# STOCHASTIC EQUILIBRIA OF AIMD COMMUNICATION NETWORKS

F. WIRTH*, R. STANOJEVIĆ†, R. SHORTEN‡, AND D. LEITH§

**Abstract.** In this paper we develop tools to analyse a recently proposed random matrix model of communication networks that employ AIMD congestion control algorithms. We analyse properties of the Markov process describing the evolution of the window sizes of network users. Using paracontractivity properties of the matrices involved in the model, it is shown that the process has a unique invariant probability and the support of this probability is characterized. Based on these results we obtain a weak law of large numbers for the average distribution of resources between the users of a network. This shows that under reasonable assumptions such networks have a well defined stochastic equilibrium. NS-simulation[1] results are given to demonstrate the efficacy of the obtained formulae.

**Key Words:** Positive matrices; Infinite products of positive matrices; AIMD Congestion Control; Communication networks; Markov e-chain; weak law of large numbers.

## 1. Introduction.

Recent years have witnessed increased attention in the dynamics of communication networks. Networks of devices that employ *Additive-Increase Multiplicative Decrease* (AIMD) congestion control algorithms, such as the widely deployed *Transmission control protocol* (TCP), have become the focus of much of this activity. Typically, the approach adopted by the community is to model such networks by means of a fluid analogy and to employ techniques from control theory and convex optimization in their analysis; see the recent book by Srikant [26] and the references therein for an overview of this work. Recently, several authors have proposed an alternative model of TCP dynamics using products of random matrices [3, 2, 25]. The basic approach followed in these papers is to use ideas from hybrid systems theory to model the dynamics of AIMD networks as a switched, or time-varying, discrete time linear system. The approach adopted in [25] is particularly useful as it enables techniques from the theory of nonnegative matrices and Markov chains to be employed in the analysis of these networks. The application of these techniques to the study of such networks is the principal contribution of this paper.

In Section 2 we begin our discussion by giving an overview of AIMD congestion control and by briefly reviewing the random matrix model of AIMD network dynamic first derived in [25]. In Section 3 a number of basic results are presented relating to the set of matrices used in the model. It is shown that on a jointly invariant subspace the matrices are paracontractive, which is used to show that with probability one, left products of the matrices approach the set of rank one column stochastic matrices. This ergodicity property plays a vital role in all the subsequent considerations. Section 4 is devoted to the analysis of the Markov cain model of the AIMD process. It is shown that the chain in question is an e-chain. Using the results of Section 3 we obtain that this chain has positive and aperiodic states. From this we obtain the unique existence of an invariant probability and weak law of large number statements. Finally, the support of the invariant probability is characterized. In Section 5 we give the main results of this paper; here we collect and derive a number of results that are useful in characterising the stochastic equilibria of various types of communication networks that employ AIMD congestion control mechanisms. Finally, in Section 6 we apply these results to the study of networks employing TCP congestion control. It is shown that the model is able to predict the average behaviour of TCP flows very accurately.

*Hamilton Institute, NUI Maynooth, Maynooth, Co. Kildare, Ireland, Email: fabian.wirth@may.ie

†Email: rade.stanojevic@may.ie

‡Email: robert.shorten@may.ie

§Email: doug.leith@may.ie

[1]The simulation program *NS*, or *Network simulator*, is an industry standard for the simulation of internet dynamics.

**2. Column stochastic matrices and AIMD congestion control.** A communication network consists of a number of sources and sinks connected together via links and routers. In this paper we assume that these links can be modelled as a constant propagation delay together with a queue, that the queue is operating according to a drop-tail discipline, and that all of the sources are operating a *Additive-Increase Multiplicative Decrease* (AIMD) -like congestion control algorithm. AIMD congestion control operates a window based congestion control strategy. More specifically, each source maintains an internal variable $w_i$ (the window size) which tracks the number of sent unacknowledged packets that can be in transit at any time. When the window size is exhausted, the source must wait for an acknowledgement before sending a new packet. Congestion control is achieved by dynamically adapting the window size according to an additive-increase multiplicative-decrease law. Roughly speaking, the basic idea is for a source to gently probe the network for spare capacity by increasing the rate at which packets are inserted into the network, and to rapidly back-off the number of packets transmitted through the network when congestion is detected through the loss of packets of date. More specifically, an individual source sends packets of data through the network to a destination, and the transmission is deemed complete, if an acknowledgement issued by the destination upon receipt of the packet is received by the source. As long as transmission is successful, that is as long as all acknowledgements are received, the source increments $w_i(t)$, by a fixed amount $\alpha_i$ upon receipt of an acknowledgement. If an acknowledgement for a certain packet does not arrive at the sender, it is assumed, that there has been a packet loss due to congestion in the network. As a consequence, the variable $w_i(t)$ is reduced in multiplicative fashion to $\beta_i w_i(t)$.

**2.1. A model for AIMD dynamics.** Networks of synchronised sources and drop-tail queues have also been the subject of several other studies [16, 3, 5, 1, 12]. The novelty of our approach lies in the fact that we use positive matrices to model their behaviour. We shall see that this fact will enable results for such matrices to be employed to make predictions concerning the behaviour of AIMD networks.

In [24] a model has been presented which assumes that (i) at congestion every source experiences a packet drop; and (ii) each source has the same round-trip-time (RTT)[2].

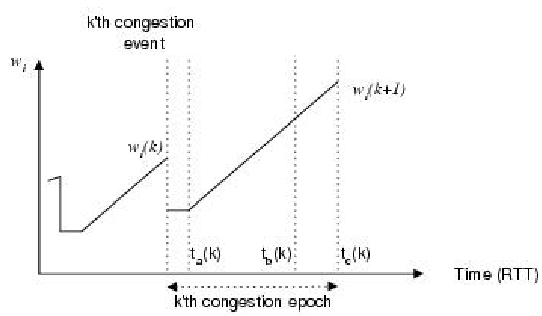



FIG. 2.1. *Evolution of window size*

We now describe an augmented model without the assumption of synchronisation, i.e at a congestion event not all sources are necessarily informed of this congestion. For the moment uniform RTT is still assumed, we will weaken this assumption later on. Let $w_i(k)$ denote the congestion window size of source $i$ immediately before the $k$'th network congestion event is detected by the source. Over the $k$'th congestion epoch three important events can be discerned: $t_a(k), t_b(k)$ and $t_c(k)$; as depicted in Figure 2.1. The time $t_a(k)$ denotes the instant at which the number of unacknowledged packets in flight equals $\beta_i w_i(k)$; $t_b(k)$ is the time at which the bottleneck queue is full; and $t_c(k)$ is the time at which packet drop is detected by

[2]One RTT is the time between sending a packet and receiving the corresponding acknowledgement when there are no packet drops.

some of the sources, where time is measured in units of RTT[3]. It follows from the definition of the *AIMD* algorithm that the window evolution is completely defined over all time instants by knowledge of the $w_i(k)$ and the event times $t_a(k), t_b(k)$ and $t_c(k)$ of each congestion epoch. We therefore only need to investigate the behaviour of these quantities.

We assume that source that loose a package at congestion are informed of this loss one RTT after the queue at the bottleneck link becomes full; that is $t_c(k) - t_b(k) = 1$. Also,

$$w_i(k) \geq 0, \quad \text{and} \quad \sum_{i=1}^{n} w_i(k) = P + \sum_{i=1}^{n} \alpha_i, \ \forall k > 0, \tag{2.1}$$

where $P$ is the maximum number of packets which can be in transit in the network at any time; $P$ is usually equal to $q_{max} + BT_d$ where $q_{max}$ is the maximum queue length of the congested link, $B$ is the service rate of the congested link in packets per second and $T_d$ is the round-trip time when the queue is empty. At the $(k+1)$th congestion event

$$w_i(k+1) = \begin{cases} \beta_i^s w_i(k) + \alpha_i[t_c(k) - t_a(k)] & \text{if the source } i \text{ experiences congestion,} \\ w_i(k) + \alpha_i[t_c(k) - t_a(k)] & \text{else.} \end{cases} \tag{2.2}$$

and we set

$$\beta_i(k) \in \{\beta_i^s, 1\}, \tag{2.3}$$

corresponding to whether the source experiences a packet loss or not. Then summing the equations in (2.2) and using (2.1) we obtain

$$t_c(k) - t_a(k) = \frac{1}{\sum_{i=1}^{n} \alpha_i} \left[ P - \sum_{i=1}^{n} \beta_i(k) w_i(k) \right] + 1. \tag{2.4}$$

and using (2.2)–(2.4), it follows that

$$w_i(k+1) = \beta_i(k) w_i(k) + \frac{\alpha_i}{\sum_{j=1}^{n} \alpha_j} \left[ \sum_{i=1}^{n} (1 - \beta_i(k)) w_i(k) \right]. \tag{2.5}$$

Thus the dynamics of an entire network of such sources is given by

$$W(k+1) = A(k) W(k), \tag{2.6}$$

where $W^T(k) = [w_1(k), \cdots, w_n(k)]$, and

$$A(k) = \begin{bmatrix} \beta_1(k) & 0 & \cdots & 0 \\ 0 & \beta_2(k) & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \beta_n(k) \end{bmatrix} + \frac{1}{\sum_{j=1}^{n} \alpha_j} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdots \\ \alpha_n \end{bmatrix} \begin{bmatrix} 1 - \beta_1(k) & \cdots & 1 - \beta_n(k) \end{bmatrix}. \tag{2.7}$$

As the entries of $W(k)$ are nonnegative for all $k \geq 0$ the equations (2.6) defines a positive linear system [4]. Using $b_i(s) \in (0, 1], i = 1, \ldots, n$, we also see that all possible matrices that appear are column stochastic. In the sequel we will call column stochastic matrices of the form (2.7) *AIMD matrices*.

