Theta)_x\|_{\mathrm{op}} \cdot \left\| \mathrm{D}\left(v \mapsto \frac{v}{\|v\|} \right)_{x+r\Theta} \right\|_{\mathrm{op}} \\ &= 1 \cdot \frac{1}{\|x + r\Theta\|}.\end{aligned}$$

Hence, for a certain constant C_1 independent of Θ and r,

$$\left| \frac{v_x \omega_{\widehat{x+r\Theta}}(\Theta^{\perp}, v_1^{\perp}, \ldots, v_k^{\perp})}{\|x + r\Theta\|^{k+1}} \right| \leqslant \frac{\mathrm{C}_1\|v\|}{\|x + r\Theta\|^{k+2}}.$$

We can now deal with the second term in the sum. Following a similar argument, we have

$$\begin{aligned}\left\| \mathrm{D}\left(x \mapsto \frac{1}{\|x + r\Theta\|^{k+1}} \right)_x \right\|_{\mathrm{op}} &= \|\mathrm{D}(x \mapsto \|x + r\Theta\|)_x\|_{\mathrm{op}} \cdot \left\| \mathrm{D}\left(y \mapsto \frac{1}{y^{k+1}} \right)_{\|x+r\Theta\|} \right\|_{\mathrm{op}} \\ &= 1 \cdot \frac{k+1}{\|x + r\Theta\|^{k+2}}\end{aligned}$$

Thus, for a finite constant C_2 independent of Θ and r,

$$\left| \omega_{\widehat{x+r\Theta}}(\Theta^{\perp}, v_1^{\perp}, \ldots, v_k^{\perp}) v_x \frac{1}{\|x + r\Theta\|^{k+1}} \right| \leqslant \frac{\mathrm{C}_2\|v\|}{\|x + r\Theta\|^{k+2}}.$$

Putting everything together, we can deduce that

$$\left| v_x \left(\frac{\omega_{\widehat{x+r\Theta}}((\Theta)^{\perp}, v_1^{\perp}, \ldots, v_k^{\perp})}{\|x + r\Theta\|^{k+1}} \right) \right| \leqslant \frac{(\mathrm{C}_1 + \mathrm{C}_2)\|v\|}{\|x + r\Theta\|^{k+2}},$$

which converges absolutely when integrated over $0 \leqslant r < \infty$ and $\Theta \in \mathrm{S}^{\mathrm{N}-1}$ under our hypotheses and the assumption that $x \neq 0$. This follows easily from Lemma 4.5.

According to § 4.3, $d\bar{\mu} = \bar{\omega}$ on $\mathbb{R}^{\mathrm{N}} \setminus \{0\}$. The restriction of $\bar{\mu}$ to the sphere will verify $d\bar{\mu}|_{\mathrm{S}^{\mathrm{N}-1}} = \bar{\omega}|_{\mathrm{S}^{\mathrm{N}-1}}$ and, therefore, will be the function μ that we were looking for.

If we now take spherical coordinates around 0 in the original integral — that is, if we perform the change of coordinates $y = r\Theta$ for $r > 0$ and $\Theta \in \mathrm{S}^{\mathrm{N}-1}$, — we can reach the expression

$$\begin{aligned}\mu_x(v_0, \ldots, v_k) &= \frac{1}{\operatorname{vol} \mathrm{S}^{\mathrm{N}-1}} \int_{\mathrm{S}^{\mathrm{N}-1}} \int_0^{\infty} \frac{\omega_{\Theta}((x - r\Theta)^{\perp}, v_1^{\perp}, \ldots, v_k^{\perp})}{r^{k+1}\|x - r\Theta\|^{\mathrm{N}}} r^{\mathrm{N}-1}\, dr\, d\Theta \\ &= \frac{1}{\operatorname{vol} \mathrm{S}^{\mathrm{N}-1}} \int_{\mathrm{S}^{\mathrm{N}-1}} \omega_{\Theta}\left(\int_0^{\infty} \frac{(x - r\Theta)^{\perp}}{\|x - r\Theta\|^{\mathrm{N}}} r^{\mathrm{N}-k-2}\, dr, v_1, \ldots, v_k \right) d\Theta.\end{aligned}$$

