PROJECTIVE INDECOMPOSABLE MODULES, SCOTT MODULES AND THE FROBENIUS-SCHUR INDICATOR

JOHN MURRAY

Abstract. Let Φ be a principal indecomposable character of a finite group G in characteristic 2. The Frobenius-Schur indicator $\nu(\Phi)$ of Φ is shown to equal the rank of a bilinear form defined on the span of the involutions in G. Moreover, if the principal indecomposable module corresponding to Φ affords a quadratic geometry, then $\nu(\Phi) > 0$. This result is used to prove a more precise form of a theorem of Benson and Carlson on the existence of Scott components in the endomorphism ring of an indecomposable G-module, in case the module affords a G-invariant symmetric form.

1. Statement of results

This paper continues our investigation, begun in [7] and [8], into unusual properties of the group algebra of a finite group over a field of characteristic 2. Most of our techniques are not available, and the obvious analogues of our results are false, if the characteristic is odd. The characteristic 2 theory appear to be particularly fertile due to the rich interactions between involutions in the group, the Frobenius-Schur indicator, quadratic forms, and the contragradient operation on the group algebra.

We make extensive use of the modular representation theory of finite groups, as described in [9]. In particular we fix a finite group G and let $(\mathcal{O}, F, k)$ be a 2-modular system for G. So $\mathcal{O}$ is a complete discrete valuation ring, with field of fractions F, unique maximal ideal $J(\mathcal{O})$ and residue field $\mathcal{O}/J(\mathcal{O}) = k$ that has characteristic 2. For convenience we assume that both F and k are algebraically closed. We use the symbol R for either of the rings $\mathcal{O}$ or k. To avoid trivialities, we assume that $|G|$ is even.

The Frobenius-Schur indicator of a generalized character χ of G is $\nu(\chi) := |G|^{-1}\sum_{g\in G}\chi(g^2)$, which turns out to be an integer. If χ is the character of an irreducible FG-module M, then $\nu(\chi) = 1, -1$ or 0, depending on whether M is of quadratic, symplectic or not selfdual type, respectively. G. Frobenius and I. Schur first noted this and the fact that $|\Omega| = \sum\limits_{\chi\in\mathrm{Irr}(G)}\chi(1)\nu(\chi)$, where

$$\Omega := \{g \in G \mid g^2 = 1_G\}.$$

Suppose that e is a primitive idempotent in kG. Then there exists a primitive idempotent $\hat{e}$ in $\mathcal{O}G$ such that e is the image of $\hat{e}$, modulo $J(\mathcal{O})G$. The module ekG is called a *principal indecomposable* kG-module, while $\hat{e}\mathcal{O}G$ is called a

Date: March 21, 2006.

1991 *Mathematics Subject Classification.* 20C20.

Key words and phrases. Green correspondence, quadratic form, symmetric bilinear form, Frobenius-Schur indicator, Broué-Robinson form, Scott module.

princpal indecomposable $\mathcal{O}G$-module. The character Φ of $F \otimes_{\mathcal{O}} \hat{P}$ is called the *principal indecomposable character* of G corresponding to $e, \hat{e}, ekG$ or $\hat{P}$. We may write $\Phi = \sum d_{\chi,\Phi}\chi$, where χ ranges over the ordinary irreducible characters of G. The non-negative integers $d_{\chi,\Phi}$ are known as the *decomposition numbers* of Φ. G. R. Robinson observed in [10] that $\nu(\Phi) = \sum d_{\chi,\Phi}\nu(\chi)$ is non-negative. This is false when the characteristic is odd.

Each $x \in RG$ is an R-linear combination of the elements of G. We use $\lambda(x)$ to denote the coefficient of 1_G in this sum. The map $\lambda : RG \to R$ is called the *standard symmetrizing form* on RG. The *contragradient operator* ^{o} is an involutary algebra anti-automorphism of RG that maps each $g \in G$ to its inverse. We use RS to denote the span of a subset S of G in RG. Our main result is:

Theorem 1.1. *Let e be a primitive idempotent in kG and let Φ be the corresponding principal indecomposable character of G. Then $\nu(\Phi)$ is the rank of the bilinear form*

$$\lambda_e : k\Omega \times k\Omega \to k, \quad \text{where} \quad \lambda_e(s,t) := \lambda(e^o set), \quad \text{for all } s,t \in \Omega.$$

A conjugacy class of G is said to be *real* if it contains the inverse of each of its elements, and said to be *strongly real* if each of its elements is inverted by an involution. The R-lattice spanned by a conjugacy class is an RG-permutation module. Theorem 1.1 allows us to add condition (iv) below to the main result of [8]:

Corollary 1.2. *Let B be a 2-block of kG. Then the following are equivalent:*
- (i) *B is real and has a strongly real defect class;*
- (ii) $\sum_{\chi \in \mathrm{Irr}(B)} \chi(1)\nu(\chi) \neq 0_F$;
- (iii) *$k\Omega$ has a composition factor that belongs to B;*
- (iv) *$\lambda(e^o tes) \neq 0_k$ for some primitive idempotent $e \in B$ and some $s,t \in \Omega$.*

In conformity with [7] and [8], we call a 2-block that satisfies any one of these equivalent conditions a *strongly real* 2*-block* of G.

Our interest in Theorem 1.1 arose as follows. Let K be a field. A KG-module M is said to have a *quadratic geometry* if there exists a G-invariant K-valued quadratic form Q on M whose polarization $b(m_1, m_2) := Q(m_1+m_2)-Q(m_1)-Q(m_2)$, $\forall m_1, m_2 \in M$, is non-degenerate. If $\mathrm{char}(K)$ is odd, there is a characterization, due separately to W. Willems and J. G. Thompson, of the quadratic type of a principal indecomposable G-module (and its irreducible head) that makes use of the Frobenius-Schur indicator of any one of the irreducible characters of G whose multiplicity in $\hat{P}$ is odd [12, Proposition 2.2 and Theorem 2.8]. This result does not hold if $\mathrm{char}(K) = 2$. In particular in characteristic 2 there is no known connection between the type of a principal indecomposable module and the type of its irreducible head. Using Theorem 1.1 and the approach adopted by R. Gow and W. Willems in [3], we prove:

Theorem 1.3. *Let e be a primitive idempotent in kG and let Φ be the corresponding principal indecomposable character of G. Suppose that ekG has a quadratic geometry. Then $\nu(\Phi) > 0$. In particular, e belongs to a strongly real 2-block of G.*

A more precise module theoretic form of this result is given in Corollary 6.5.

Example 1.4. *Let G be a finite group of Lie type defined over a field of characteristic 2 and let Φ be a principal indecomposable character of G that is real valued.*

We claim that $\nu(\Phi) > 0$. *For, let* P *be the principal indecomposable* kG*-module that corresponds to* Φ. *Then* P *is of quadratic type, by a result of Gow and Willems* [12, 3.9]. *Our claim then follows from Theorem 1.3.*

Example 1.5. *Let* $G = H \wr C_2$, *where* H *is the unique nonabelian group of order 12 that is not isomorphic to* A_4 *or a dihedral group. Then* [3, 2.12] *shows that* kG *has a principal indecomposable module that does not have a quadratic geometry. However the character* Φ *of this module satisfies* $\nu(\Phi) = 2$. *So the converse of Theorem 1.3 is false.*

Theorem 7.2 is a refinement, for modules that possess a G-invariant symmetric bilinear form, of a result of D. Benson and J. Carlson on the existence of Scott components in the endomorphism ring of a kG-module.

Theorem A.5 is concerned with bilinear forms and projective (in the sense of Schur) modules. This result is needed to prove 7.2. Since Theorem A.5 has a different character to the rest of the paper, we consign its proof to the appendix.

2. Bilinear forms and adjoints

Just as in [7] and [8] we let Σ be a cyclic group of order 2, generated by an involution σ. The wreath product $G \wr \Sigma$ of G with Σ is a split extension of the base group $G \times G$ by Σ. Here σ acts on $G \times G$ via $(g_1, g_2)^\sigma = (g_2, g_1)$, for all $g_1, g_2 \in G$. If H is a subgroup of G then the diagonal subgroup of $G \wr \Sigma$ is $\underline{H} := \{(h, h) \mid h \in H\}$.

Throughout the paper M will be a right RG-module: the image of $m \in M$ under $g \in G$ is written $m \cdot g$. We write endomorphisms and linear forms on the right, but most other functions on the left. We use $M{\downarrow_H}$ for the restriction of M to H, and $N{\uparrow^G}$ for the induced RG-module $N \otimes_{RH} RG$, whenever N is an RH-module. A theorem of J. A. Green ([4]) states that if M is an indecomposable $\mathcal{O}G$-module, with F-character χ then

(1) $\quad \chi(g) = 0, \quad$ if the 2-part of $g \in G$ is not contained in some vertex of M.