[3]Note that measuring time in units of RTT results in a linear rate of increase for each of the congestion window variables between congestion events.

So far we have worked with the assumption of uniform RTT, which is quite restrictive (although it may, for example, be valid in some long-distance networks [29]). It is therefore of great interest to extend our approach to more general network conditions. As we will see the model that we obtain shares many structural and qualitative properties of the model described above. To distinguish variables, we will from now on denote the nominal parameters of the sources used in the previous section by $\alpha_i^s, \beta_i^s$, $i = 1, \ldots, n$. Here the index $s$ may remind the reader that these are the parameters that are chosen by each *source*.

Consider the general case of a number of sources competing for shared bandwidth in a generic dumbbell topology (where sources may have different round-trip times and drops need not be synchronised). The evolution of the window size $w_i$ of a typical source as a function of time, over the $k'th$ congestion epoch, is depicted in Figure 2.2. As before a number of important events may be discerned, where we now measure

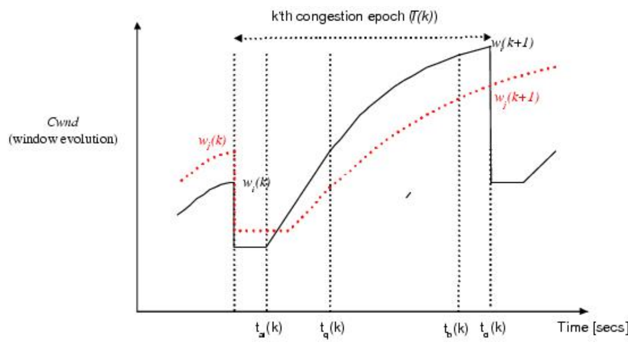



FIG. 2.2. *Evolution of window size over a congestion epoch. $T(k)$ is the length of the congestion epoch in seconds.*

time in seconds, rather than units of $RTT$. Denote by $t_{ai}(k)$ the time at which the number of packets in flight belonging to source $i$ is equal to $\beta_i^s w_i(k)$; $t_q(k)$ is the time at which the bottleneck queue begins to fill; $t_b(k)$ is the time at which the bottleneck queue is full; and $t_{ci}(k)$ is the time at which the $i$'th source is informed of congestion. In this case the evolution of the $i$'th congestion window variable does not evolve linearly with time after $t_q$ seconds due to the effect of the bottleneck queue filling and the resulting variation in RTT; namely, the $RTT$ of the $i$'th source increases according to $RTT_i(t) = T_{d_i} + q(t)/B$ after $t_q$, where $T_{d_i}$ is the $RTT$ of source $i$ when the bottleneck queue is empty and $0 \leq q(t) \leq q_{\max}$ denotes the number of packets in the queue. Note also that we do not assume that every source experiences a drop when congestion occurs. For example, a situation is depicted in Figure 2.2 where the $i$'th source experiences congestion at the end of the epoch whereas the $j$'th source does not.

Given these general features it is clear that the modelling task is more involved than in the synchronised case. Nonetheless, it is possible to relate $w_i(k)$ and $w_i(k+1)$ using a similar approach to the synchronised case by accounting for the effect of non-uniform RTT's and unsynchronised packet drops as follows.

Due to the variation in round trip time, the congestion window of a flow does not evolve linearly with time over a congestion epoch. Nevertheless, we may relate $w_i(k)$ and $w_i(k+1)$ linearly by defining an average rate $\alpha_i(k)$ depending on the $k$'th congestion epoch:

$$\alpha_i(k) := \frac{w_i(k+1) - \beta_i(k)w(k)}{T(k)}, \tag{2.8}$$

where $T(k)$ is the duration of the $k$'th epoch measured in seconds. Equivalently we have

$$w_i(k+1) = \beta_i(k)w_i(k) + \alpha_i(k)T(k)\,. \tag{2.9}$$

In the case when $q_{max} << BT_{d_i}$ , $i = 1, \ldots, n$, the average $\alpha_i$ are (almost) independent of $k$ and given by

$\alpha_i(k) \approx \alpha_i^s / T_{d_i}$ for all $k \in \mathbb{N}, i = 1, \ldots, n$. The situation when

$$\alpha_i \approx \frac{\alpha_i^s}{T_{d_i}}, \quad i = 1, \ldots, n \tag{2.10}$$

is of considerable practical importance and such networks are the principal concern of this paper. This corresponds to the case of a network whose bottleneck buffer is small compared with the delay-bandwidth product for all sources utilising the congested link. Such conditions prevail on a variety of networks; for example networks with large delay-bandwidth products, and networks where large jitter and/or latency cannot be tolerated.

In view of (2.3) and (2.9) a convenient representation of the network dynamics is obtained as follows. At congestion the bottleneck link is operating at its capacity $B$, i.e.,

$$\sum_{i=1}^{n} \frac{w_i(k) - \alpha_i}{RTT_{i,max}} = B, \tag{2.11}$$

where $RTT_{i,max}$ is the RTT experienced by the $i$'th flow when the bottleneck queue is full. Note, that $RTT_{i,max}$ is independent of $k$. Setting $\gamma_i := (RTT_{i,max})^{-1}$ we have that

$$\sum_{i=1}^{n} \gamma_i w_i(k) = B + \sum_{i=1}^{n} \gamma_i \alpha_i \,. \tag{2.12}$$

Using steps similar to the ones performed in (2.2)–(2.4) we obtain the model

$$w_i(k+1) = \beta_i(k) w_i(k) + \frac{\alpha_i}{\sum_{j=1}^{n} \gamma_j \alpha_j} \left( \sum_{j=1}^{n} \gamma_j (1 - \beta_j(k)) w_j(k) \right) \tag{2.13}$$

and the dynamics of the entire network of sources at the $k$-th congestion event are again described by $W(k+1) = A(k)W(k)$, where

$$A(k) = \begin{bmatrix} \beta_1(k) & 0 & \cdots & 0 \\ 0 & \beta_2(k) & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \beta_n(k) \end{bmatrix} + \frac{1}{\sum_{j=1}^{n} \gamma_j \alpha_j} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdots \\ \alpha_n \end{bmatrix} [\gamma_1(1 - \beta_1(k)), \ldots, \gamma_n(1 - \beta_n(k))] \tag{2.14}$$

and where $\beta_i(k)$ is either 1 or $\beta_i^s$. The non-negative matrices $A_2, .., A_m$ are constructed by taking the matrix $A_1$,

$$A_1 = \begin{bmatrix} \beta_1^s & 0 & \cdots & 0 \\ 0 & \beta_2^s & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \beta_n^s \end{bmatrix} + \frac{1}{\sum_{j=1}^{n} \gamma_j \alpha_j} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdots \\ \alpha_n \end{bmatrix} \begin{bmatrix} \gamma_1(1 - \beta_1^s), & \ldots, & \gamma_n(1 - \beta_n^s) \end{bmatrix}$$

and setting some, but not all, of the $\beta_i$ to 1. This gives rise to $m = 2^n - 1$ matrices associated with the system (2.13) that correspond to the different combinations of source drops that are possible. These matrices are not AIMD matrices in the sense we have defined above. However, by a small transformation we come back to our original situation.

By considering the evolution of $W_\gamma^T(k) = [\gamma_1 w_1(k), \gamma_2 w_2(k), \ldots, \gamma_n w_n(k)]$ we obtain the following description of the network dynamics:

$$W_\gamma(k+1) = \bar{A}(k) W_\gamma(k) \tag{2.15}$$

with $\bar{A}(k) \in \bar{\mathcal{A}} = \{\bar{A}_1, ..., \bar{A}_m\}$, $m = 2^n - 1$ and where the $\bar{A}_i$ are obtained by the diagonal similarity transformation associated with the change of variables. As before the non-negative matrices $\bar{A}_2, .., \bar{A}_m$ are constructed by taking the matrix $\bar{A}_1$ and setting some, but not all, of the $\beta_i^s$ to 1. It is easy to see, that all of the matrices in the set $\bar{\mathcal{A}}$ are now AIMD matrices; for convenience we use this representation of the network dynamics to prove the main mathematical results presented in this paper. Note furthermore, that the similarity transformation used to bring the matrices in AIMD form only depends on the round trip times $RTT_i$ and not on the $\alpha_i, \beta_i$.

**2.2. Networks of flows whose parameters vary in time.** Before proceeding with our analysis we note that for some applications it is convenient to allow the parameters of the matrix $A(k)$ to vary in more general a manner than that described in the previous two sections. Our model may be extended trivially to model networks whose AIMD parameters vary with time: $\alpha_i(k)$; $\beta_i(k)$. Such situations may arise in applications where the protocol adapts its parameters to reflect prevailing network conditions or in applications where variations in network delays lead to a consequent variation in the AIMD parameters (for example due to routing changes or in wireless networks) [22]; in fact a number of AIMD networks of this type have recently been proposed by a number of authors in the context of high-speed long-distance networks [8, 29]. We account for such behaviour in this paper by defining the set $\mathcal{M}$ to be the union of a finite number of matrix sets $\bar{\mathcal{A}}_j$ each of which are defined as above but which correspond to fixed AIMD parameters $\{\alpha_1^j, ...., \alpha_n^j\}$ and $\{\beta_1^j, ...., \beta_n^j\}$, $1 \le j \le h$, with $\mathcal{M} = \bigcup_{j=1}^h \bar{\mathcal{A}}_j$, where $h$ is some fixed integer.