Since $(x - r\Theta)^{\perp\Theta} = x^{\perp\Theta}$, this concludes the proof. ■

4.7 Theorem. Let K be a natural number and let $\mathrm{N} := 2\mathrm{K}+1$. If $f : \mathrm{S^N} \longrightarrow \mathrm{S^K}$ is a smooth function and $p > \mathrm{N}+1$ is a real number, then

$$h(f) \leqslant \rho \cdot \mathrm{E}_{p\mathrm{K}}(f)^{1/p} \cdot \mathrm{E_K}(f).$$

where $h(f)$ denotes the Hopf invariant of f and

$$\rho := \frac{1}{(\mathrm{vol\,S^K})^2\,\mathrm{vol\,S^N}}\left[\int_{\mathrm{S^N}}\left(\|x^{\perp\Theta}\|\int_0^\infty \frac{r^{\mathrm{N-K}}}{\|x-r\Theta\|^{\mathrm{N+1}}}\,dr\right)^{\frac{p}{p-1}} d\Theta\right]^{\frac{p-1}{p}}$$

for any $x \in \mathrm{S^N}$. Moreover, $\rho < \infty$.

Proof. Using Whitehead's integral formula [§ II-1.14], we know that the Hopf invariant of f can be computed as

$$h(f) = \frac{1}{(\mathrm{vol\,S^K})^2}\int_{\mathrm{S^N}} f^*(*1_{\mathrm{S^K}}) \wedge d^{-1}f^*(*1_{\mathrm{S^K}}),$$

where $*1_{\mathrm{S_K}}$ denotes the volume form in the K-dimensional sphere. Thus, if we take $\|-\|$ to represent the operator norm, we must have

$$\begin{aligned} h(f)(\mathrm{vol\,S^K})^2 &\leqslant \int_{\mathrm{S^N}} \|f^*(*1_{\mathrm{S^K}}) \wedge d^{-1}f^*(*1_{\mathrm{S_K}})\|\,dx \\ &\leqslant \int_{\mathrm{S^N}} \|f^*(*1_{\mathrm{S^K}})\| \cdot \|d^{-1}f^*(*1_{\mathrm{S^K}})\|\,dx = (\square). \end{aligned}$$

Since the operator norm of a volume form is 1, this means that

$$\begin{aligned} (\square) &\leqslant \int_{\mathrm{S^N}} \|\mathrm{D}f\|^{\mathrm{K}} \cdot \left\|\frac{1}{\mathrm{vol\,S^N}}\int_{\mathrm{S^N}} f^*(*1_{\mathrm{S^K}})_\Theta\left(\int_0^\infty \frac{r^{\mathrm{N-K}}\cdot x^{\perp\Theta}}{\|x-r\Theta\|^{\mathrm{N+1}}}\,dr,\dots\right)d\Theta\right\|\,dx \\ &\leqslant \frac{1}{\mathrm{vol\,S^N}}\int_{\mathrm{S^N}}\|\mathrm{D}f_x\|^{\mathrm{K}}\cdot\int_{\mathrm{S^N}}\|\mathrm{D}f_\Theta\|^{\mathrm{K}}\cdot\int_0^\infty \frac{r^{\mathrm{N-K}}\cdot\|x^{\perp\Theta}\|}{\|x-r\Theta\|^{\mathrm{N+1}}}\,dr\,d\Theta\,dx. \end{aligned}$$