Let $\mu : RG \to \mathrm{End}_R(M)$ be the ring homomorphism associated with M. Then the dual space $M^* = \mathrm{Hom}(M, R)$ is an RG-module via $f \cdot g := \mu(g^{-1})f$, for all $f \in M^*$ and $g \in G$. Also $\mathrm{End}_R(M)$ is an $RG \times G$-module, via $f \cdot (g_1, g_2) := \mu(g_1^{-1}) f \mu(g_2)$, for all $f \in \mathrm{End}_R(M)$ and $g_1, g_2 \in G$. In particular for the restricted module $\mathrm{End}_k(M){\downarrow_{\underline{G}}}$, the action of $\underline{g} \in \underline{G}$ is conjugation $f \cdot (g, g) := \mu(g^{-1}) f \mu(g)$ by the unit $\mu(g) \in \mathrm{End}_k(M)$. We identify $RG \times G$-modules $M^* \otimes_R M = \mathrm{End}_R(M)$. The space $\mathrm{Bil}_R(M)$ of all bilinear forms on M is an $RG \times G$-module via $(b \cdot (g_1, g_2))(m_1, m_2) := b(m_1 \cdot g_1^{-1}, m_2 \cdot g_2^{-1})$, for all bilinear forms b, and all $m_1, m_2 \in M$. We identify $RG \times G$-modules $M^* \otimes_R M^* = \mathrm{Hom}(M, M^*) = \mathrm{Bil}_R(M)$. Note also the natural isomorphism $M \otimes_R M \cong \mathrm{Bil}_R(M)^*$.

The equality $\mathrm{Bil}_R(M) = \mathrm{Hom}(M, M^*)$, identifies a bilinear form b with the map $M \to M^*$ that sends $m_2 \in M$ to the linear form $m_1 \to b(m_1, m_2)$, for all $m_1 \in M$. We say that b is *non-degenerate* if this map is an R-isomorphism, and we say that b is *G-invariant* if this map is an RG-homomorphism. Now M is said to be *self-dual* if $M \cong M^*$ as RG-modules. So M is self-dual if and only there exists a non-degenerate G-invariant bilinear form on M. For example, the form $B_1(x, y) := \lambda(xy^o)$, on the regular RG-module, is non-degenerate and G-invariant. So RG is a self-dual RG-module.

Let N be an RG-module and let $f \in \mathrm{Hom}(M, N^*)$. Then $f^t \in \mathrm{Hom}(N, M^*)$ is defined by $m(nf^t) := n(mf)$, for all $m \in M$ and $n \in N$. If $N = M^*$, then

$\mathrm{Hom}(M, N^*) = \mathrm{End}_R(M)$. In this case we call $f^t \in \mathrm{End}_R(M)$ the *transpose* of f. In terms of tensors, $(\alpha \otimes \beta)^t = \beta \otimes \alpha$, for all $\alpha, \beta \in M^*$.

We extend $\mathrm{Bil}_R(M)$ to a $G \wr \Sigma$-module by defining $b \cdot \sigma := b^t$, for each $b \in \mathrm{Bil}_R(M)$. Thus $b \cdot \sigma(m_1, m_2) := b(m_2, m_1)$, for all $m_1, m_2 \in M$. B. Külshammer uses the notation $M^{\otimes 2}$ for the extension of $M \otimes_R M$ to $G \wr \Sigma$, such that $m_1 \otimes m_2 \cdot \sigma := m_2 \otimes m_1$, for all $m_1, m_2 \in M$. Clearly $\mathrm{Bil}_R(M) \cong (M^*)^{\otimes 2}$, as $RG \wr \Sigma$-modules. It is shown in [6] that $M^{\otimes 2}$ is indecomposable if M is indecomposable. Moreover, if M is indecomposable with vertex V, then $M^{\otimes 2}$ has vertex $V \wr \Sigma$. If $R = \mathcal{O}$ and $F \otimes_{\mathcal{O}} M$ has character χ, then $M^{\otimes 2}$ has character $\chi^{\otimes 2}$, where

$$\chi^{\otimes 2}((g_1, g_2)\sigma) := \chi(g_1 g_2), \quad \text{for all } g_1, g_2 \in G.$$

Let b be a non-degenerate bilinear form on M and let $f \in \mathrm{End}_R(M)$. Then there is a unique endomorphism f^β of M such that $b(m_1 f^\beta, m_2) = b(m_1, m_2 f)$, for all $m_1, m_2 \in M$. We call f^β the *adjoint* of f with respect to b. Clearly the adjoint map $f \to f^\beta$ is an R-algebra anti-automorphism of $\mathrm{End}_R(M)$. Our next lemma shows that a non-degenerate G-invariant bilinear form can be recovered from its adjoint.

Lemma 2.1. *The map sending a non-degenerate form b to its adjoint β establishes a bijection between the rank 1-subspaces of $\mathrm{Bil}_R(M)$ that contain a non-degenerate G-invariant form and the algebra anti-automorphisms of $\mathrm{End}_R(M)$ that invert each $\mu(g)$, with $g \in G$. If $R = k$ then b is symmetric if and only if β is an involution.*

Proof. Let b be a non-degenerate G-invariant bilinear form on M, with adjoint map β. The G-invariance of b implies that $\mu(g)^\beta = \mu(g^{-1})$, for all $g \in G$. Note that if $\lambda \in R$, then λb is non-degenerate if and only if λ is a unit in R. Also if λb is non-degenerate then it has adjoint β.

Conversely let γ be an R-algebra anti-automorphism of $\mathrm{End}_R(M)$ such that $\mu(g)^\gamma = \mu(g^{-1})$, for each $g \in G$. Choose a primitive idempotent ϵ in $\mathrm{End}_R(M)$. Then ϵ^γ is also a primitive idempotent in $\mathrm{End}_R(M)$. Choose an R-isomorphism $\phi : \epsilon\mathrm{End}_R(M)\epsilon^\gamma \to R$. Now $\epsilon\mathrm{End}_R(M)$ is an irreducible $\mathrm{End}_R(M)$-module that is isomorphic to M as G-module. Define an R-bilinear form c on $\epsilon\mathrm{End}_R(M)$ by setting $c(\epsilon f_1, \epsilon f_2) = \phi(\epsilon f_1 f_2^\gamma \epsilon^\gamma)$, for all $f_1, f_2 \in \mathrm{End}_R(M)$. Then c is non-degenerate, as its kernel is a proper $\mathrm{End}_R(M)$-submodule of $\epsilon\mathrm{End}_R(M)$. Clearly c has adjoint map γ. In addition, c is G-invariant, as $\mu(g)^\gamma = \mu(g^{-1})$, for each $g \in G$.

Let b and c be non-degenerate G-invariant bilinear forms on M, whose adjoints coincide with β. Let $B : M \to M^*$, $C : M \to M^*$ be the G-module isomorphisms corresponding to b, respectively c. Then $f^\beta = Bf^*B^{-1}$ and also $f^\beta = Cf^*C^{-1}$, for all $f \in \mathrm{End}_R(M)$. So $B^{-1}fB = C^{-1}fC$, for all $f \in \mathrm{End}_R(M)$. It follows that CB^{-1} is a central unit in $\mathrm{End}_R(M)$, whence $C = \lambda B$, for some unit λ in R. This shows that the correspondence $Rb \leftrightarrow \beta$ is bijective.

If b is symmetric then β is easily seen to be an involution. Suppose that $R = k$ and that β is an involution. Then β acts as an involutary anti-automorphism on the 1-dimensional k-space $\epsilon\mathrm{End}_k(M)\epsilon^\beta$. As $\mathrm{char}(k) = 2$, this map must be the identity. We conclude from this that b is symmetric. □

Proposition 2.2. *Suppose that M affords a nondegenerate G-invariant symmetric bilinear form b. Let β be the adjoint of b. Then $\mathrm{End}_R(M)$ can be extended to a $G \wr \Sigma$-module by letting σ act as β on $\mathrm{End}_R(M)$. Moreover $\mathrm{End}_R(M) \cong \mathrm{Bil}_R(M)$, as $RG \wr \Sigma$-modules.*

Proof. It is easily checked that $f \cdot \sigma := f^\beta$, for all $f \in \mathrm{End}(M)$, extends the $G \times G$-action to $G \wr \Sigma$. The required $G \wr \Sigma$-module isomorphism sends $f \in \mathrm{End}_R(M)$ to $B_f \in \mathrm{Bil}_R(M)$, where $B_f(m_1, m_2) := b(m_1 f, m_2)$, for all $m_1, m_2 \in M$. □

3. A Scott multiplicity formula

Let H be a subgroup of G. We use M^H to denote the space of H-fixed points in M, but we also use the alternatives $\mathrm{Bil}_{RH}(M)$ and $\mathrm{End}_{RH}(M)$. The *relative trace map* $\mathrm{Tr}_H^G : M^H \to M^G$ is defined by $\mathrm{Tr}_H^G(m) := \sum m \cdot g$, for all $m \in M^H$. Here g ranges over a set of representatives for the right cosets of H in G. Set $\mathrm{Tr}_H^G(M^H) := \{\mathrm{Tr}_H^G(m) \mid m \in M^H\}$. We shall identify the groups $\underline{H} \times \Sigma$ and $H \times \Sigma$ in expressions involving the relative trace map on $\underline{G} \times \Sigma$-modules. For instance $\mathrm{Tr}_{<g\sigma>}^{G\times\Sigma}$ is the trace map from $< (g, g)\sigma >$ to $\underline{G} \times \Sigma$.

The *Scott module* $S_G(H)$ is the only component of $R_H{\uparrow}^G$ that has a trivial submodule or a trivial factor module (c.f. [9, 4.8.4]). It is known that each Sylow 2-subgroup of H is a vertex of $S_G(H)$. J. A. Green proved the following in [5, (1.3)]:

Lemma 3.1. *The multiplicity of the Scott module with vertex $V \leq G$ as a component of M is the rank of the bilinear form $\rho_{V,M} : \mathrm{Tr}_V^G(M^V) \times \mathrm{Tr}_V^G((M^*)^V) \to k$, where*

$$\rho_{V,M}(m, f) = mf_1 = m_1 f,$$

whenever $m = \mathrm{Tr}_V^G(m_1)$ for $m_1 \in M^V$, and $f = \mathrm{Tr}_V^G(f_1)$ for $f_1 \in (M^*)^V$.