COMMENT 2.1. *Before proceeding we note that networks of unsynchronised sources have also been the subject of wide study in the TCP community: see [18, 20, 21, 19, 22, 13, 27, 17, 6, 14, 15, 14] and the accompanying references for further details. While most of this work has concentrated on developing and analysing TCP models that are based upon fluid analogies, several authors have recently developed hybrid systems models of networks with a single bottleneck link which employ AIMD congestion control mechanisms: most notably by Hespanha [11] and Baccelli and Hong [2]. We note that the model derived in [2] is similar to the model presented here. However, whereas the model derived by Baccelli and Hong is also a random matrix model, their model is both affine and the homogeneous (linear) part is characterised by general matrices (namely, not by non-negative matrices).*

**3. Preliminaries.** The principal objective of this paper it to collect and develop analytic tools to analyse models of the form derived in Section 2. We will see in Section 5 that it is possible to characterise the stochastic behaviour of the random variable $W(k)$ under certain assumptions. The derivation of these results is somewhat technical and to ease exposition we introduce here a number of definitions and technical results.

**3.1. Basic notation.** The following results are based on the theory of nonnegative matrices. A matrix $A$ or a vector $x$ is said to be nonnegative if each of its entries is a nonnegative real number and matrices or vectors are called positive if all their entries are positive. We write $A \succ B$ or $A \succeq B$ if $A - B$ is positive, respectively nonnegative. The set of nonnegative matrices in $\mathbb{R}^{n \times n}$ is denoted by $\mathbb{R}_+^{n \times n}$. The componentwise absolute value of $A = (a_{ij}) \in \mathbb{R}^{n \times m}$ is defined by $|A| := (|a_{ij}|) \in \mathbb{R}_+^{n \times m}$.

A special subset of $\mathbb{R}_+^{n \times n}$ are the column stochastic matrices. A matrix $A \in \mathbb{R}_+^{n \times n}$ is called column stochastic if for each of its columns the sum of the corresponding elements is equal to 1. Denoting $y := [1, 1, \ldots, 1]^T$, it follows that $y^T$ is a left eigenvector of a column stochastic matrix corresponding to the eigenvalue 1. We denote by $\mathcal{R} \subset \mathbb{R}^{n \times n}$ the set of all column stochastic matrices of rank 1 and the distance between a matrix

$P \in \mathbb{R}^{n\times n}$ and the set $\mathcal{R}$ by $\operatorname{dist}(P, \mathcal{R}) = \inf\{\|P - C\| : \ C \in \mathcal{R}\}$. Finally, the $j$-th standard unit vector is denoted by $e_j$.

**3.2. Basic assumptions.** Our basic objective is to model the evolution of the vector $W(k)$ for networks of AIMD flows. We consider a set of AIMD matrices $\mathcal{M} = \{M_1, .., M_\mu\}$. Associated to this set we consider the deterministic system

$$x(k+1) \in \{Mx(k) \mid M \in \mathcal{M}\}\,, \tag{3.1}$$

and a Markov chain model

$$W(k+1) = A(k)W(k)\,, \tag{3.2}$$

where for each $k$ the $A(k)$ is a random variable with values in $\mathcal{M}$. We recall that by (2.1) the sum $\sum_i w_i(k)$ is a constant. We may thus restrict our attention to the simplex

$$\Sigma := \left\{ x \in \mathbb{R}^n_+ \mid \sum_{i=1}^{n} x_i = 1 \right\}\,,$$

and we will study the evolution of (3.2) on $\Sigma$. We assume that the random variables $A(k), k = 0, 1, \ldots$ are independent and identically distributed (i.i.d.) and denote

$$P(A(k) = M_i) = \rho_i\,, \quad i = 1, \ldots, \mu\,.$$

As we are dealing with probabilities, necessarily, we assume $\sum_i \rho_i = 1$. With this setup the sequence $\{W(k)\}_{k\in\mathbb{N}}$ is a Markov process.

The random variable of a product of length $k$ is denoted by $\Pi(k) = A(k)A(k-1)\ldots A(0)$.

Clearly, $W(k) = \Pi(k)W(0)$, and consequently that the behaviour of $W(k)$, as well as the network fairness and convergence properties, are governed by the asymptotic properties of the matrix product $\Pi(k)$ as $k \to \infty$.

It has been documented by many authors that networks of many AIMD flows exhibit extremely complex behaviour. Consequently, it is convenient to analyse such networks for a probabilistic viewpoint. In this paper we take a first step in this direction and analyse networks for which the following assumptions hold. We shall see in Section 6 that these assumptions seem to capture essential characteristics of some real networks and therefore are somewhat less restrictive that they might appear.

Assumption 3.1. *Let $\mathcal{M} = \{M_1, .., M_\mu\}$ be a set of matrices of the form* (2.7)*. We assume that the probability that $A(k) = M_i$ in (3.2), is independent of $k$ and equals $\rho_i > 0$.*

Comment 3.2. *In other words Assumption 3.1 says that the probability that the network dynamics are described by $W(k+1) = A(k)W(k), A(k) = M_i$ over the $k$-th congestion epoch is $\rho_i$ and that the random variables $A(k), k \in \mathbb{N}$ are i.i.d. Furthermore, we assume that we only have matrices in the set $\mathcal{M}$ which occur with positive probability. The reason for this assumption is, that without it, there is little insight to be gained into the dynamics of the Markov chain* (3.2) *by studying the deterministic system* (3.1)*. The assumption is of course easy to guarantee this assumption by simply removing matrices with* 0 *probability from the set $\mathcal{M}$.*

Given the probabilities $\rho_i$ for $M_i \in \mathcal{M}$, one may then define the probability $\lambda_j$ that source $j$ experiences a backoff at the $k$-th congestion event as follows:

$$\lambda_j = \sum \rho_i\,,$$

where the summation is taken over those $i$ which correspond to a matrix in which the $j$-th source sees a drop. Or to put it another way, the summation is over those indices $i$ for which the matrix $M_i$ is defined with a value of $\beta_j \neq 1$.

Assumption 3.3. *Let $\mathcal{M} = \{M_1, .., M_\mu\}$ be the set of AIMD matrices defining* (3.2) *and assume that $P(A(k) = M_i) = \rho_i, i = 1, \dots, \mu$. We assume that $\lambda_j > 0$ for all $j \in \{1, ..., n\}$.*

Simply stated, Assumption 3.3 states that almost surely all flows must see a drop at some time (provided that they live for a long enough time).

**3.3. Column Stochastic Matrices.** Column stochastic matrices will play a central role in the discussion in Section 5. We begin by collecting some results. The following two are immediate consequences of the definition of a column stochastic matrix.

Lemma 3.4. *A matrix $A \in \mathbb{R}_+^{n\times n}$ is column stochastic if and only if $y^T A = y^T$. Any product of a finite number of column stochastic matrices is a column stochastic matrix (i.e., the set of column stochastic matrices is a semigroup).*

It is sometimes convenient to consider the subspace orthogonal to $y$, which we denote by

$$S := \{z \in \mathbb{R}^n \mid y^T z = 0\} .$$

The subspace $S$ is an invariant subspace for all column stochastic matrices. Given a column stochastic matrix $A$ we denote by $\tilde{A} : S \to S$ the linear operator obtained by restricting $A$ to $S$. Furthermore, we denote by $\|\cdot\|$ the 1-norm and the corresponding induced matrix norm.

Lemma 3.5. *For any column stochastic matrix $A$ it holds that $\|A\| \leq 1$ and that $\|\tilde{A}\| \leq 1$. If $A$ is positive, then $\|\tilde{A}\| < 1$ .*

*Proof.* The first claim is immediate from the standard characterization of the induced 1-norm as the column-sum norm. The second claim follows as $\|\tilde{A}\| \leq \|A\|$ using the definition of induced norms. Finally, if $A$ is positive, then for a vector $z \in S, \|z\| = 1$ it holds that $-A|z| \prec |Az| \prec A|z|$ as $z$ has positive and negative entries due to $y^T z = 0$. This implies for $z \in S, \|z\| = 1$ that

$$\|\tilde{A}z\| = \|Az\| = \|\,|Az|\,\| < \|A|z|\,\| = 1 .$$

This shows the assertion. □

A feature in the proof of our main results is the observation that products of column stochastic matrices converge in a certain sense to a subset of the rank-1 idempotent matrices. We use the following lemma to estimate the distance of a matrix product from the set $\mathcal{R}$ defined at the beginning of this section.

Lemma 3.6. *Let $A \in \mathbb{R}_+^{n\times n}$ be column stochastic, then* $\operatorname{dist}(A, \mathcal{R}) \leq 2\|\tilde{A}\|$.