Using Hölder's inequality [§ B.1] taking p and its conjugate p^*, we have that the quantity above will be bounded by

$$\frac{1}{\mathrm{vol\,S^N}}\int_{\mathrm{S^N}}\|\mathrm{D}f_x\|^{\mathrm{K}}\cdot\mathrm{E}_{p\mathrm{K}}(f)^{1/p}\left(\int_{\mathrm{S^N}}\left[\int_0^\infty \frac{r^{\mathrm{N-K}}\cdot\|x^{\perp\Theta}\|}{\|x-r\Theta\|^{\mathrm{N+1}}}\,dr\right]^{p^*} d\Theta\right)^{1/p^*} dx,$$

As in previous proofs, we have used the fact that $\|-\| \leqslant \|-\|_{\mathrm{HS}}$.

For any $x \in \mathrm{S^N}$, let us analyse the convergence of

$$\int_{\mathrm{S^N}}\left(\int_0^\infty \frac{r^{\mathrm{N-K}}\cdot\|x^{\perp\Theta}\|}{\|x-r\Theta\|^{\mathrm{N+1}}}\,dr\right)^{p^*} d\Theta.$$

The limit as $r \to \infty$ is not problematic since $\mathrm{N-K-(N+1)} \leqslant -2$, but we still need to deal with the asymptote that we have when $x - r\Theta = 0$, i.e., at $\Theta = x$ and

$r = 1$. To this end, we should notice that, since $p^* = p/(p-1) \geqslant 1$, the inequality $(a+b)^{p^*} \leqslant 2^{p^*}(a^{p^*} + b^{p^*})$ holds for any pair of non-negative real numbers a and b. Thus, the integral above can be bounded by

$$2^{p^*}\left[\int_{\mathrm{S^N}}\left(\int_0^2 \frac{r^{\mathrm{N-K}} \cdot \|x^{\perp\Theta}\|}{\|x - r\Theta\|^{\mathrm{N+1}}}\,dr\right)^{p^*} d\Theta + \int_{\mathrm{S^N}}\left(\int_2^\infty \frac{r^{\mathrm{N-K}} \cdot \|x^{\perp\Theta}\|}{\|x - r\Theta\|^{\mathrm{N+1}}}\,dr\right)^{p^*} d\Theta\right].$$

Taking into account that the asymptote that we want to take care of is found when $r = 1$, we only need to consider the first integral in the sum. Furthermore, as $p^* > 1$, the function $(-)^{p^*}$ is convex, hence

$$\int_{\mathrm{S^N}}\left(\int_0^2 \frac{r^{\mathrm{N-K}} \cdot \|x^{\perp\Theta}\|}{\|x - r\Theta\|^{\mathrm{N+1}}}\,dr\right)^{p^*} d\Theta \leqslant \int_{\mathrm{S^N}} \frac{2^{p*}}{2}\int_0^2\left(\frac{r^{\mathrm{N-K}} \cdot \|x^{\perp\Theta}\|}{\|x - r\Theta\|^{\mathrm{N+1}}}\right)^{p^*} dr\,d\Theta,$$

so it will suffice for us to study the convergence of this integral.

If we now switch to cartesian coordinates and change our domain of integration to the closed ball of radius 2 centred at 0 in $\mathbb{R}^{\mathrm{N+1}}$, we know that

$$\int_{\mathrm{S^N}}\int_0^2\left(\frac{r^{\mathrm{N-K}} \cdot \|x^{\perp\Theta}\|}{\|x - r\Theta\|^{\mathrm{N+1}}}\right)^{p^*} dr\,d\Theta = \int_{\mathrm{B}(0,2)\subseteq\mathbb{R}^{\mathrm{N+1}}}\left(\frac{\|y\|^{\mathrm{N-K}} \cdot \|(x-y)^{\perp\hat{y}}\|}{\|x - y\|^{\mathrm{N+1}}}\right)^{p^*} \|y\|^{-\mathrm{N}}\,dy,$$