Remark 3.2. *The form $\rho_{V,M}$ is well-behaved with respect to direct products. Specifically, suppose that $M = M_1 \oplus M_2$ as kG-modules, that $m_1 \in \mathrm{Tr}_V^G(M^V) \cap M_1$, and that $\rho_{V,M}(m_1, f) \neq 0_k$, where $f \in \mathrm{Tr}_V^G((M^*)^V)$. Write $f = f_1 + f_2$, where f_i is the projection of f onto M_i^*. Then $\rho_{V,M}(m_1, f) = \rho_{V,M}(m_1, f_1) \neq 0_k$. In particular, in this situation M_1 has a Scott component with vertex V.*

The next result is a consequence of Mackey's formula.

Lemma 3.3. *Suppose that V and W are 2-subgroups of G such that no G-conjugate of W contains V. Then*

$$m_1 f = 0_k, \quad \text{if } m_1 \in M^V \text{ and } f \in \mathrm{Tr}_W^G((M^*)^W);$$
$$mf_1 = 0_k, \quad \text{if } m \in \mathrm{Tr}_V^G(M^V) \text{ and } f_1 \in (M^*)^W.$$

We note also that:

Lemma 3.4. *Suppose that no component of M has a vertex that properly contains $V \leq G$. Then $\rho_{V,M}$ extends to a bilinear form $\hat{\rho}_{V,M}$ on $M^G \times (M^*)^V$, such that $\hat{\rho}_{V,M}(m, \mathrm{Tr}_V^G(f_1)) = mf_1$, for all $m \in M^G$ and $f_1 \in (M^*)^V$. The rank of $\hat{\rho}_{V,M}$ equals the rank of $\rho_{V,M}$.*

Now suppose that A is a symmetric G-algebra, with symmetrizing form t, and let D be a 2-subgroup of G. M. Broué and G. R. Robinson [2, (1.2)] define the symmetric bilinear form $\rho_D = \rho_{D,G}^{A,t}$ on $\mathrm{Tr}_D^G(A^D)$ as

$$\rho_D(x, y) = t(x_1 y) = t(x y_1),$$

whenever $x = \mathrm{Tr}_D^G(x_1)$, or $y = \mathrm{Tr}_D^G(y_1)$, with $x_1, y_1 \in A^D$. Using Green's result, Lemma 3.1, they show that the rank of ρ_D coincides with the multiplicity of the Scott module with vertex D as a component of A.

Now take $A = \mathrm{End}_k(M)$ and regard $\mathrm{End}_k(M)$ as a G-algebra via the restriction of the $G \times G$-module $\mathrm{End}_k(M)$ to $\underline{G}$. Let $t = \mathrm{tr}$ denote the usual trace form on $\mathrm{End}_k(M)$. Set $\rho_V := \rho_{V,G}^{\mathrm{End}_k(M),\mathrm{tr}}$.

Proposition 3.5. *Suppose that M affords a nondegenerate G-invariant symmetric bilinear form b, and that $\mathrm{End}_k(M)$ is extended to a $G \wr \Sigma$-module, according to Proposition 2.2. Let $\hat{D}$ be a 2-subgroup of $\underline{G} \times \Sigma$. Set $\underline{D} = \hat{D} \cap \underline{G}$. Then the multiplicity of the Scott module with vertex $\hat{D}$ as a component of $\mathrm{End}_k(M){\downarrow_{G\times\Sigma}}$ is equal to the rank of the restriction $\rho_{\hat{D}}$ of ρ_D to $\mathrm{Tr}_{\hat{D}}^{G\times\Sigma}(\mathrm{End}_{k\hat{D}}(M))$.*

Proof. We may assume that $\hat{D} \neq \underline{D}$. Note that the restriction makes sense. For, $\hat{D} = \underline{D} < \underline{t}\sigma >$, where t is any element of $\hat{D} \backslash \underline{D}$. Any set of representatives for the cosets of $\underline{D}$ in $\underline{G}$ is also a set of representatives for the cosets of $\underline{D} < \underline{t}\sigma >$ in $\underline{G} \times \Sigma$.

We adapt the proof of Proposition (1.3) in [2]. Let $\{m_i\}$ be a basis of M, with b-dual basis $\{n_i\}$. So $b(m_i, n_j) = \delta_{ij}$, for all i and j. As b is symmetric, $\{m_i\}$ is the b-dual basis of $\{n_i\}$. Now for $f \in \mathrm{End}_k(M)$ we have $\mathrm{tr}(f) = \sum_i b(m_i f, n_i)$. Thus

$$\mathrm{tr}(f^\beta) = \sum_i b(m_i f^\beta, n_i) = \sum_i b(m_i, n_i f) = \sum_i b(n_i f, m_i) = \mathrm{tr}(f).$$

For $f_2 \in \mathrm{End}_k(M)$, define $f_2 T \in \mathrm{End}_k(M)^*$ by $f_1(f_2 T) = \mathrm{tr}(f_1 f_2)$, for all $f_1 \in \mathrm{End}_k(M)$. Then T is a $\underline{G}$-module isomorphism $\mathrm{End}_k(M) \to \mathrm{End}_k(M)^*$. Also

$$f_1((f_2T)^\beta) = f_1^\beta(f_2T) = \mathrm{tr}(f_1^\beta f_2) = \mathrm{tr}(f_2^\beta f_1) = \mathrm{tr}(f_1 f_2^\beta) = f_1(f_2^\beta T),$$

for all $f_1 \in \mathrm{End}_k(M)$. So $(f_2T)^\beta = f_2^\beta T$ and hence T is even a $\underline{G} \times \Sigma$-module isomorphism. In particular, if $H \leq \underline{G} \times \Sigma$ then the image of $\mathrm{End}_{kH}(M)$ under T is $\mathrm{End}_{kH}(M)^*$.

By Lemma 3.1 the multiplicity of the Scott module with vertex $\hat{D}$ as a component of $\mathrm{End}_k(M){\downarrow_{G\times\Sigma}}$ is the rank of the bilinear form $\rho_{\hat{D}}$ on $\mathrm{Tr}_{\hat{D}}^{G\times\Sigma}(\mathrm{End}_{k\hat{D}}(M))$, where $\rho_{\hat{D}}(x, y) = \mathrm{tr}(x_1 y) = \mathrm{tr}(x y_1)$, whenever $x = \mathrm{Tr}_{\hat{D}}^{G\times\Sigma}(x_1)$, or $y = \mathrm{Tr}_{\hat{D}}^{G\times\Sigma}(y_1)$, with $x_1, y_1 \in \mathrm{End}_{k\hat{D}}(M)$. The lemma now follows from the observation that $\rho_{\hat{D}}$ coincides with the restriction of ρ_D to $\mathrm{Tr}_{\hat{D}}^{G\times\Sigma}(\mathrm{End}_{k\hat{D}}(M))$. □

4. Bilinear forms on the group algebra

Recall that $\lambda : RG \to R$, with $\lambda(\sum \mu_g g) = \mu_1$, is a symmetrizing form on RG. The corresponding bilinear form $B_1(x, y) := \lambda(xy^o)$ is G-invariant, symmetric and non-degenerate. So $\mathrm{End}_{RG}(RG) \cong \mathrm{Bil}_R(RG)$, as $G \wr \Sigma$-modules. Concretely, $x \cdot (g_1, g_2) := g_1^{-1} x g_2$ and $x \cdot \sigma := x^o$, for each $x \in RG$ and $g_1, g_2 \in G$. We use the isomorphism $RG \otimes_R RG \cong \mathrm{Bil}_R(RG)$, without further comment.

Lemma 4.1. *Each non-projective component of $\mathrm{Bil}_R(RG)$ has vertex Σ and takes the form $P^{\otimes 2}$, for some principal indecomposable RG-module P; the multiplicity of $P^{\otimes 2}$ equals the dimension of the corresponding irreducible kG-module.*

Proof. Let $1_G = e_1 + \ldots + e_d + \ldots + e_m$ be a decomposition of 1_G into a sum of pairwise orthogonal primitive idempotents in RG. Then

$$RG \otimes_R RG = \sum_i (e_i RG)^{\otimes 2} + \sum_{i<j} (e_i RG \otimes e_j RG + e_j RG \otimes e_i RG).$$

Each term in the second sum is a projective $G \wr \Sigma$-module. The lemma follows from this. □

Lemma 4.2. $\mathrm{Bil}_R(RG) \cong R_\Sigma\uparrow^{G\wr\Sigma}$.