*Proof.* Let $A_1 = A - 1/nAyy^T$. Note that $1/nAyy^T$ is a rank-1 column stochastic matrix. Then $\operatorname{dist}(A, R) = \inf\{\|A - C\| :\ C \in \mathcal{R}\} \leq \|A - 1/nAyy^T\| = \|A_1\|$. We are proving that $\|A_1\| \leq 2\|\tilde{A}\|$. So let $x = z + ty$, where $z \in S, t \in \mathbb{R}$ are arbitrary. Then

$$A_1 x = (A - 1/nAyy^T)(z + ty) = Az = \tilde{A}z ,$$

so

$$\|A_1 x\| \leq \|\tilde{A}z\| \leq \|\tilde{A}\|\|z\| .$$

To complete the proof we show that $\|z\| \leq 2\|z + ty\|$. Indeed, if $z_1, z_2, \dots, z_n$ are the components of $z$ ordered such that: $z_1 \geq z_2 \geq \cdots \geq z_r \geq 0 > z_{r+1} \geq \cdots \geq z_n$. Then $\|z\| = |z_1| + |z_2| + \cdots + |z_n| =$

$2(|z_1| + |z_2| + \cdots + |z_r|)$. On the other hand for $t \geq 0$,

$$\|z + ty\| = \sum_{j=1}^{n} |z_j + t| \geq \sum_{j=1}^{r} |z_j + t| \geq \sum_{j=1}^{r} |z_j| = \frac{1}{2}\|z\|\,,$$

thus $\|z\| \leq 2\|z + ty\|$. For $t < 0$ a similar argument applies. □

Recall also that the matrix set $\mathcal{M}$ is defined in Section 2. We note that each matrix $M \in \mathcal{M}$ can be written in form

$$M = \mathrm{diag}(\beta_1, \beta_2, ..., \beta_n) + \frac{1}{\sum_{j=1}^{n} \alpha_j \gamma_j} [\alpha_1\gamma_1, \ldots, \alpha_n\gamma_n]^T [(1-\beta_1), \ldots, (1-\beta_n)] \tag{3.3}$$

where $\alpha_j$ are positive, and all $\beta_j$ are positive and not greater than 1. The parameters $\gamma_j$ are also positive and fixed for each matrix in $\mathcal{M}$. Thus the matrices in $\mathcal{M}$ are column stochastic, and properties of $W(k+1) = A(k)A(k-1)\cdots A(0)W(0)$ are determined by product $A(k)A(k-1)\cdots A(0)$ and that the $j$-th column of $P \in \mathcal{M}$ is nonpositive then that column is equal to $e_j$. Given these facts, and using the assumptions given in 3.2, we now aim to prove certain convergence results for the restriction of $A(k)A(k-1)\cdots A(1)$ to $S$. To this end we employ the notion of paracontractivity [10] from the theory of nonhomogeneous matrix products. A linear operator $A$ on $\mathbb{R}^n$ is called *paracontractive* with respect to the norm $\|\cdot\|$, if

$$Ax \neq x \Rightarrow \|Ax\| < \|x\|. \tag{3.4}$$

We will employ the following three results to show that almost surely products of matrices from $\mathcal{M}$ converge to the set $\mathcal{R}$. The following result is proved in [7].

THEOREM 3.7. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ and let $\mathcal{F} \subset \mathbb{R}^{n\times n}$ be a finite set of linear operators which are paracontractive with respect to $\|\cdot\|$. Then for any sequence $\{A_k\}_{k\in\mathbb{N}} \subset \mathcal{F}^{\mathbb{N}}$, the sequence of left products $\{A_kA_{k-1}\ldots A_1\}_{k\in\mathbb{N}}$ converges.*

The second result shows that all matrices from $\mathcal{M}$ are paracontractive with respect to the 1-norm on $S$.

LEMMA 3.8. *Let $A \in \mathcal{M}$. Then $\tilde{A}$ is paracontractive on $S$ with respect to the* 1*-norm.*

*Proof.* As before, let $\|\cdot\|$ denote the 1-norm. For $x \in S$ we want to show (3.4). We know that any matrix from $\mathcal{M}$ can be written in form (3.3), where $\beta_i \in (0,1], i = 1, \ldots, n$ and not all of them are equal to 1. Also $\alpha_i > 0$ and $\gamma_i > 0$ for $i = 1, 2, \ldots, n$. Now we can, without loss of generality, suppose that $\beta_1 = \beta_2 = \cdots = \beta_q = 1$ for $q < n$ and $\beta_i < 1, i = q+1, \ldots, n$. In this case our matrix $A$ is of the form

$$A = \begin{bmatrix} I_q & A_{12} \\ 0 & A_{22} \end{bmatrix}\,,$$

where $I_q$ is the identity matrix of order $q$ and where $A_{12}, A_{22} \succ 0$ are such that the elements of each column sums to 1. If we partition $x = \begin{bmatrix} x_1^T & x_2^T \end{bmatrix}^T$ accordingly, we have

$$Ax = \begin{bmatrix} x_1 + A_{21}x_2 \\ A_{22}x_2 \end{bmatrix}.$$

By Lemma 3.5 we have $\|Ax\| \leq \|x\|$. If $\|Ax\| = \|x\|$, then in each entry of $Ax$ the summands have the same sign, because otherwise $\|Ax\| < \|A|x|\| \leq \||x|\| = \|x\|$, a contradiction. For $1 \leq j \leq q$, this implies that for $(Ax)_j = x_j + a_{jq+1}x_{q+1} + \ldots + a_{jn}x_n$ the signs of the summands coincide. Similarly for $q+1 \leq j \leq n$ the signs of the summands of $(Ax)_j = a_{jq+1}x_{q+1} + \ldots + a_{jn}x_n$ coincide. This implies $x_ix_j \geq 0$ for all $i = 1, 2, \ldots, n$ and all $j = q+1, q+2, \ldots, n$. If we fix $j \geq q+1$ we have

$$0 = x_j(x_1 + \cdots + x_n) \geq x_j^2\,, \tag{3.5}$$

as $x \in S$, i.e. $y^T x = 0$. From (3.5) we conclude that $x_j = 0$ for all $j \in \{q+1, q+2, \dots, n\}$ which also means that $Ax = x$. Thus for $x \in S$, we have $\|Ax\| \le \|x\|$ with equality if and only if $Ax = x$, as desired. □

Our third result is purely technical and is stated a separate lemma to aid exposition of Theorem 3.10.

COROLLARY 3.9. *If $A \in \mathcal{M}$ is such that its nonpositive columns are indexed by $i_1, i_2, \dots, i_q$ and $x \in S$ is such that $Ax = x$, then $x$ lies in the subspace spanned by the vectors $e_{i_1}, e_{i_2}, \dots, e_{i_q}$.*

*Proof.* This follows from the previous proof, as we have seen that $Ax = x$ implies that $x_j = 0$ for $j = q+1, \dots, n$. In other words, $x \in \operatorname{span}\{e_1, \dots, e_q\}$. The general statement follows by permutation. □

Given the three previous result it is now possible to show that almost all products of matrices from $\mathcal{M}$ approach the set $\mathcal{R}$.

THEOREM 3.10. *Let $\{A_k\}_{k\in\mathbb{N}}$ be a sequence of matrices from $\mathcal{M}$. Assume that for all $i \in \{1, 2, \dots, n\}$ there is a matrix $T_i \in \mathcal{M}$ with positive $i$-th column which occurs infinitely often in $\{A_k\}_{k\in\mathbb{N}}$. Then*

$$\lim_{k\to\infty} \{\tilde{A}_k \tilde{A}_{k-1} \cdots \tilde{A}_1\} = 0\,.$$

*In particular under Assumption 3.3, we have for the stochastic process $\{A(k)\}_{k\in\mathbb{N}}$ that $\lim_{k\to\infty} \tilde{A}(k)\tilde{A}(k-1)\cdots\tilde{A}(0) = 0$ almost surely.*

*Proof.* By Lemma 3.8, the matrices $\tilde{A}_k, k \in \mathbb{N}$ are paracontractive with respect to $\|\cdot\|$. Using Theorem 3.7 it follows that $\{\tilde{A}_k \tilde{A}_{k-1} \cdots \tilde{A}_1\}_{k\in\mathbb{N}}$ is convergent. To prove that the limit is 0 let $s \in S$. Then there exist $y \in S$ such that $y = \lim_{k\to\infty} A_k A_{k-1} \cdots A_1 s$. We will prove that $y = 0$ from which the first assertion follows. For fixed $i$ let $\{A_{n_k}\}_{k\in\mathbb{N}}$ be a subsequence of $\{A_k\}_{k\in\mathbb{N}}$ with $A_{n_k} = T_i$. Then

$$y = \lim_{k\to\infty} A_{n_k} A_{n_k-1} \cdots A_1 s = T_i \lim_{k\to\infty} A_{n_k-1} \cdots A_1 s = T_i y.$$

Thus $T_i y = y \in S$ since $s \in S$. By Corollary 3.9 the $i$-th coordinate of $y$ is zero. Since $i$ is arbitrary, it follows that $y = 0$.

By Assumption 3.3 for each $j \in \{1, \dots, n\}$ the probability that matrices with positive $j$-th column occur infinitely often in a realization of the process is equal to 1. Thus $\lim_{k\to\infty} \tilde{A}(k)\tilde{A}(k-1)\cdots\tilde{A}(0) = 0$ with probability 1. □

The next result shows that the expected distance between $A(k)A(k-1)\cdots A(1)$ and $\mathcal{R}$ decreases exponentially; a fact of independent interest.

PROPOSITION 3.11. *Let $\{A(k)\}_{k\in\mathbb{N}}$ be a sequence of random variables satisfying Assumptions 3.1 and 3.3. Let $d(k) := E(\operatorname{dist}(A(k)A(k-1)\cdots A(1), \mathcal{R}))$. Then there exist $\eta < 1$ and $C \ge 1$ such that for all $k$ it holds that*

$$d(k) \le C\eta^k. \tag{3.6}$$

*Proof.* Let $\mu = 1 - min_{j=1,n}\lambda_j < 1$ and let $l$ be an integer such that $1 > n\mu^l$.