where we have used the fact that $x^{\perp\hat{y}} = (x-y)^{\perp\hat{y}}$. This function has an asymptote of order $\mathrm{N} - (\mathrm{N-K})p^* < \mathrm{N}$ at $y = 0$ and — as we can deduce from the proof of Theorem 4.6 — one of order $\mathrm{N}p^*$ at $y = x$. Thus, by § 4.5, we only need to show that $p^* < (\mathrm{N}+1)/\mathrm{N}$. Indeed, according to our hypotheses,

$$\begin{aligned} p > \mathrm{N}+1 &\implies p(\mathrm{N}+1-\mathrm{N}) > \mathrm{N}+1 \implies p\mathrm{N} < (\mathrm{N}+1)p - (\mathrm{N}+1) \\ &\implies p\mathrm{N} < (\mathrm{N}+1)(p-1) \implies p^* = \frac{p}{p-1} < \frac{\mathrm{N}+1}{\mathrm{N}}. \end{aligned}$$

Now that we have full certainty about the convergence of our integral, we should notice that, by symmetry, the function

$$x \in \mathrm{S^N} \longmapsto \left[\int_{\mathrm{S^N}}\left[\int_0^\infty \frac{r^{\mathrm{N-K}} \cdot \|x^{\perp\Theta}\|}{\|x - r\Theta\|^{\mathrm{N+1}}}\,dr\right]^{p^*} d\Theta\right]^{1/p^*}$$

is constant, and we can denote the constant value that it takes as ρ_0.

Thus, putting everything together, we have that

$$h(f) \cdot (\operatorname{vol} \mathrm{S^K})^2 \leqslant \frac{\rho_0 \cdot \mathrm{E}_{p\mathrm{K}}(f)^{1/p}}{\operatorname{vol} \mathrm{S^N}} \int_{\mathrm{S^N}} \|\mathrm{D}f_x\|^{\mathrm{K}}\,dx \leqslant \frac{\rho_0 \cdot \mathrm{E}_{p\mathrm{K}}(f)^{1/p} \cdot \mathrm{E_K}(f)}{\operatorname{vol} \mathrm{S^N}},$$

just as we wanted to show. ■

Appendix

A Topology

A.1 A subset of a topological space is *sequentially compact* if every sequence in the subset has a converging subsequence, and it is *precompact* if its closure is compact. Consequently, a subset is said to be *sequentially precompact* if its closure is sequentially compact.

In metric spaces, precompactness and sequential precompactness are always equivalent, but that is not the case in a general topological space. That is why we had to use the Eberlein–Šmulian Theorem [§ II-3.2].

Given a normed space X and a normed space A included in X, we say that A is *compactly embedded* into X if there exists a constant scalar C such that, for every $a \in \mathrm{A}$, $\|a\|_{\mathrm{A}} \leqslant \mathrm{C}\|a\|_{\mathrm{X}}$, and if every bounded sequence in A has a Cauchy subsequence in X.

A.2 Definition. Given any metric space (X, d), any point $x_0 \in \mathrm{X}$ and any $\delta > 0$, we write $\mathrm{B}^\circ(x_0, \delta)$ to denote the open ball of radius δ centred at x_0: the set of points $x \in \mathrm{X}$ such that $d(x, x_0) < \delta$. We denote the closure of this open ball (i.e., the closed ball) by $\mathrm{B}(x_0, \delta)$.

A.3 Definition. A collection F of functions between two metric spaces X and Y is said to be *equicontinuous* if, for every $x_0 \in \mathrm{X}$ and every $\varepsilon > 0$, there exists a $\delta > 0$ such that, for every $f \in \mathrm{F}$ and every $x \in \mathrm{X}$, if $d(x_0, x) < \delta$, then $d(f(x_0), f(x)) < \varepsilon$.

A.4 Filters and ultrafilters. A *filter* on a set X is a collection $\mathcal{F}$ of subsets of X that is closed under finite intersections, that does not contain the empty set, and such that, whenever a subset $\mathrm{A} \subseteq \mathrm{X}$ belongs to $\mathcal{F}$, so does any superset of A included in X. For example, we can define the *Fréchet filter* on the natural numbers by considering the collection of subsets of $\mathbb{N}$ with finite complement.