Proof. Clearly $\{g_1\otimes g_2 \mid g_1, g_2 \in G\}$ is a $G\wr\Sigma$-permutation basis for $RG^{\otimes 2}$. Moreover $G \wr \Sigma$ acts transitively on this basis and the stabilizer of $1_G \otimes 1_G$ is Σ. □

For $g \in G$, define $g^* \in (RG)^*$ by $gg^* = 1_R$ and $hg^* = 0_R$, for $g \neq h \in G$. Then $\{g_1^* \otimes g_2^* \mid g_1, g_2 \in G\}$ forms a basis for $\mathrm{Bil}_R(RG)$. Now for $x \in G$, we have $g^*\cdot x = (gx)^*$, in the dual G-module $(RG)^*$. From this it follows that $\mathrm{Tr}_1^G(g_1^*\otimes g_2^*) = B_{g_1 g_2^{-1}}$, where $B_a(x, y) := \lambda(axy^o)$, for all $a, x, y \in RG$. Thus $\{B_g \mid g \in G\}$ is a basis for the space $\mathrm{Bil}_{RG}(RG)$ of G-invariant bilinear forms on RG. Clearly B_a is a symmetric form if and only if $a = a^o$. Let $(G\backslash\Omega)^\pm$ be a set of representatives for the subsets $\{g, g^{-1}\}_{g\in G}$ of $G\backslash\Omega$. Then $\{B_t \mid t \in \Omega\} \cup \{B_{g+g^{-1}} \mid g \in (G\backslash\Omega)^\pm\}$ is a basis for the space $\mathrm{Bil}_{RG\times\Sigma}(RG)$ of G-invariant symmetric bilinear forms on RG. Also if e is an idempotent in RG, then

$$\mathrm{Bil}_{RG}(eRG) = \{B_{e^o ae} \mid e^o ae \in e^o RGe\}. \tag{2}$$

Now let $R = k$ and choose $t \in \Omega$. Let $\mathcal{T}$ be the conjugacy class of G that contains t. Recall that $\mathrm{Bil_k(kG)}^* \cong kG \otimes_k kG$. For $< \underline{t}\sigma >$-fixed points

$$\begin{aligned} &(kG \otimes_k kG)^{k<t\sigma>} \quad \text{has } k\text{-basis} \\ &\quad \{gt \otimes g \mid g \in G\} \cup \{g_1 t \otimes g_2 + g_2 t \otimes g_1 \mid g_1 \neq g_2 \in G\}. \end{aligned} \tag{3}$$

The analogous basis of $\mathrm{Bil}_{k<t\sigma>}(kG)$ enables one to show that

$$\begin{aligned} &\mathrm{Tr}_{<t\sigma>}^{G\times\Sigma} (\mathrm{Bil}_{k<t\sigma>}(kG)) \quad \text{has } k\text{-basis} \\ &\quad \{B_s \mid s \in \mathcal{T}\} \ \cup\ \{B_{g+g^{-1}} \mid g \in (G\backslash\Omega)^\pm\}. \end{aligned} \tag{4}$$

Lemma 4.3. *Let e be a primitive idempotent in kG, let $t \in \Omega$ and let $\mathcal{T}$ be the conjugacy class of G that contains t. Then the multiplicity of the Scott module with vertex $< \underline{t}\,\sigma >$ as a component of $\mathrm{Bil}_k(ekG)\downarrow_{\underline{G}\times\Sigma}$ coincides with the rank of the symmetric bilinear form*

$$\lambda_{e,\mathcal{T}} : k\mathcal{T} \times k\mathcal{T} \to k, \quad \textit{where} \quad \lambda_{e,\mathcal{T}}(r, s) := \lambda(e^o res), \quad \textit{for all } r, s \in \mathcal{T}.$$

Proof. Let $s \in \mathcal{T}$ and $g \in G$. Then $B_{e^o se}(egt \otimes eg) = \lambda(e^o segtg^{-1})$. The result now follows from (2), (3) and (4), and Lemmas 3.1 and 3.3. □

Let $\mathcal{T}_0 = \{1_G\}$ and let $\mathcal{T}_1 \ldots, \mathcal{T}_n$ be the conjugacy classes of involutions in G. We extend each $\lambda_{e,\mathcal{T}_i}$ to a symmetric bilinear form on $k\Omega$ by setting $\lambda_{e,\mathcal{T}_i}(s, t) = 0_k$, whenever $s \notin \mathcal{T}_i$ or $t \notin \mathcal{T}_i$. Recall the definition of λ_e, from the statement of Theorem 1.1.

Proposition 4.4. *Let e be a primitive idempotent in kG. Then*

$$\lambda_e = \lambda_{e,\mathcal{T}_0} \perp \lambda_{e,\mathcal{T}_1} \perp \ldots \perp \lambda_{e,\mathcal{T}_n}.$$

The rank of λ_e is the number of nonprojective Scott components in $\mathrm{Bil}_k(kG)\downarrow_{\underline{G}\times\Sigma}$.

Proof. Suppose that $s, t \in \Omega$ are not conjugate in G. Then $< \underline{t}\sigma >$ is not contained in any $\underline{G} \times \Sigma$-conjugate of $< \underline{s}\sigma >$. So $\lambda(e^o tes) = B_{e^o te}(es \otimes e) = 0_k$, using Lemma 3.3. The result now follows from Lemmas 4.1 and 4.3. □

5. Scott multiplicities from the Frobenius-Schur indicator

In this section we aim to interpret a result of G. R. Robinson on principal indecomposable modules in characteristic 2. We give a Scott-multiplicity formula in terms of the restriction of a projective character to the centralizer of an involution. We then relate this to the Frobenius-Schur indicator of the projective character.

Let $t \in \Omega$ and set $T :=< \underline{t}\sigma >$. Then $C_G(t) \cong (\underline{C_G(t)} \times \Sigma)/T$. Suppose that Q is a principal indecomposable $RC_G(t)$-module. Denote by $\hat{Q}$ the inflation of Q, regarded as an $\underline{C_G(t)} \times \Sigma/T$-module, to $\underline{C_G(t)} \times \Sigma$. Then $\hat{Q}$ is indecomposable with vertex T and its kernel contains T. Conversely, each indecomposable $\underline{C_G(t)} \times \Sigma$-module that has vertex T and kernel containing T has the form $\hat{P}$, for some principal indecomposable $RC_G(t)$-module P. We use fQ to denote the Green correspondent, with respect to $(\underline{G} \times \Sigma, T, \underline{C_G(t)} \times \Sigma)$, of $\hat{Q}$. So fQ is an $R\underline{G} \times \Sigma$-module that has trivial source and vertex T. Moreover, fQ is the unique non-projective component of $\hat{Q}{\uparrow}^{\underline{G}\times\Sigma}$, and $\hat{Q}$ is the unique component of $fQ{\downarrow}_{\underline{C_G(t)}\times\Sigma}$ that has vertex T. Note that for each involution $s \in \underline{C_G(t)}$ that is G-conjugate to t, the restricted module $fQ{\downarrow}_{\underline{C_G(t)}\times\Sigma}$ has at least one component with vertex $< \underline{s}\sigma >$.

Given $g \in G$ we may write $g = g_2 g_{2'} = g_{2'} g_2$, for a unique 2-element g_2 and a unique $2'$-element $g_{2'}$ in G. The *Frobenius twist* M^{Fr} of M is the RG-module with the same underlying R-module M, where $g \in G$ acts on M^{Fr} as $g_2 g_{2'}^2$ acts on M. If M has (Brauer or ordinary) character ϕ then M^{Fr} has character $\phi^{\mathrm{Fr}} : g \to \phi(g_2 g_{2'}^2)$, for all g in the domain of ϕ.

We use Φ_Q to denote the character of $F \otimes_{\mathcal{O}} Q$ whenever H is a subgroup of G and Q is an $\mathcal{O}H$-module. Our next result is more general than required here.

Lemma 5.1. *Let P be a principal indecomposable $\mathcal{O}G$-module, let $t \in \Omega$, and let $\{Q\}$ range over the isomorphism classes of principal indecomposable $\mathcal{O}C_G(t)$-modules. Then*

$$P^{\mathrm{Fr}}{\downarrow}_{C_G(t)} = \sum a_Q Q, \qquad \textit{if and only if} \qquad P^{\otimes 2}{\downarrow}_{\underline{G}\times\Sigma} = \sum a_Q fQ.$$

Proof. Suppose that Q is a principal indecomposable $\mathcal{O}C_G(t)$-module. Then $\hat{Q}$ is the unique component of $fQ{\downarrow}_{\underline{C_G(t)}\times\Sigma}$ that has a vertex containing $< \underline{t}\sigma >$, and $< \underline{t}\sigma >$ is contained in the kernel of $\hat{Q}$. It then follows from (1) that $\Phi_{fQ}(\underline{gt}\sigma) = \Phi_{\hat{Q}}(\underline{gt}\sigma) = \Phi_Q(g)$, for each $2'$-element $g \in C_G(t)$. Thus

$$\left(\Phi^{\otimes 2} - \sum a_Q \Phi_{fQ}\right)(\underline{gt}\sigma) = \Phi(g^2) - \sum a_Q \Phi_Q(g) = \left(\Phi^{\mathrm{Fr}}{\downarrow}_{C_G(t)} - \sum a_Q \Phi_Q\right)(g) = 0,$$

for each 2-regular element g in $C_G(t)$.