At first, note that the $j$-th column of the product of several matrices from $\mathcal{M}$ is positive if and only if one of these matrices has positive $j$-th column, otherwise it is equal to $e_j$. Consider the products of length $l$: $\Pi(l) = A(l)A(l-1)\cdot A(1)$. The probability that the $j$-th column of $\Pi(l)$ is nonpositive is $o_j = (1-\lambda_j)^l \le \mu^l$. For the probability $q_l$ that at least one column of $\Pi(l)$ is nonpositive we have that $q_l \le o_1 + o_2 + \cdots + o_n \le n\mu^l$. Thus the probability $p_l$ that $\Pi(l)$ is positive satisfies $p_l = 1 - q_l \ge 1 - n\mu^l > 0$. Let $k = dl + r$ where $0 \le r < l$. We can split the product $\Pi(k) = A(k)A(k-1)\cdots A(1)$ into product of first $r$ terms

$D_0 = A(k)A(k-1)\cdots A(k-r+1)$ and product of $d$ blocks of length $l$: $D_i = A(il)A(il-1)\cdots A(l(i-1)+1)$, for, $i = 1,2,\ldots,d$. So $\Pi(k) = D_0 D_d \cdots D_1$. Note, that, for all $i = 0,1,\ldots,d$, $D_i$, as product of column stochastic matrices is column stochastic, and therefore, $\|D_i\| \le 1$, and $\|\tilde{D}_i\| \le 1$. With this notation we have

$$\mathrm{dist}\,(\Pi(k),\mathcal{R}) \le 2\|\tilde{\Pi}(k)\| = 2\|\tilde{D}_0\tilde{D}_d\cdots\tilde{D}_1\| \le 2\|\tilde{D}_d\cdots\tilde{D}_1\|\,.$$

Define

$$\delta := \max\{\|\tilde{T}\| :\; T = A_l A_{l-1}\ldots A_1 > 0, A_1, A_2, \ldots, A_l \in \mathcal{M}\} < 1\,. \tag{3.7}$$

Since the set in (3.7) is finite, the maximum exist and is strictly less then 1 by Lemma 3.5. For any $j \in \{0,1,2,\ldots,d\}$ the probability that exactly $j$ of the matrices $D_1, D_2, \ldots, D_d$ are positive is equal to $z_j = \binom{d}{j} p_l^j (1-p_l)^{d-j}$. We also know that if $j$ of matrices $D_1, D_2, \ldots, D_d$ are positive then $\|(D_d D_{d-1}\cdots D_1)\,\tilde{}\,\| = \|\tilde{D}_d\cdots\tilde{D}_1\| \le \|\tilde{D}_d\|\cdots\|\tilde{D}_1\| \le \delta^j$. Thus we obtain

$$d(k) \le 2E(\|\tilde{D}_d\|\cdots\|\tilde{D}_1\|) \le 2\sum_{j=0}^{d} z_j\delta^j =$$

$$= 2\sum_{j=0}^{d}\binom{d}{j}(p_l\delta)^j(1-p_l)^{d-j} = 2(1+p_l\delta-p_l)^d \le C\eta^k\,,$$

where for the last inequality we choose

$$\eta = (1-p_l+p_l\delta)^{1/l} < 1 \quad\text{and}\quad C = 2/\eta^l\,. \tag{3.8}$$

This shows the assertion. □

**4. Invariant measures.** In this section we study the existence of invariant measures of the Markov process $\{W(k)\}_{k\in\mathbb{N}}$. Throughout we assume that Assumptions 3.1 and 3.3 are satisfied. Our considerations are based on the results presented in [23] to which we refer the reader for further background material. We briefly present some basic properties for the Markov chain $\{W(k)\}_{k\in\mathbb{N}}$ on the simplex $\Sigma$. By $\mathcal{B}(\Sigma)$ we denote the Borel $\sigma$-algebra of $\Sigma$.

Associated to our Markov chain there is a transition kernel $P(x,X)$ for $x\in\Sigma, X\in\mathcal{B}(\Sigma)$, which gives the probability to reach the set $X$ from the point $x$. This transition kernel acts on continuous functions $h:\Sigma\to\mathbb{R}$ through

$$Ph(x) = \int_\Sigma h(y)P(x,dy) = \sum_{i=1}^{\mu}\rho_i h(M_i x)\,. \tag{4.1}$$

It is obvious that $Ph$ is continuous for continuous $h$, so that $P$ is *(weak) Feller*. Furthermore we have $\|A_i\| \le 1, i = 1,\ldots,\mu$, so that $\|A_i(x-y)\| \le \|x-y\|$. Using the uniform continuity of $h$ it follows that for any continuous function $h:\Sigma\to\mathbb{R}$, the sequence

$$P^k h\,,\quad k\in\mathbb{N}\,,$$

defined inductively through repeated application of (4.1), is equicontinuous. Markov chains whose transition kernel have this property are called *e-chains*, see [23].

An important notion in the study of Markov chains are invariant probabilities. Recall, that a probability measure $\pi$ is called *invariant* for a Markov process, if

$$\pi(X) = \int_{\Sigma} P(x, X) d\pi(x) , \quad \forall X \in \mathcal{B}(\Sigma)$$

that is, intuitively, the distribution of mass on $\Sigma$ given by the probability measure $\pi$ is not changed if it is rearranged according to the evolution of the Markov process.

As we are considering an e-chain we obtain from [23, Theorem 12.0.1] that an invariant probability exists in our case. We aim to show its uniqueness. To this end we first study the possible support of invariant measures. We introduce the set of sequences

$$\mathcal{L} := \{\{A_k\}_{k\in\mathbb{N}} \in \mathcal{M}^{\mathbb{N}} \mid \{A_k\}_{k\in\mathbb{N}} \text{ satisfies the conditions of Theorem 3.10}\}$$

By Theorem 3.10 we know that the left products of a sequence $\{A_k\}_{k\in\mathbb{N}} \in \mathcal{L}$ approach the set of rank one, column stochastic matrices. We define the set of limit points of such sequences by

$$\mathcal{R}_{\mathcal{L}} := \{R \in \mathcal{R} \mid \exists \{A_k\}_{k\in\mathbb{N}} \in \mathcal{L}, k_l \to \infty : \lim_{l\to\infty} \Pi(k_l) = R\} .$$

As the matrices $R \in \mathcal{R}$ are column stochastic and of rank 1 they can be represented in the form $R = zy^T$, where $z \succeq 0$ and $\|z\| = 1$. Thus the set $\mathcal{R}_{\mathcal{L}}$ naturally defines a subset of the simplex $\Sigma$ by

$$\mathcal{C} := \{z \in \Sigma \mid zy^T \in \mathcal{R}_{\mathcal{L}}\} . \tag{4.2}$$

We note the following properties of $\mathcal{C}$.

PROPOSITION 4.1. *Consider a finite set of AIMD matrices $\mathcal{M}$ and the associated deterministic system* (3.1) *and the Markov chain* (3.2)*. Let $\mathcal{C}$ be defined by* (4.2)*, then*

*(i) $\mathcal{C}$ is compact,*
*(ii) $\mathcal{C}$ is forward invariant under* (3.1)*,*
*(iii) for any solution $\{x(k)\}_{k\in\mathbb{N}}$, $x(0) \in \Sigma$ of* (3.1) *the distance*

$$\mathrm{dist}\,(x(k), \mathcal{C})$$

*is nonincreasing,*
*(iv) for any $z \in \mathcal{C}$ and any open neighborhood $U \subset \Sigma$ of $z$ there is a $k_0 > 0$ such that $P^k(x, U) > \delta > 0$, for all $k \geq k_0$ and all $x \in \Sigma$,*
*(v) For any initial condition $W_0 \in \Sigma$ we have almost surely*

$$\lim_{k\to\infty} \mathrm{dist}\,(W(k), \mathcal{C}) = 0 .$$

*Proof.* (i) This is clear.

(ii) Let $x \in \mathcal{C}, B \in \mathcal{M}$. By definition there exists a sequence $\{A_k\}_{k\in\mathbb{N}} \in \mathcal{L}$ and $k_l \to \infty$ such that

$$\Pi(k_l) = A_{k_l} A_{k_l - 1} \dots A_1 \to zy^T .$$

We write $\Pi(k_l) = zy^T + \Delta_k$, where $\|\Delta_k\| \to 0$. Now define a new sequence by repeating our initial sequence sequence and inserting $B$, i.e. we consider the sequence

$$\{A_1, A_2, \dots, A_{k_1}, B, A_1, A_2, \dots, A_{k_2}, B, A_1, \dots, A_{k_3}, B, A_1, \dots\} .$$

Denoting products of length $l$ of this sequence by $\Psi(l)$ we have

$$\Psi(l+\sum_{j=1}^{l} k_j) = B\Pi(k_l)\Psi((l-1)+\sum_{j=1}^{l-1} k_j) = B(zy^T+\Delta_k)\Psi((l-1)+\sum_{j=1}^{l-1} k_j)$$
$$= Bzy^T + B\Delta_k\Psi((l-1)+\sum_{j=1}^{l-1} k_j)\,,$$

where we have used that all matrices are column stochastic in the last step. As $\|\Delta_k\| \to 0$, this implies that $\Psi(l+\sum_{j=1}^{l} k_j) \to Bzy^T$ as $l \to \infty$. The constructed sequence clearly lies in $\mathcal{L}$ so that $Bz \in \mathcal{C}$, which we wanted to show.

(iii) Let $x \in \Sigma$. By (i) there is a $z \in \mathcal{C}$ such that $\mathrm{dist}\,(x,\mathcal{C}) = \|x-z\|$. Then for $A \in \mathcal{M}$ it follows using (ii) that

$$\mathrm{dist}\,(Ax,\mathcal{C}) \le \|Ax - Az\| \le \|x-z\| = \mathrm{dist}\,(x,\mathcal{C})\,.$$

This shows the assertion.