A filter $\mathcal{U}$ on a set is said to be an *ultrafilter* if no filter on that set strictly includes $\mathcal{U}$. It is a consequence of the axiom of choice that every filter is included in an ultrafilter.

A family $\{x_i\}_{i \in \mathrm{I}}$ in a topological space X *converges* to a point $x_0 \in \mathrm{X}$ according to a filter $\mathcal{F}$ on I if, for every neighbourhood V of x, the collection of indices i such that $x_i \in \mathrm{V}$ belongs to $\mathcal{F}$. We will denote this as

$$\lim_{i \to \mathcal{U}} x_i = x_0,$$

or simply as $x_i \longrightarrow x_0$ if there is no risk of ambiguity.

(i) A function $f : \mathrm{X} \longrightarrow \mathrm{Y}$ between topological spaces is continuous at a point x_0 if and only if, for every ultrafilter $\mathcal{U}$ on an infinite set of indices I, when a sequence $\{x_i\}_{i\in\mathrm{I}}$ in X converges to x_0 according to $\mathcal{U}$, the sequence $\{f(x_i)\}_{i\in\mathrm{I}}$ converges to $f(x_0)$ according to $\mathcal{U}$.

(ii) A topological space X is compact if and only if, for every ultrafilter $\mathcal{U}$ on an infinite set of indices I, every sequence in X is convergent with respect to $\mathcal{U}$ to some limit in X.

Filters, ultrafilters and their properties are discussed in detail in section I.6 of *General topology: Chapters 1–4* [**4**].

A.5 Here we present an alternative proof of Lemma II-2.5.

We should first notice that, thanks to the compactness of C, our functions are well-defined. We will first prove the contrapositive of our result for $f := f_{\max}$. For our convenience — and since we are not assuming X and C to be metric — we will consider, throughout the whole proof, convergence according to an arbitrary ultrafilter $\mathcal{U}$ over an infinite set of indices [§ A.4].

If we assume f not to be continuous at a point $x_0 \in \mathrm{X}$, we may fix a sequence $x_n \longrightarrow x_0$ such that $f(x_n) \not\longrightarrow f(x_0)$, all according to $\mathcal{U}$. In addition, we may 'choose' a sequence of elements $c_n \in \mathrm{C}$ such that $\mathrm{F}(x_n, c_n) = f(x_n)$ for every index n.

Let us first assume the limit of $f(x_n)$ not to exist. As C is compact, c_n must converge to some $c_\mathrm{L} \in \mathrm{C}$ according to $\mathcal{U}$, so $(x_n, c_n) \longrightarrow (x_0, c_\mathrm{L})$ according to $\mathcal{U}$, but $f(x_n) = \mathrm{F}(x_n, c_n)$ would not converge, meaning that F would not be continuous. We can hence assume the limit of $f(x_n)$ to exist.

Let $c_0 \in \mathrm{C}$ be any element such that $f(x_0) = \mathrm{F}(x_0, c_0)$. We will first suppose that $\lim_{n\to\mathcal{U}} f(x_n) < f(x_0)$. By the definition of f, we know that $\mathrm{F}(x_n, c_0) \leqslant f(x_n)$ for every index n. Consequently, we will have

$$\lim_{n\to\mathcal{U}} \mathrm{F}(x_n, c_0) \leqslant \lim_{n\to\mathcal{U}} f(x_n) < f(x_0) = \mathrm{F}(x_0, c_0),$$

which will imply that $\lim_{n\to\mathcal{U}} \mathrm{F}(x_n, c_0) \neq \mathrm{F}(x_0, c_0)$ and, thus, that F is not continuous.