The functions Φ_Q are linearly independent on the $2'$-elements of $C_G(t)$. It follows that the functions Φ_{fQ} are linearly independent on the 2-section of $\underline{G} \times \Sigma$ that contains $\underline{t}\sigma$. Moreover, if an indecomposabe $\mathcal{O}\underline{G} \times \Sigma$-module has a character that does not vanishes on the 2-section of $\underline{G} \times \Sigma$ that contains $\underline{t}\sigma$ then by (1) that module has a vertex that contains $\underline{t}\sigma$. The proposition now follows from the previous paragraph. □

Corollary 5.2. *Let P be a principal indecomposable $\mathcal{O}G$-module and let Φ be the character of $F \otimes_{\mathcal{O}} P$. Then for $t \in \Omega$, the Scott module with vertex $< \underline{t}\sigma >$ occurs with multiplicity $< \Phi{\downarrow}_{C_G(t)}, 1_{C_G(t)} >$ as a component of $P^{\otimes 2}{\downarrow}_{\underline{G}\times\Sigma}$.*

Proof. Let Q be the projective cover of the trivial $\mathcal{O}C_G(t)$-module. Then Q is the Scott module with trivial vertex for $C_G(t)$ and $\hat{Q}$ is the Scott module with vertex $< \underline{t}\sigma >$ for $\underline{C_G(t)} \times \Sigma$. Green correspondence preserves Scott modules. So fQ is the Scott module with vertex $< \underline{t}\sigma >$ for $\underline{G} \times \Sigma$. The trival $FC_G(t)$-module occurs with multiplicity 1 as a submodule of $F \otimes_{\mathcal{O}} Q$, and with multiplicity 0 as a submodule of $F \otimes_{\mathcal{O}} Q'$, for any principal indecomposable $\mathcal{O}C_G(t)$-module $Q' \not\cong Q$. It follows that Q occurs with multiplicity $< \Phi^{\mathrm{Fr}}{\downarrow_{C_G(t)}}, 1_{C_G(t)} >=< \Phi{\downarrow_{C_G(t)}}, 1_{C_G(t)} >$ as a component of $P^{\mathrm{Fr}}{\downarrow_{C_G(t)}}$. The result now follows from Lemma 5.1. □

Proof of Theorem 1.1. Recall that Ω is a union of the G-conjugacy classes $\bigcup_{i=0}^{n} \mathcal{T}_i$. Choose $t_i \in \mathcal{T}_i$, for $i = 0, \dots, n$. It follows from Corollary 5.2 and Proposition 4.4 that λ_e has rank equal to $\sum_{i=0}^{n} < \Phi{\downarrow_{C_G(t_i)}}, 1_{C_G(t_i)} >$. The proof is now a consequence of G. R. Robinson's observation [10, Lemma 1] that $\nu(\Phi) = \sum_{i=0}^{n} < \Phi{\downarrow_{C_G(t_i)}}, 1_{C_G(t_i)} >$. □

6. Quadratic forms and the Frobenius-Schur indicator

In this section we adopt the approach of Gow and Willems to quadratic forms on principal indecomposable RG-modules in order to prove Theorem 1.3. We highlight two results from [3] that will be important for our purposes.

Lemma 6.1. *No principal indecomposable $\mathcal{O}G$-module has a symplectic geometry.*

Proof. Suppose for the sake of contradiction that P is a principal indecomposable $\mathcal{O}G$-module that has a nondegenerate G-invariant symplectic bilinear form b. Then b induces a symplectic form, also denoted by b, on $F \otimes_{\mathcal{O}} P$. Proposition 1.1 of [3] implies that there is an irreducible FG-module M such that M is of quadratic type and M occurs with odd multiplicity in $F \otimes_{\mathcal{O}} P$. Then Lemma 3.6 of [12] shows that there is a component M' of $F \otimes_{\mathcal{O}} P$, that is isomorphic to M, such that the restriction of b to M' is nondegenerate. Thus M is of quadratic type and also of symplectic type, a contradiction. □

Lemma 6.2. *Let P be a principal indecomposable kG-module. Then each non-degenerate G-invariant quadratic form on P can be extended to a non-degenerate G-invariant quadratic form on kG. If in addition P is not the projective cover of k_G, then each G-invariant symmetric form on P is the polarization of a G-invariant quadratic form on P.*

Proof. This follows from Propositions 2.2 and 2.6 in [3]. □

We say that $a \in RG$ is *symmetric* if $a = a^o$, and say that it is *even* if $\lambda(a) \in 2R$. When dealing with quadratic forms on RG it is useful to fix (arbitrarily) a total order $<$ on the elements of G. Suppose that $a = \sum_{g \in G} a_g g \in RG$ is even and symmetric. Then for each $s \in R$, define a quadratic form $Q_{s,a}$ on RG via $Q_{s,a}(\sum\limits_{g \in G} x_g g) := s \sum\limits_{g \in G} x_g^2 + \sum\limits_{h<i \in G} x_h x_i a_{ih^{-1}}$. This is well defined because $a = a^o$. Moreover it is known that

$$\{Q_{s,a} \mid s \in R, \text{ and } a \in RG, \text{ even and symmetric}\}$$

gives all G-invariant quadratic forms on RG. If $R = k$ then

$$B_a(x, y) = Q_{s,a}(x + y) - Q_{s,a}(x) - Q_{s,a}(y), \quad \text{for all } x, y \in kG, \tag{5}$$

is the polarisation of $Q_{s,a}$.

Corollary 6.3. *Let e be a primitive idempotent in kG. Then ekG has a quadratic geometry if and only if there exists $a \in kG$, even and symmetric, such that the restriction of B_a to ekG is nondegenerate.*

Proof. Suppose first that ekG is the projective cover of the trivial module. Then ekG has multiplicity 1 as a component of kG. It follows from this that if $t \in \Omega$ then the restriction of B_t is a nondegenerate G-invariant symmetric bilinear form on ekG.

Now suppose that ekG is not the projective cover of the trivial module. Then the desired conclusion follows from Lemma 6.2 and the above description of the G-invariant quadratic forms on kG. □

The proof of the following result is adapted from that of Lemma 3.2 in [3]:

Lemma 6.4. *Let e be a primitive idempotent in kG. Suppose that $a \in kG$ is even and symmetric and that the restriction of B_a to ekG is nondegenerate. Then there exists $t \in \Omega$ such that $\lambda(at) \neq 0_k$, and the restriction of B_t to ekG is nondegenerate.*

Proof. As $\mathrm{Soc}(ekG)$ is irreducible, the degeneracy of a bilinear form on ekG depends on whether or not $\mathrm{Soc}(ekG)$ is contained in its kernel. It follows that if $a = c + d$ where $c, d \in kG$, then the restriction of one of B_c or B_d to ekG is nondegenerate.

Write $a = c + d$ where $c = \sum_{t\in\Omega\setminus\{1\}} \lambda(at)t$ and $d = \sum_{g\in(G\setminus\Omega)^{\pm}} \lambda(ag)(g + g^{-1})$. We claim that B_d is degenerate. Suppose otherwise. Set $\hat{d} := \sum_{g\in(G\setminus\Omega)^{\pm}} \widehat{\lambda(ag)}(g - g^{-1}) \in \mathcal{O}G$, where $\widehat{\lambda(ag)} \in \mathcal{O}$ has image $\lambda(ag)$ modulo $J(\mathcal{O})$. Then $\hat{d}^o = -\hat{d}$ and d is the image of $\hat{d}$ modulo $J(\mathcal{O})G$. As $\hat{d}$ is skew-symmetric, $B_{\hat{d}}$ is a non-degenerate G-invariant symplectic form on the lift $\widehat{ekG}$ of ekG to $\mathcal{O}G$. This contradicts Lemma 6.1, and proves our claim. It now follows from the first paragraph that there exists $t \in \Omega$ such that $\lambda(at) \neq 0_k$ and the restriction of B_t to ekG is nondegenerate. □

Proof of Theorem 1.3. Corollary 6.3 implies that there exists $a \in kG$ such that a is even and symmetric and the restriction of B_a to ekG is nondegenerate. It then follows from Lemma 6.4 that there exists $t \in \Omega$ such that the restriction of B_t to ekG is nondegenerate. Now the restriction of B_t to ekG coincides with the restriction of B_{e^ote} to ekG. So, again using Lemma 6.4, there exists $s \in \Omega$ such that $\lambda((e^o te)s) \neq 0_k$. We conclude from Theorem 1.1 that $\nu(\Phi) > 0$. □

Theorem 3.1 of [3] states that a principal indecomposable RG-module P has a quadratic geometry if and only if there exists a primitive idempotent $e \in RG$, and an element $t \in \Omega$, such that $P \cong eRG$ and $e^o = tet$. We note the following consequence of our methods:

Corollary 6.5. *Let e be a primitive idempotent in kG and let $t \in \Omega$ be such that $e^o = tet$. Then the irreducible kG-module $ekG/J(ekG)$ occurs as a composition factor in $k_{C_G(t)}\uparrow^G$.*

Proof. The essential work in the proof of Lemma 3.5 of [3] is to show that if $e^o = tet$, then the restriction of B_t to ekG is nondegenerate. As above, this means that there exists $s \in \Omega$ such that $\lambda(e^o tes) \neq 0_k$. Proposition 4.4 forces s to be G-conjugate to t. We deduce from this and Lemma 4.3 that the Scott module with vertex $< \underline{t}\sigma >$ is a component of $ekG^{\otimes 2}\downarrow_{\underline{G}\times\Sigma}$. It then follows from Corollary 5.2 that the projective cover of the trivial $kC_G(t)$-module is a component of the restriction $ekG{\downarrow_{C_G(t)}}$. Then by Frobenius-Nakayama reciprocity [9, 3.1.27(i)] the irreducible module $ekG/J(ekG)$ is a composition factor of $k_{C_G(t)}{\uparrow^G}$. □

7. Extension of a theorem of Benson and Carlson

In this section M is an indecomposable kG-module that affords a non-degenerate G-invariant symmetric bilinear form b. The adjoint β of b is an involutary k-algebra anti-automorphism of $\mathrm{End}_k(M)$, such that $\mu(g)^\beta = \mu(g^{-1})$, for all $g \in G$. Proposition 2.2 implies that $\mathrm{End}_k(M) \cong M^{\otimes 2}$, as $kG \wr \Sigma$-modules. Here $f \cdot \sigma = f^\beta$, for all $f \in \mathrm{End}_k(M)$. Using the methods of Section 6, we prove an analogue of a theorem of Benson and Carlson on the existence of Scott components in $\mathrm{End}_k(M)$.