(iv) Fix $z \in \mathcal{C}$ and let $U \subset \Sigma$ be an open neighborhood of $z$. Then we may choose $\epsilon > 0$ such that $x \in \Sigma, \|x-z\| < \epsilon$ implies $x \in U$. By definition of $\mathcal{C}$ there exists a $k_0$ and a product $\Pi(k_0)$ such that $\|\Pi(k_0) - zy^T\| < \epsilon$. This implies for any $x \in \Sigma$ that

$$\|\Pi(k_0)x - z\| = \|(\Pi(k_0) - zy^T)x\| < \epsilon\,,$$

so that $\Pi(k_0)x \in U$ and consequently, $P^{k_0}(x,U) > \delta > 0$ for all $x \in \Sigma$. As this probability is independent of $x$ we see in particular, that $P^k(z,U) > \delta > 0$ for all $k \ge k_0$, by considering the transition from $k-k_0$ to $k$.

(v) This is an immediate consequence of Theorem 3.10. ▯

In the terminology of Markov chains, we have proved in Proposition 4.1 (iv) that each $z \in \mathcal{C}$ is *positive* and *aperiodic* for the Markov chain $\{W(k)\}_{k\in\mathbb{N}}$. For a general definition of positive and aperiodic states of an e-chain, see [23, p. 456, p. 459]. Using the existence of positive and aperiodic states we obtain the following fundamental statement from [23, Theorem 18.0.2].

THEOREM 4.2. *Consider a finite set of AIMD matrices $\mathcal{M}$ and the associated Markov chain* (3.2). *Then*

*(i) there exists a unique invariant probability $\pi$,*

*(ii) for every $x \in \Sigma$ and every continuous function $h : \Sigma \to \mathbb{R}$ we have that if $W(0) = x$, then*

$$\lim_{k\to\infty} \frac{1}{k}\sum_{j=0}^{k-1} h(W(j)) = \int_\Sigma h(y)d\pi(y)\,, \qquad \textit{in probability,}$$

*(iii) for every $x \in \Sigma$ and every continuous function $h : \Sigma \to \mathbb{R}$ we have*

$$\int_\Sigma h(y)P^k(x,dy) \to \int_\Sigma h(y)d\pi(y)\,, \qquad \textit{as } k \to \infty\,.$$

The previous result can be sharpened by using the special structure of the set of AIMD matrices $\mathcal{M}$.

THEOREM 4.3. *Consider a finite set of AIMD matrices $\mathcal{M}$ and the associated Markov chain* (3.2) *with its unique invariant probability $\pi$. Then*

$$\mathrm{supp}\,\pi = \mathcal{C}\,.$$

*Proof.* We first show that $\mathcal{C} \subset \operatorname{supp} \pi$. Assume to the contrary that $x \in \mathcal{C} \setminus \operatorname{supp} \pi$. Then there exists an open neighborhood $V$ of $x$. By Proposition 4.1 (iv) it follows that $P^k(x, V) > 0$ for some $k$ large enough, this contradicts $x \notin \operatorname{supp} \pi$.

To show the converse let $\epsilon > 0$ and consider the set

$$U_\epsilon := \{x \in \Sigma \mid \operatorname{dist}(x, \mathcal{C}) > \epsilon\}.$$

As the distance of $W(k)$ to $\mathcal{C}$ is nonincreasing for every sample path by Proposition 4.1 (iii), this shows that $P(x, U_\epsilon) > 0$ implies $x \in U_\epsilon$. Thus

$$\pi(U_\epsilon) = \int_\Sigma P(x, U_\epsilon) d\pi(x) = \int_{U_\epsilon} P(x, U_\epsilon) d\pi(x).$$

If $\pi(U_\epsilon) > 0$ this shows that with probability 1 any evolution starting in $U_\epsilon$ stays in $U_\epsilon$. This is a contradiction to $\operatorname{dist}(W(k), \mathcal{C}) \to 0$ with probability 1, which we know by Proposition 4.1 (v). This shows $\pi(U_\epsilon) = 0$ and as $\epsilon > 0$ was arbitrary we obtain the assertion. □

The interesting point of the previous result is that the support of the invariant probability $\pi$ is determined by the set of matrices $\mathcal{M}$ and only the distribution of mass on that set changes under variation of the probabilities $\rho_i$. In the next section we show that the expected values of the average can be elegantly expressed in terms of the data, without the knowledge of the invariant probability $\pi$.

**5. Main Results.** We now present the main results of the paper. For the system defined in Section 3.2 we show that following statements are true.

(i) The expectation of $\Pi(k)$ converges to a fixed rank-1 matrix. A consequence of this result is that the random variable $W(k)$ always converges in expectation to a well defined stochastic equilibrium.
(ii) The stochastic process $\{W(k)\}$ satisfies the weak law of large numbers. A important consequence of this result is that the vector of window sizes $W(k)$, averaged over time, converges in probability to a well defined stochastic equilibrium.

It is prudent at this point to note that it follows from the discussion that the expectation of the random variable $A(k)$ is independent of $k$, and is equal to:

$$E(A(k)) = E(A(1)) = \sum_{i=1}^{\mu} \rho_i M_i. \tag{5.1}$$

Given Assumption 3.3, this immediately implies that matrix $E(A(1))$ is a positive column stochastic matrix and consequently has a unique Perron eigenvector $x_p$ given by $E(A(1))x_p = x_p$, $x_p^T y = 1$[4]. Using the independence of the random variables $A(k)$, this shows the following statement.

THEOREM 5.1. *Consider a finite set of AIMD matrices $\mathcal{M}$ and let $\{A(k)\}_{k\in\mathbb{N}}$ be an i.i.d. stochastic process satisfying Assumptions 3.1 and 3.3. Let $\Pi(k)$ be the random variable defined by*

$$\Pi(k) = A(k-1)A(k-2)....A(0).$$

*Then, the expectation of $\Pi(k)$ is given by*

$$E(\Pi(k)) = (\sum_{i=1}^{\mu} \rho_i M_i)^k; \tag{5.2}$$

[4]Recall that for any positive column stochastic matrix $V$ with Perron eigenvector $x_p$, it follows that $\lim_{k\to\infty} V^k = x_p y^T$ [4]

*and the asymptotic behaviour of $E(\Pi(k))$ satisfies*

$$\lim_{k\to\infty} E(\Pi(k)) = x_p y^T, \tag{5.3}$$

*where the vector $x_p \succ 0$ is uniquely determined by*

$$(\sum_{i=1}^{\mu} \rho_i M_i) x_p = x_p\,, \quad x_p^T y = 1\,. \tag{5.4}$$

We are now interested in the random variable $\overline{W}(k)$ defined by

$$\overline{W}(k) := \frac{1}{k+1}\sum_{i=0}^{k} W(i) = \left(\frac{1}{k+1}\sum_{i=0}^{k}\Pi(i)\right) W(0) = \overline{\Pi(k)}W(0),$$

where

$$\Pi(k) = A(k-1)A(k-2)\dots A(0) \ \text{ and } \ \overline{\Pi(k)} = \frac{1}{k+1}\sum_{i=0}^{k}\Pi(i).$$

COROLLARY 5.2. *Consider a finite set of AIMD matrices $\mathcal{M}$, let $\{A(k)\}_{k\in\mathbb{N}}$ be an i.i.d. stochastic process satisfying Assumptions 3.1 and 3.3. Then the expectation of $\overline{W}(k)$ is given by*

$$E(\overline{W}(k)) = \frac{1}{k+1}(I + E(A(1)) + E(A(1))^2 + \cdots + E(A(1))^k)W(0)$$

*and*

$$\lim_{k\to\infty} E(\overline{W}(k)) = x_p y^T W(0)\,,$$

*with $x_p$ defined by* (5.4)*.*

*Proof.* This follows since $E(A(1))^k \to x_p y^T$ as $k \to \infty$. □

The following theorem shows how the average distribution of network capacities can be characterized in probability.

THEOREM 5.3. *Consider a finite set of AIMD matrices $\mathcal{M}$ and the associated Markov chain* (3.2)*. Let Assumptions 3.1 and 3.3 be satisfied. Then, for every $\varepsilon > 0$,*

$$\lim_{k\to\infty} P(\|\overline{W}(k) - x_p y^T W(0)\| > \varepsilon) = 0, \tag{5.5}$$

*where the vector $x_p$ is defined by* (5.4)*.*

*Proof.* This is a consequence of Theorem 4.2 and Corollary 5.2. To be precise by Theorem 4.2 (ii) we have that if $W(0) \in \Sigma$ then

$$\overline{W}(k) \to \int_{\Sigma} W d\pi(W) =: E_\pi(W)\,,$$

in probability. (To obtain the desired result for vectors from the scalar results presented in Theorem 4.2, it suffices to consider the projections onto each coordinate.) If $W(0) \succeq 0$ is not in $\Sigma$ this equation scales by $y^T W(0)$ by linearity. Thus in particular $E(\overline{W}(k)) \to E_\pi(W) y^T W(0)$. As by Corollary 5.2 we have $E(\overline{W}(k)) \to x_p y^T W(0)$, this implies (5.5). □

Summarizing, the previous result says that the average distribution of the resources of the network is given by the vector $x_p$ which can be simply obtained by finding the dominant eigenvalue of $\sum \rho_i M_i \succ 0$.

**5.1. Stochastic equilibria of AIMD networks.** Theorems 5.1 and 5.3 provide remarkable insights into the behaviour of communication networks employing AIMD congestion control. In principle, they relate the asymptotic properties of such networks to the Perron eigenvector of $E(A(1))$. Since $E(A(1))$ is easily computable, it is not only possible to predict, but also to control, the asymptotic properties of such networks through judiciously manipulating the AIMD parameters and/or the probabilities $\rho_i$. In this context it is natural to ask whether the Perron eigenvector of $E(A(1))$ can be directly related to the AIMD parameters of the network.