On the other hand, if $f(x_0) < \lim_{n\to\mathcal{U}} f(x_n)$ and we once again take $c_\mathrm{L} := \lim_{n\to\mathcal{U}} c_n$, we will have

$$\mathrm{F}(x_0, c_\mathrm{L}) = \lim_{n\to\mathcal{U}} \mathrm{F}(x_n, c_n) = \lim_{n\to\mathcal{U}} f(x_n) > f(x_0),$$

which is a contradiction and thus shows that F cannot be continuous, for $f(x_0)$ cannot be strictly smaller than $\mathrm{F}(x_0, c_\mathrm{L})$.

In regard to the function $f_{\min}$, it suffices to notice how

$$\min_{c\in\mathrm{C}} \mathrm{F}(x, c) = -\max_{c\in\mathrm{C}} -\mathrm{F}(x, c),$$

so the result follows trivially.

B Real analysis

B.1 Theorem (Hölder)**.** In any measure space, let f and g be measurable real-valued functions. Given any $p \in [1,\infty]$, we define its *conjugate* to be the only p^* such that

$$\frac{1}{p} + \frac{1}{p^*} = 1.$$

In particular, $(1)^* = \infty$ and $(\infty)^* = 1$. If $f \in \mathrm{L}^p$ and $g \in \mathrm{L}^{p^*}$, then $f \cdot g \in \mathrm{L}^1$ and

$$\|f \cdot g\|_1 \leqslant \|f\|_p \cdot \|g\|_{p^*},$$

where we are considering the norms $\| - \|_q$ of the different L^q spaces.

Proof. Theorem 1 in Chapter 19 of *Real analysis* [**14**]. ■

B.2 Theorem (Dominated Convergence)**.** In any measure space, let $\{f_n\}_n$ be a sequence of measurable real-valued functions converging point-wise to some measurable function f. If there exists a non-negative integrable function g such that $|f_n| \leqslant g$ for every f_n in the sequence, then f is integrable and

$$\lim_{n\to\infty} \int f_n = \int f.$$

Proof. The Lebesgue Dominated Convergence Theorem (page 376), in Chapter 18 of *Real analysis* [**14**]. ■

B.3 Theorem (Leibniz Integral Rule)**.** Let $\mathrm{F}(x,t)$ be a $C^1([\alpha,\beta)\times[a,b])$ function, where we may have $\beta = \infty$. If $|\mathrm{F}|$ and $|\partial_t \mathrm{F}|$ can each be bounded by a function $h(x)$ (with no dependence on t) integrable over $[\alpha,\beta[$, then

$$\frac{d}{dt}\int_\alpha^\beta \mathrm{F}(x,t)\,dt = \int_\alpha^\beta \frac{\partial \mathrm{F}}{\partial t}\,dx.$$

Proof. Theorem 15 in Chapter 8 of *Intermediate calculus* [**13**]. ■

Conclusions

1 The two main contributions of this thesis are the precompactness proof that we introduced in Theorem III-3.7 (and Corollary III-3.8) and the estimates that we discussed in Sections IV-3 and IV-4. Our estimates for the p-energy of functions $S^N \longrightarrow S^K$ are summarised in the following table.

Estimate	Conditions on p	Conditions on N and K
Degree [§ IV-2.4]	$p \geqslant N$	$N = K$
General [§ IV-3.5]	$p > N$	None
Hopf [§ IV-4.7]	$p > (N+1)K$	$N = 2K + 1$

2 There is some work to be done in two different areas.

In regard to the precompactness result, it yet remains to be proved whether the limit of a uniformly convergent subsequence of any sequence of functions with decreasing energy is smooth and can therefore be an energy minimiser. Continuity is guaranteed by uniform convergence and so is the fact that, if all the elements of the sequence belong to a certain homotopy class, so will the limit.

Regarding the lower bounds for the energy, there is still room for improvement in the estimate in terms of the Hopf invariant, possibly relaxing the conditions on p.

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