Fix a vertex V of M, and a V-source S of M. Then $V \times V$ is a vertex of $\mathrm{End}_k(M)$, as $G \times G$-module. By Mackey's formula, each component of $\mathrm{End}_k(M){\downarrow_{\underline{G}}}$ has a vertex contained in $\underline{V}$. D. Benson and J. Carlson prove in [1, 2.4] that
(6)

$\mathrm{End}_k(M){\downarrow_{\underline{G}}}$ has a Scott component with vertex $\underline{V}$ if and only if $\dim(S)$ is odd.

Now $V \wr \Sigma$ is a vertex of $\mathrm{End}_k(M)$, as $G \wr \Sigma$-module. Again by Mackey's formula, each component of $\mathrm{End}_k(M){\downarrow_{\underline{G}\times\Sigma}}$, has a vertex contained in a group of the form $\underline{V} < \underline{n}\sigma >$, where $n \in N_G(V)$ is such that $n^2 \in V$. In view of (6), we ask

Question 7.1. *Does* $\mathrm{End}_k(M){\downarrow_{\underline{G}\times\Sigma}}$ *have a Scott component with vertex* $\underline{V} < \underline{n}\sigma >$ *for some* $n \in N_G(V)$ *with* $n^2 \in V$*?*

If the answer is 'yes', then in particular $\mathrm{End}_k(M){\downarrow_{\underline{G}}}$ has a Scott component with vertex $\underline{V}$. So $\dim(S)$ is odd. We therefore assume from now on that $\dim(S)$ is odd.

Proposition 3.5 shows that Question 7.1 can be answered by studying the restriction of the Broué-Robinson form to a certain subspace of $\mathrm{Tr}_V^G(\mathrm{End}_{kV}(M))$.

L. Puig defines a *point* of an algebra A to be an $A^\times$-conjugacy class of primitive idempotents of A. The theory of points and the related notions of defect points, multiplicity modules and multiplicity algebras is comprehensively explained in [11]. We borrow heavily from Thevenáz book.

Let δ_1 be the defect point of the G-algebra $\mathrm{End}_k(M)$ corresponding to the V-source S of M. So $Me \cong S$ as V-modules, for any idempotent $e \in \delta_1$. The *inertial group* of S or of δ_1 in $N_G(V)/V$ is $I := \{g \in N_G(V) \mid S^g \cong S\}/V$. Let $\mathfrak{M}_1$ be the unique maximal ideal of $\mathrm{End}_{kV}(M)$ that does not contain any idempotent in δ_1. The simple quotient algebra $\mathrm{End}_{kV}(M)/\mathfrak{M}_1$ is called a *defect multiplicity algebra* of $\mathrm{End}_k(M)$. By Wedderburn's theorem, this algebra is the endomorphism algebra of a *defect multiplicity module* P_1 of $\mathrm{End}_k(M)$. It is known that P_1 is a projective indecomposable module for a twisted group algebra of I.

Now σ acts on $\mathrm{End}_{kV}(M)$. Set $\delta_2 := \{e^\sigma \mid e \in \delta_1\}$. Then δ_2 is a defect point of $\mathrm{End}_{kV}(M)$ and $Me \cong S^*$ as V-modules, for each idempotent $e \in \delta_2$. Let P_2 be the defect multiplicity module of $\mathrm{End}_k(M)$ corresponding to δ_2. Its endomorphism ring is $\mathrm{End}_{kV}(M)/\mathfrak{M}_2$, where $\mathfrak{M}_2 := \mathfrak{M}_1^\sigma$.

Define the *extended inertial group* of S or of δ_1 in $N_G(V)/V$ as

$$J := \{g \in N_G(V) \mid S^g \cong S \text{ or } S^g \cong S^*\}/V.$$

Note that $I \leq J$ and that $[J : I] = 1$ or 2. For the moment we assume that $[J : I] = 2$.

Set $P := P_1 \oplus P_2$ and let $1 = e_1 + e_2$ be the corresponding orthogonal decomposition of the identity in $\mathrm{End}_k(P)$. Then $\mathrm{End}_{kJ}(P)$ is local, and the trivial group is a defect group of 1_P in J. Moreover, $\{e_1\}$ and $\{e_2\}$ are the only source points of the J-algebra $\mathrm{End}_k(P)$. These points are conjugate in J, and each has stabilizer I. Let $\rho_V := \rho_{V,G}^{\mathrm{End}_k(M),\mathrm{tr}}$ and $\rho_1 := \rho_{1,J}^{\mathrm{End}_k(P),\mathrm{tr}}$ be Broué-Robinson bilinear forms. Applying (1) of Proposition (1.8) of [2] twice, first to ρ_V and then to ρ_1, we get

$$\rho_V(f_1, f_2) = \rho_1(\theta(f_1), \theta(f_2)), \quad \text{for all } f_1, f_2 \in \mathrm{Tr}_V^{G\times\Sigma}(\mathrm{End}_{kV}(M)). \tag{7}$$

Here θ is the composition $\mathrm{End}_{kV}(M) \twoheadrightarrow \mathrm{End}_k(P_1) \times \mathrm{End}_k(P_2) \hookrightarrow \mathrm{End}_k(P)$.

The group $J \times \Sigma$ acts on $\mathrm{End}_k(P_1) \times \mathrm{End}_k(P_2)$, with σ acting as an involutary anti-automorphism. In addition, $e_1^\sigma = e_2$ and $e_2^\sigma = e_1$. We are in the situation of Theorem A.5 of our Appendix; there is a unique involutary k-algebra anti-automorphism $\hat\sigma$ of $\mathrm{End}_k(P)$ whose restriction to $\mathrm{End}_k(P_1) \times \mathrm{End}_k(P_2)$ coincides with σ. Moreover, there exists a central extension H of J by a finite cyclic $2'$-group Z and a commutative diagram of groups:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z & \xrightarrow{\text{inc}} & H & \xrightarrow{\pi} & J & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\rho} & & \\ 1 & \longrightarrow & \nabla(k) & \xrightarrow{\text{inc}} & C(\hat\sigma) & \xrightarrow{\pi} & C(\sigma) & \longrightarrow & 1 \end{array} \tag{8}$$

In particular $\hat\sigma$ is the adjoint of a nondegenerate H-invariant symmetric bilinear form $\hat b$ on the kH-module P. For notational simplicity we will use σ for $\hat\sigma$.

Theorem 7.2. *Let M be an indecomposable kG-module that affords a nondegenerate G-invariant symmetric bilinear form. Let V be a vertex of M. Then there exists $n \in N_G(V)$ with $n^2 \in V$, such that $\mathrm{End}_k(M)\downarrow_{\underline{G}\times\Sigma}$ has a Scott component with vertex $\underline{V} < \underline{n}\sigma >$ if and only if a source of M has odd dimension.*

Proof. We keep the notation and assumptions of this section. In particular we assume that $\dim(S)$ is odd. We initially suppose that $S \not\cong S^*$. So $[J : I] = 2$. The restriction of P to the inverse image of I in H is a sum of P_1 and its dual P_2. Thus P is not the projective cover of the trivial kH-module. However P is a self-dual principal indecomposable kH-module.

Set $\mathrm{Bil}_k(P)_0 := P_1^* \otimes P_2^* + P_2^* \otimes P_1^*$ and $\mathrm{Bil}_k(P)_1 := \mathrm{Bil}_k(P_1) + \mathrm{Bil}_k(P_2)$. Then $\mathrm{Bil}_k(P) = \mathrm{Bil}_k(P)_0 + \mathrm{Bil}_k(P)_1$ is a direct sum decomposition as $k\underline{H} \times \Sigma$-modules. As $e_1^\sigma = e_2$ and $e_2^\sigma = e_1$, the form $\hat b$ vanishes on $P_1 \times P_1$ and also on $P_2 \times P_2$. Thus $\hat b$ belongs to $\mathrm{Bil}_k(P)_0$.

Identify P with ekH, where e is a primitive idempotent in kH. Lemma 6.2 implies that there exists $a \in kH$ such that a is even and symmetric and $\hat b$ agrees with the restriction of B_a to ekH. By Lemma 6.4, there exists $t \in \Omega(H)$ such that $\hat b(et \otimes e) \neq 0_k$. But $\hat b$ belongs to $\mathrm{Bil}_k(ekH)_0^{G\times\Sigma}$, while $et \otimes e$ belongs to $(\mathrm{Bil}_k(ekH)^*)^{<t\sigma>}$. We conclude from remark 3.2 and Lemma 3.4 that $\mathrm{Bil}_k(P)_0$ has a Scott component with vertex $< t\sigma >$.