(i) **Time-invariant networks :** Here $\mathcal{M} = \bar{\mathcal{A}}_1$ and the set of AIMD parameters has one element: $((\alpha_1, \ldots, \alpha_n), (\beta_1, \ldots, \beta_n))$. In this case it is readily shown that

$$E(A(1)) = \mathrm{diag}(\delta_1, \ldots, \delta_n) + \frac{1}{\sum_{i=1}^n \alpha_i \gamma_i} [\alpha_1\gamma_1, \ldots, \alpha_n\gamma_n]^T [1-\delta_1, \ldots, 1-\delta_n], \quad (5.6)$$

where $\delta_i = 1 - \lambda_i(1-\beta_i)$. Further, it follows directly by inspection that the Perron eigenvector of E(A(1)) is given by

$$x_p = [\frac{\alpha_1\gamma_1}{\lambda_1(1-\beta_1)}, \ldots, \frac{\alpha_n\gamma_n}{\lambda_n(1-\beta_n)}].$$

Consequently, the network convergence properties and the rates of convergence of $E(W(k))$ can be controlled directly by manipulating the network parameters $(\alpha_i, \beta_i, \rho_i)$. Clearly, such networks are of great interest since most practical wireline networks (including those employing TCP) fall into this category. A more detailed discussion of such network types can be found in [25].

(ii) **Time-varying networks :** In this case it is convenient to consider two cases: (a) networks where the the $\alpha_i$ are fixed in time and the $\beta_i$ vary; and (b) networks where both $\alpha_i$ and $\beta_i$ vary in time. In the first case it is again readily shown that

$$E(A(1)) = \mathrm{diag}(\delta_1, \ldots, \delta_n) + \frac{1}{\sum_{i=1}^n \alpha_i \gamma_i} [\alpha_1\gamma_1, \ldots, \alpha_n\gamma_n]^T [1-\delta_1, \ldots, 1-\delta_n], \quad (5.7)$$

where $\delta_i = E(\beta_i) < 1$. As before $x_p$ can be found by inspection and is given by

$$x_p = [\frac{\alpha_1\gamma_1}{1-\delta_1}, \ldots, \frac{\alpha_n\gamma_n}{1-\delta_n}]. \quad (5.8)$$

In the more general case it appears to be difficult to derive explicit formulae for $x_p$. One simplification occurs when the following situation prevails. The matrix $E(A(1))$ can be written

$$E(A(1)) = \sum_{j=1}^{h} \sum_{M_i \in \bar{\mathcal{A}}_j} \rho_i M_i = \sum_{j=1}^{h} Z_j. \quad (5.9)$$

In the case when the $Z_j$ are positive matrices with a common perron eigenvector $x_p$ it follows that $x_p$ is also the Perron eigenvector of $E(A(1))$ and the stochastic equilibria of the corresponding communication network is defined by $x_p$. Hence, it follows that time-varying networks constructed by switching between networks with a common equilibrium results in a constituent network with the same equilibrium state (although the rate of convergence to this equilibrium is difficult to bound).

**6. Experimental results.** The mathematical results derived in Section 5 are surprisingly simple when one considers the potential mathematical complexity of the unsynchronised network model (2.6). The simplicity of these results is a direct consequence of Assumptions 3.1 and 3.3. The objective of this section is therefore twofold; (i) to validate the unsynchronised model (2.6) in a general context; and (ii) to validate the analytical predictions of the model and thereby confirm that the aforementioned assumptions are appropriate in practical situations.

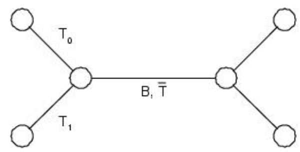



FIG. 6.1. *Dumbbell topology.*

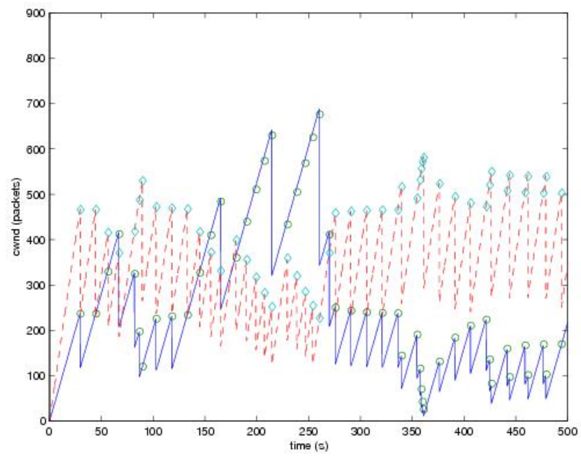



FIG. 6.2. *Predictions of the network model compared with packet-level NS simulation results. Key: ○ flow 1 (model), ◇ flow 2 (model), - flow 1 (NS), – flow 2 (NS). Network parameters: B=100Mb, $q_{max}$=80 packets, $\bar{T}$=20ms, $T_0$=102ms; $T_1$=42ms; no background web traffic.*

**6.1. Networks of two Unsynchronised Flows: Ensemble averages.** We first consider the behaviour of two TCP flows in the dumbbell topology shown in Figure 6.1. Our analytic results are based upon two fundamental assumptions: (i) that the dynamics of the evolution of the source congestion windows can be accurately modelled by equation (2.6); and (ii) the allocation of packet drops amongst the sources at congestion can be described by random variables. We consider each of these assumptions in turn.

(i) *Accuracy of dynamic model*. A comparison of the predictions the model (2.6) against the output of a packet-level $NS$ simulation is depicted in Figure 6.2. Here, the pattern of packet drops observed in the simulation is used to select the appropriate matrix $A(k)$ from the set $\mathcal{M}$ at each congestion event when evaluating (2.6). As can be seen, the model output is very accurate. Also plotted in Figure 6.3 is the evolution of the linear combination $\sum_{i=1}^{n} \gamma_i w_i$ where the $\gamma_i$ are defined in Equation (2.12). It can be seen that $\sum_{i=1}^{n} \gamma_i w_i$ has the same value at each congestion event thereby validating the constraint (2.12) used in the model.

(ii) *Validity of random drop model*. It is well known that networks of TCP flows with drop-tail queues can exhibit a rich variety of deterministic drop-behaviours [9]. However, most real networks carry at least a small amount web traffic. In Figure 6.4 we plot $NS$ simulation results where the mean congestion window of long-lived flows as the level of background web traffic is varied (background information on the web traffic generator in $NS$ is described in [28]). To illustrate the impact of small amounts of web traffic, these results are given for a network condition where phase effects are particularly pronounced: the congestion window time histories with no web traffic are shown in Figure 6.5. It can be seen that the time histories appear to jump between two persistent regimes. In the first regime flow 1, which has a propagation delay of 122ms, achieves a larger congestion window than flow 2, which has a propagation time of only 62ms, see Figure 6.5(b). The reverse

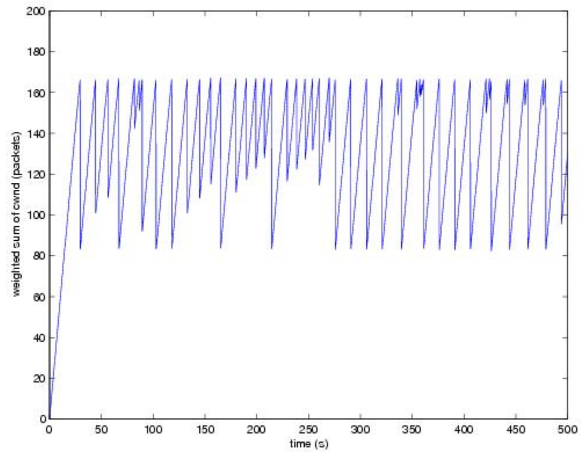



FIG. 6.3. *Evolution of* $\sum_{i=1}^{n} \gamma_i w_i$. *Network parameters: B=100Mb,* $q_{max}$*=80 packets,* $\bar{T}$*=20ms,* $T_0$*=102ms;* $T_1$*=42ms; no background web traffic.*

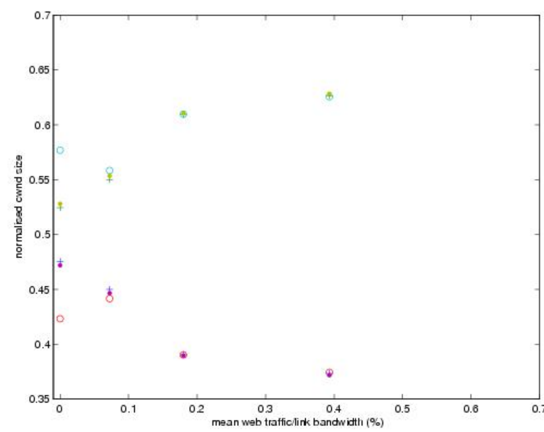
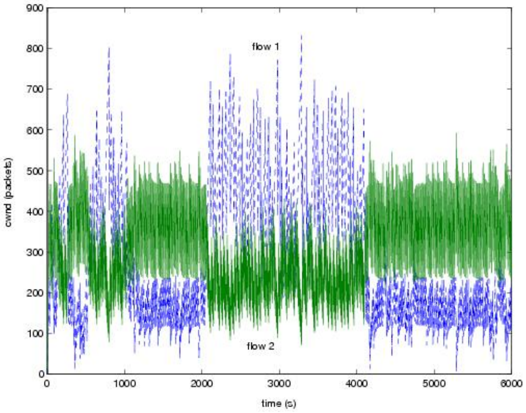