Recall that $B : \mathrm{End}_k(P) \to \mathrm{Bil}_k(P)$, such that $B_f(u, v) := \hat b(uf, v)$, for $f \in \mathrm{End}_k(P)$ and $u, v \in P$, is a $H \wr \Sigma$-module isomorphism. Under this isomorphism the $\underline{H} \times \Sigma$-submodule $\mathrm{End}_k(P_1) + \mathrm{End}_k(P_2)$ is mapped onto $\mathrm{Bil}_k(P)_0$. So $\mathrm{End}_k(P_1) + \mathrm{End}_k(P_2)$ has a Scott component with vertex $< \underline{t}\sigma >$, as $\underline{H} \times \Sigma$-module. Let n be an element of $N_G(V)/V$ whose image $\overline{n}$ in $N_G(V)/V$ coincides with the image of t in $J = H/Z$. In particular $n^2 \in V$. Now Z is a normal $2'$-subgroup of H that acts trivially on $\mathrm{End}_k(P_1)+\mathrm{End}_k(P_2)$. It follows that $\mathrm{End}_k(P_1)+\mathrm{End}_k(P_2)$ has a Scott component with vertex $<\overline{n}\sigma>$, as $J\times\Sigma$-module.

The previous paragraph shows that there exist $f_1, f_2 \in \mathrm{End}_{kG\times\Sigma}(M)$ such that $\theta(f_1), \theta(f_2) \in \mathrm{Tr}^{J\times\Sigma}_{<t\sigma>}(\mathrm{End}_{k<t\sigma>}(P))$ and $\rho_1(\theta(f_1),\theta(f_2)) \neq 0_k$. Since $\mathrm{End}_k(M)$ has vertex $V\wr\Sigma$, as $G\wr\Sigma$-module, we may write $f_1=\sum_u f_{1u}$ and $f_2=\sum_u f_{2u}$, where u ranges over certain elements of $N_G(V)$ with $u^2\in V$, and $f_{1u}, f_{2u} \in \mathrm{Tr}^{G\times\Sigma}_{V<u\sigma>}(\mathrm{End}_{kV<u\sigma>}(M))$. Let $\overline{u}$ denote the image of u in $N_G(V)/V$. Then

$$\theta(\mathrm{Tr}^{G\times\Sigma}_{V<u\sigma>}(\mathrm{End}_{kV<u\sigma>}(M))) \subseteq \mathrm{Tr}^{J\times\Sigma}_{<\overline{u}\sigma>}(\mathrm{End}_{k<\overline{u}\sigma>}(P)).$$

Using Lemma 3.3 twice, we get

$$\rho_1(\theta(f_{1n}),\theta(f_{2n})) = \rho_1(\theta(f_1),\theta(f_{2n})) = \rho_1(\theta(f_1),\theta(f_2)).$$

We deduce from this and Equation (7) that

$$\rho_V(f_{1n}, f_{2n}) \neq 0_k.$$

But $f_{1n}, f_{2n} \in \mathrm{Tr}^{G\times\Sigma}_{V<n\sigma>}(\mathrm{End}_{kV<n\sigma>}(M))$. We conclude from this and Proposition 3.5 that $\mathrm{End}_k(M){\downarrow}_{G\times\Sigma}$ has a Scott component with vertex $\underline{V}<\underline{n}\sigma>$.

The arguments are simpler when S is self-dual and $J=I$. In particular we can reach the desired conclusion without appealing to Theorem A.5. We leave the details to the reader. □

8. Acknowledgement

The ideas in the first half of this paper originated from conversations with R. Gow and W. Willems, whilst the latter was visiting University College Dublin in January 2005.

Appendix A. Anti-automorphisms and G-algebras

The aim of this appendix is to prove Theorem A.5. This enables us to lift projective representations of a group in a way that is compatible with an involutary algebra anti-automorphism.

If A is a k-algebra, we let $\mathrm{Aut}(A)$ denote the group of all automorphisms of A and we let $\mathrm{Aut}^*(A)$ denote the group of all automorphisms and anti-automorphisms of A. So each $\alpha\in\mathrm{Aut}^*(A)$ is a k-linear isomorphism of A such that either $(ab)^\alpha = a^\alpha b^\alpha$ for all $a,b\in A$, or $(ab)^\alpha = b^\alpha a^\alpha$ for all $a,b\in A$.

Fix an even dimensional k-vector space V and a decomposition $V=V_1\oplus V_2$, where $\dim(V_1)=\dim(V_2)$. Let $1_E=\epsilon_1+\epsilon_2$ be the corresponding orthogonal idempotent decomposition in $E=\mathrm{End}_k(V)$. Now $\epsilon_i E\epsilon_j$ can be identified with $E_{ij} := \mathrm{Hom}_k(V_i,V_j)$. In this way E has a matrix representation $E=\begin{bmatrix} E_1 & E_{12}\\ E_{21} & E_2\end{bmatrix}$, where for notational simplicity E_i denotes E_{ii}.

The general linear group $\mathrm{GL}(V)$ of V is the group units in E. We identify $\mathrm{GL}(V_1)\times\mathrm{GL}(V_2)\leq\mathrm{GL}(V)$ with the set of elements $g_1+g_2\in E$ such that g_i is a unit in E_i. The factor group $\mathrm{PGL}(V)=\mathrm{GL}(V)/k^\times 1_E$ is naturally isomorphic to $\mathrm{Aut}(E)$. If $\theta(g)$ denotes the image of $g\in\mathrm{GL}(V)$ in $\mathrm{Aut}(E)$, then $f^{\theta(g)}=g^{-1}fg$, for all $f\in E$.

Let $N(\epsilon_1,\epsilon_2)$ denote the stabilizer subgroup of the set $\{\epsilon_1,\epsilon_2\}$ in $\mathrm{Aut}(E)$, and let $\mathrm{GL}(V_1,V_2)$ be the inverse image of $N(\epsilon_1,\epsilon_2)$ in $\mathrm{GL}(V)$. As V_1 and V_2 are isomorphic subspaces of V, there is a unit τ in E such that $\epsilon_1\tau=\tau\epsilon_2$. Replacing

τ by $\epsilon_1\tau + \tau^{-1}\epsilon_1$, we can and do assume that τ is an involution. It is clear that $\mathrm{GL}(V_1, V_2) = \mathrm{GL}(V_1) \times \mathrm{GL}(V_2) :< \tau >$, a group that is isomorphic to $\mathrm{GL}_d(k) \wr \Sigma_2$.

Restriction to $E_1 \times E_2$ induces a group homomorphism $\phi : N(\epsilon_1, \epsilon_2) \to \mathrm{Aut}(E_1 \times E_2)$. Each $\alpha \in \mathrm{Aut}(E_1 \times E_2)$ satisfies $\epsilon_i^\alpha \in \{\epsilon_1, \epsilon_2\}$, for $i = 1, 2$. If $\epsilon_i^\alpha = \epsilon_{3-i}$ then $\epsilon_i^{\alpha t} = \epsilon_i$, while if $\epsilon_i^\alpha = \epsilon_i$ then we can identify α, via its restrictions to E_1 and to E_2, with an element of $\mathrm{Aut}(E_1) \times \mathrm{Aut}(E_2)$. It follows that $\mathrm{Aut}(E_1 \times E_2) = \mathrm{Aut}(E_1) \times \mathrm{Aut}(E_2) :< \phi(\tau) >$, a group that is isomorphic to $\mathrm{PGL}_d(k) \wr \Sigma_2$.

Our lemma is a consequence of this discussion:

Lemma A.1. *Every k-automorphism of $E_1 \times E_2$ extends to an inner automorphism of E. The kernel of the surjective map $\phi\theta : \mathrm{GL}(V_1, V_2) \to \mathrm{Aut}(E_1 \times E_2)$ is $k^\times\epsilon_1 + k^\times\epsilon_2$.*

We now discuss k-algebra anti-automorphisms. Fix a non-degenerate symmetric bilinear k-form b_1 on V_1. Then $Q(v) := b_1(v\epsilon_1, v\epsilon_2\tau)$, for $v \in V$, defines a quadratic form on V. Let b be the polarization of Q. So $b(u, v) = b_1(u\epsilon_1, v\epsilon_2\tau) + b_1(v\epsilon_1, u\epsilon_2\tau)$, for all $u, v \in V$. The adjoint of b is an involution $\beta \in \mathrm{Aut}^*(E)\backslash\mathrm{Aut}(E)$ such that $\tau^\beta = \tau$ and $\epsilon_1^\beta = \epsilon_2$ and $\epsilon_2^\beta = \epsilon_1$. Also $\mathrm{Aut}^*(E) = \mathrm{Aut}(E) :< \beta >$, as the product of two anti-automorphisms is an automorphism.

Let $g \in \mathrm{GL}(V)$ and $f \in E$. Then $f^{\beta\theta(g)\beta} = (g^{-1}f^\beta g)^\beta = g^\beta f g^{-\beta}$. So

$$\theta(g)^\beta = \theta(g^{-\beta}) \quad \text{in } \mathrm{Aut}(E). \tag{9}$$

For instance, $\theta(\tau)^\beta = \theta(\tau)$, as τ is an involution.