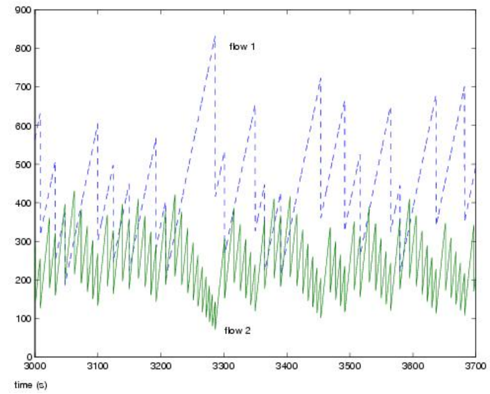

FIG. 6.4. *Variation of mean* $w_i(k)$ *with level of background web traffic in dumbbell topology of Figure 6.1. Key:* +NS *simulation result;* $\cdot$ *mathematical model (2.6);* $\circ$ *Theorem 5.1. Network parameters: B=100Mb,* $q_{max}$*=80 packets,* $\bar{T}$*=20ms,* $T_0$*=102ms;* $T_1$*=42ms.*

reverse is true in the second regime, see Figure 6.5(c). The impact of background web traffic is evident from Figure 6.6: despite its small volume, the effect of this traffic is enough to disrupt the coherent structure associated with phase effects and other complex phenomena previously observed in simulations of unsynchronised networks [9]. This is confirmed by statistical tests of this measured data, which confirm the validity of Assumptions 3.1 and 3.3. Space considerations, however, prevent the inclusion of detailed test results in the present paper.

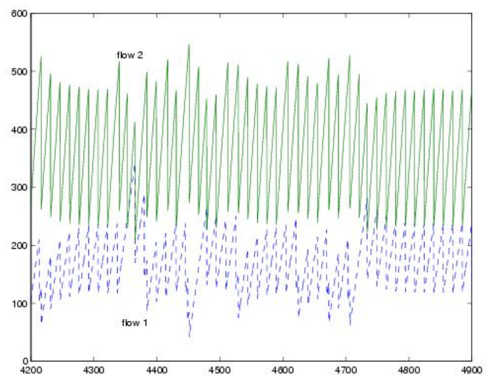
By performing repeated packet-level simulations with different random seed values for the web traffic generator, the ensemble average congestion window can be estimated. We can also determine from the simulation results the proportion of congestion events corresponding to both flows simultaneously seeing a packet drop, flow 1 seeing a drop only, and flow 2 seeing a drop only. Using these estimates of the probabilities $\rho_i$, the ensemble average congestion window can also be estimated from Theorem 5.1. An example of the resulting estimates are shown in Figure 6.7. Here,




(a)



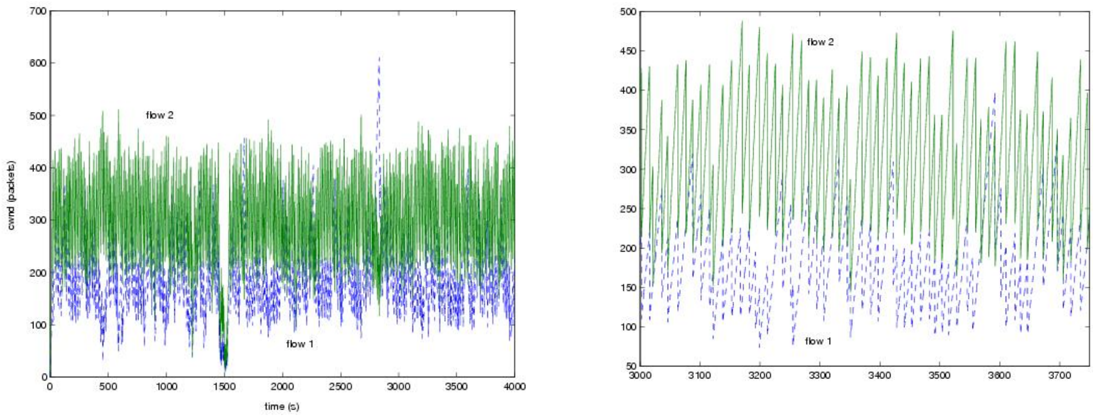

(b) $t = 3000s$ to $t = 3750s$




(c) $t = 4200s$ to $t = 4900s$

Fig. 6.5. *Congestion window time history corresponding to results in Figure 6.4 with no web traffic. Network parameters: B=100Mb, $q_{max}$=80 packets, $\bar{T}$=20ms, $T_0$=102ms; $T_1$=42ms.*

we run simulations for 250s with one flow started at 0 seconds and a second TCP flow started after 50 seconds (giving the first flow the opportunity to reach its steady state). A small amount of bi-directional background web traffic is also included and slow-start is switched off to allow us to focus on the congestion avoidance behaviour. The average congestion window evolution, estimated from 200 runs of the simulation, is plotted in Figure 6.7 together with the predictions of Theorem 5.1. It can be seen that the agreement is remarkably good. Not only is the long-term average accurately captured, but also the manner in which the flows converge to this long-term average. That is, the model accurately describes the dynamic evolution over time, on average, of the TCP flows and thereby is useful for the analysis of both short and long-lived flows. The results shown in Figure 6.7 are for a single choice of network conditions, but the model remains accurate for other conditions, see for example Figures 6.8-6.9. As can be seen from the figures, the predictions of Theorem 5.1 and the $NS$-simulations are consistently in close agreement.




(a) (b) $t = 3000s$ to $t = 3750s$

FIG. 6.6. *Congestion window time history corresponding to results in Figure 6.4 with 0.4% web traffic. Network parameters: B=100Mb, $q_{max}$=80 packets, $\bar{T}$=20ms, $T_0$=102ms; $T_1$=42ms.*

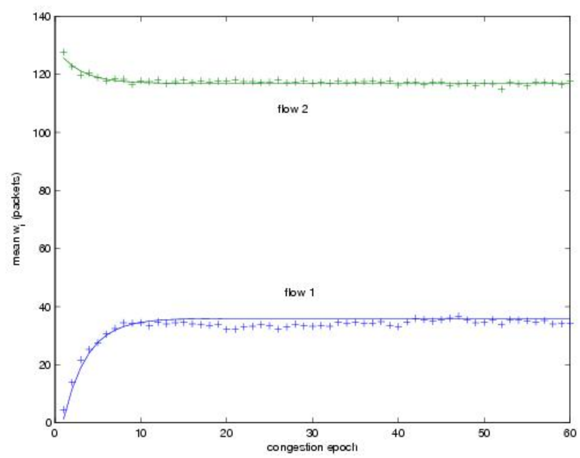



FIG. 6.7. *Variation of ensemble mean $w_i(k)$ with congestion epoch in dumbbell topology of Figure 6.1. Key: +*NS *simulation result (average over 200 runs); solid line Theorem 5.1. Network parameters: B=50Mb, $q_{max}$=50 packets, $\bar{T}$=20ms, $T_0$=102ms, $T_1$=2ms; approximately 0.5% bidirectional background web traffic.*

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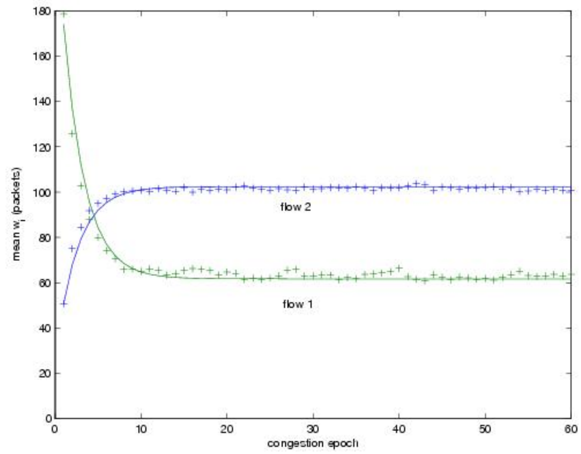


Fig. 6.8. *Variation of ensemble mean $w_i(k)$ with congestion epoch in dumbbell topology of Figure 6.1. Key:* +NS *simulation result (average over 200 runs); solid line Theorem 5.1. Network parameters: B=50Mb, $q_{max}$=50 packets, $\bar{T}$=20ms, $T_0$=2ms, $T_1$=42ms; approximately 0.5% bidirectional background web traffic.*

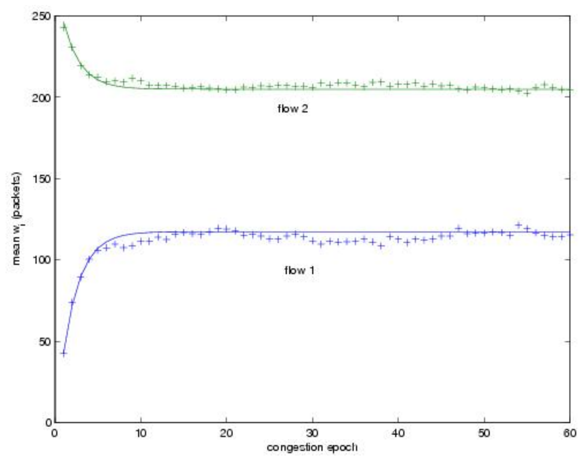


Fig. 6.9. *Variation of ensemble mean $w_i(k)$ with congestion epoch in dumbbell topology of Figure 6.1. Key:* +NS *simulation result (average over 200 runs); solid line Theorem 5.1. Network parameters: B=50Mb, $q_{max}$=50 packets, $\bar{T}$=20ms, $T_0$=102ms, $T_1$=42ms; approximately 0.5% bidirectional background web traffic.*


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