Let $N^*(\epsilon_1, \epsilon_2)$ be the stabilizer subgroup of the set $\{\epsilon_1, \epsilon_2\}$ in $\mathrm{Aut}^*(E)$. Then β belongs to $N^*(\epsilon_1, \epsilon_2)\backslash N(\epsilon_1, \epsilon_2)$. Restriction gives a group homomorphism, also denoted by ϕ, from $N^*(\epsilon_1, \epsilon_2)$ into $\mathrm{Aut}^*(E_1 \times E_2)$. Clearly $N^*(\epsilon_1, \epsilon_2) = N(\epsilon_1, \epsilon_2) :< \phi(\beta) >$ and

$$\mathrm{Aut}^*(E_1 \times E_2) = \mathrm{Aut}(E_1 \times E_2) :< \phi(\beta) >= \mathrm{Aut}(E_1) \times \mathrm{Aut}(E_2) :< \phi(\beta), \phi(\tau) > .$$

The latter group is isomorphic to $\mathrm{Aut}^*(E_1) \wr \Sigma_2$ and also to a group $\mathrm{PGL}_d(k)^2 : \mathbb{Z}_2^2$.

We summarise this discussion with:

Lemma A.2. *Every k-algebra anti-automorphism of $E_1 \times E_2$ can be extended to a k-algebra anti-automorphism of E. The extensions of a single anti-automorphism form a coset of $\theta(k^\times\epsilon_1 + k^\times\epsilon_2)$ in $N^*(\epsilon_1, \epsilon_2)$.*

For involutions in $\mathrm{Aut}^*(E_1 \times E_2)$, we even have:

Lemma A.3. *Let σ be an involutary k-algebra anti-automorphism of $E_1 \times E_2$ such that $\epsilon_1^\sigma = \epsilon_2$. Then there is a unique extension of σ to an involutary anti-automorphism $\hat{\sigma}$ of E.*

Proof. Let α be any element of $N^*(\epsilon_1, \epsilon_2)$ satisfying $\phi(\alpha) = \sigma$. Then $\alpha\beta$ is a k-algebra automorphism of $E_1 \times E_2$ and moreover $\epsilon_i^{\alpha\beta} = \epsilon_i$, for $i = 1, 2$. So $\alpha = \theta(g_1 + g_2)\beta$, for some units $g_i \in E_i$. Also $\{\alpha_\mu := \theta(\mu g_1 + g_2)\beta \mid \mu \in k^\times\}$ is the set of extensions of σ to E.

Let us denote the inverse of g_i in E_i by g_i^{-1}. As $\epsilon_i^\beta = \epsilon_{3-1}$ and β is an algebra anti-automorphism, we have $(g_i^{-1})^\beta = (g_i^\beta)^{-1}$ in E_{3-i}. We write $g_i^{-\beta}$ for this common element. For $\mu \in k^\times$, we see from (9) that

$$\alpha_\mu^2 = \theta(\mu g_1 + g_2)\theta(\mu g_1 + g_2)^\beta = \theta(\mu g_1 g_2^{-\beta} + \mu^{-1} g_2 g_1^{-\beta}).$$

As σ is an involution, α^2 acts as the identity on both E_1 and E_2. In particular $g_1 g_2^{-\beta} = \lambda\epsilon_1$, for some $\lambda \in k^\times$. It follows that $g_2^{-\beta}$ is a scalar multiple of g_1^{-1}, whence $g_2^{-\beta}$ commutes with g_1. Thus $g_2^{-\beta} g_1 = \lambda\epsilon_1$. Applying ${}^{-\beta}$ to this, we deduce that $g_2 g_1^{-\beta} = \lambda^{-1}\epsilon_2$. Thus $\alpha_\mu^2 = \theta(\mu\lambda\epsilon_1 + \mu^{-1}\lambda^{-1}\epsilon_2)$.

The last paragraph implies that the extension α_μ is an involution in $\mathrm{Aut}^*(E)$ if and only if $\mu\lambda = \mu^{-1}\lambda^{-1}$, which holds if and only if $\mu = \lambda^{-1}$. We conclude that $\hat{\sigma} := \alpha_{\lambda^{-1}}$ is the unique extension of σ to E that is an involution. □

Fix an involutary k-algebra anti-automorphism σ of $E_1 \times E_2$ such that $\epsilon_1^\sigma = \epsilon_2$. Denote by $\hat{\sigma}$ the unique involution in $N^*(\epsilon_1, \epsilon_2)$ such that $\phi(\hat{\sigma}) = \sigma$. Let $C(\sigma)$ denote the centralizer of σ in $\mathrm{Aut}(E_1 \times E_2)$ and define

$$C(\hat{\sigma}) := \{g \in \mathrm{GL}(V_1, V_2) \mid g^{\hat{\sigma}} = g^{-1}\}.$$

As $\hat{\sigma}$ is an anti-automorphism, $C(\hat{\sigma})$ is a subgroup of $\mathrm{GL}(V_1, V_2)$. Note that if $g \in C(\hat{\sigma})$, then $\theta(g)$ commutes with $\hat{\sigma}$, and hence $\phi\theta(g)$ belongs to $C(\sigma)$.

Lemma A.4. *The map $\phi\theta$ induces a group epimorphism $C(\hat{\sigma}) \twoheadrightarrow C(\sigma)$.*

Proof. Let $x \in C(\sigma)$. Choose $g \in \mathrm{GL}(V_1, V_2)$ such that $\phi\theta(g) = x$. Then $\phi\theta(gg^{\hat{\sigma}}) = \phi\theta(g)\phi\theta(g^{-1})^\sigma = 1$. It follows that $gg^{\hat{\sigma}} = \lambda_1\epsilon_1 + \lambda_2\epsilon_2$, for some $\lambda_1, \lambda_2 \in k^\times$. But $\hat{\sigma}$ is an involutary k-algebra anti-automorphism. So $gg^{\hat{\sigma}}$ is fixed by $\hat{\sigma}$. Applying $\hat{\sigma}$ to $\lambda_1\epsilon_1 + \lambda_2\epsilon_2$ we see that $\lambda_1 = \lambda_2$. We deduce from this that $g^{\hat{\sigma}} = \lambda_1 g^{-1}$. As k is perfect and has characteristic 2, there exists $\mu \in k^\times$ such that $\mu\lambda_1 = \mu^{-1}$. Then

$$(\mu g)^{\hat{\sigma}} = \mu g^s = \mu\lambda_1 g^{-1} = (\mu g)^{-1}.$$

So $\mu g \in C(\hat{\sigma})$, which completes the proof. □

Set $\nabla(k) := \{(\lambda, \lambda^{-1}) \in \mathrm{GL}(V_1) \times \mathrm{GL}(V_2)\}$, a subgroup of $\mathrm{GL}(V)$. So $\nabla(k)$ is the kernel of the restriction of $\phi\theta$ to $C(\hat{\sigma})$. We now give the main result of this section.

Theorem A.5. *Let V, E, E_i, ϵ_i be as above and let σ be an involutary anti-automorphism of $E_1 \times E_2$ such that $\epsilon_1^\sigma = \epsilon_2$, and let $\hat{\sigma}$ be the unique involutary anti-automorphism of E whose restriction to $E_1 \times E_2$ coincides with σ. Suppose that $\rho : G \to C(\sigma)$ is a group homomorphism. Then there is a commutative diagram of groups*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & Z & \xrightarrow{\text{inc}} & \hat{G} & \xrightarrow{\pi} & G & \longrightarrow & 1 \\
 & & \big\downarrow \eta & & \big\downarrow \tau & & \big\downarrow \rho & & \\
1 & \longrightarrow & \nabla(k) & \xrightarrow{\text{inc}} & C(\hat{\sigma}) & \xrightarrow{\theta} & C(\sigma) & \longrightarrow & 1
\end{array} \tag{10}$$

Here $\hat{G}$ is a finite central extension of G by a cyclic group Z of odd order. In particular, $\hat{\sigma}$ is the adjoint of a nondegenerate $\hat{G}$-invariant symmetric bilinear form on V.

Proof. This is a consequence of Lemma A.4 and standard arguments involving pull-back diagrams and cohomology. One could combine Proposition (10.5) and the methods of Example (10.8) in [11], for instance. □

References

[1] D. Benson, J. Carlson, Nilpotent elements in the Green ring, *J. Algebra* 104 (1986) 329–350.
[2] M. Broué, G. R. Robinson, Bilinear Forms on G-Algebras, *J. Algebra* 104 (1986) 377–396.
[3] R. Gow, W. Willems, Quadratic geometries, projective modules, and idempotents, *J. Algebra* 160 (1) (1993) 257–272.
[4] J. A. Green, Blocks of modular representations, *Math. Zeit.* 79 (1962) 100–115.
[5] J. A. Green, Multiplicities, Scott Modules and Lower Defect Groups, *J. London Math. Soc.* 28 (1983) 282–292.
[6] B. Külshammer, Some indecomposable modules and their vertices, *J. Pure Appl. Algebra* 86 (1993) 65–73.
[7] J. Murray, Projective modules and involutions, to appear in *J. Algebra*.
[8] J. Murray, Strongly real 2-blocks and the Frobenius-Schur indicator, to appear in *Osaka J. Math.*
[9] H. Nagao, Y. Tsushima, "Representations of Finite Groups", Academic Press, Inc. , 1989.
[10] G. R. Robinson, The Frobenius-Schur Indicator and Projective Modules, *J. Algebra* 126 (1989) 252–257.
[11] J. Thévenaz, "G-algebras and modular representation theory", Oxford Mathematical Monographs, Clarendon Press, Oxford, 1995.
[12] W. Willems, Duality and forms in representation theory, *Prog. in Math.* 95 (1991) 509–520.

Mathematics Department, National University of Ireland, Maynooth, Co. Kildare, Ireland.

E-mail address: `John.Murray@maths.nuim.ie`