COMPLEX POWERS

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§1. Introduction.

(1.1) I am most grateful to the association for this opportunity to address you again. This is the second time I have spoken to the AGM. The last time it had to do with politics. Once is enough for a mathematician to talk about politics to a mathematical audience, so I take it that you are prepared to hear mathematics this time.

My topic is a to the power b, or a^b. All of you know something about it; perhaps some of you know all about it. Certainly, many of you once knew more than you now recall. I would like to remind you of it, because it plays a central rôle in mathematics. That is not to say that the full story of a^b belongs in the syllabus for secondary schools!

(1.2) The two most famous unsolved problems in mathematics are about a^b. The first is whether "Fermat's Last Theorem" is true, and the second is whether the Riemann Hypothesis is true. The Fermat problem, which has stood for 350 years, involves only positive integral powers of positive integers. It is to decide whether or not the equation

$$x^n + y^n = z^n$$

has a solution in positive integers x,y,z,n, with $n > 2$. The Riemann problem which has stood for 100 years, involves complex powers of positive real numbers, and I will state it at the end.

§2. Theme.

(2.1) The story of powers begins with positive integral powers. As soon as we understand how to multiply two complex numbers we may define inductively

$$a^1 = a, \quad a^2 = aa, \quad a^3 = aa^2, \ldots, a^{n+1} = aa^n, \ \ldots \ .$$

For example,

$$(2+3i)^5 = 122 - 597i.$$

This operation satisfies the three laws of exponents:

$$a^{n+m} = a^n a^m,$$

$$(a^n)^m = a^{nm},$$

$$(ab)^n = a^n b^n.$$

§3. Development.

(3.1) We define the zeroth power a^0 and negative integral powers a^{-n} $(n = 1,2,3,\ldots)$ in such a way that the laws of exponents continue to hold. Thus, in order to ensure that

$$a^{1+0} = a^1 a^0,$$

we must define $a^0 = 1$; and in order to ensure that

$$a^{n-n} = a^n a^{-n},$$

we must define

$$a^{-n} = \frac{1}{a^n}(n = 1,2,3,\ldots,a \neq 0).$$

Naturally, there is a problem if $a = 0$. There is no way to define 0^{-n} so that the laws of exponents hold. (Try it!)

As an example of a negative integral power,

$$\begin{aligned}(2+3i)^{-5} &= \frac{1}{(2+3i)^5} \\ &= \frac{1}{122-597i} \\ &= \frac{122+597i}{(122)^2+(597)^2} \\ &= \frac{122}{371293} + \frac{597}{371293}i .\end{aligned}$$

We used the first law of exponents to define a^{-n}. Fortunately, the other two laws of exponents continue to hold for arbitrary integral powers.

(3.2) We define rational powers of nonzero complex numbers on the same principle, of preserving the laws of exponents. In order to ensure that

$$\left(a^{\frac{n}{m}}\right)^m = a^{(\frac{n}{m})m},$$

we must define

$$a^{\frac{n}{m}} \qquad (a \neq 0,\ n \text{ and } m \text{ integral},\ m > 0)$$

as an m-th root of the n-th power of a, or what is (fortunately) the same thing, the n-th power of an m-th root of a.

Again, there is a problem. There are m(complex) m-th roots of each non-zero number, so $a^{\frac{n}{m}}$ has more than one value. If m and n have no common factors, then $a^{\frac{n}{m}}$ has exactly m values. For instance $4^{\frac{1}{2}}$ has the values 2 and -2, and $27^{\frac{2}{3}}$ has the values

$$9,\ \frac{9}{2} + \frac{9\sqrt{3}}{2}i,\ \frac{9}{2} - \frac{9\sqrt{3}}{2}i.$$

In general, we work out these powers by using De Moivre's formula:

$$\left\{r\ (\cos\theta + i\sin\theta)\right\}^n = r^n(\cos n\theta + i\sin n\theta).$$

Take, for example $(2+3i)^{\frac{2}{3}}$. Let

$$w = r(\cos\theta + i\sin\theta)$$

be a third root of $2+3i$. Then

$$r^3(\cos 3\theta + i\sin 3\theta) = 2 + 3i,$$

$$r^3 = \sqrt{2^2+3^2} = \sqrt{13} = 3.606,$$

$$r = 1.533,$$

$$\cos 3\theta = \frac{2}{3.606} = 0.555,$$

$$\sin 3\theta = \frac{3}{3.606} = 0.832,$$

$$3\theta = 0.982 + 2n\pi, \quad (n = 0,1,-1,2,-2,\ldots)$$

$$\theta = 0.327 + \frac{2n\pi}{3}, \quad (n = 0,1,2)$$

$$\cos\theta + i\sin\theta = 0.947 + 0.\ 0.321\,i\,,$$

$$\underline{\text{or}} \quad -0.752 + 0.660\,i\,,$$

$$\underline{\text{or}} \quad -0.195 - 0.981\,i\,,$$

$$w = 1.452 + 0.492i,\ -1.153 + 1.012i, -0.299 - 1.504i\ .$$

The three values of $(2+3i)^{\frac{2}{3}}$ are now obtained by squaring the three values of w to get

$$1.866 + 1.429i,\ 0.305 - 2.334i,\ -2.173 - 0.899i.$$

The three laws of exponents hold for rational powers, if they are interpreted as equations between the <u>sets</u> of values. For instance,

$$2^{\frac{5}{6}} = 2^{\frac{1}{3}}\ 2^{\frac{1}{2}}$$

is true if we interpret it to mean that the six values of the left-hand side are obtained by multiplying in turn the three values of $2^{\frac{1}{3}}$ by the two values of $2^{\frac{1}{2}}$.

For a <u>positive</u> (real) base a and a rational exponent b, exactly <u>one</u> of the values of a^b is positive. The laws of exponents hold, as ordinary numerical equations, for positive bases and rational exponents, if we interpret a^b as the positive power. For clarity, we denote the positive value of a^b by a^{b*} in the remainder of this discussion.

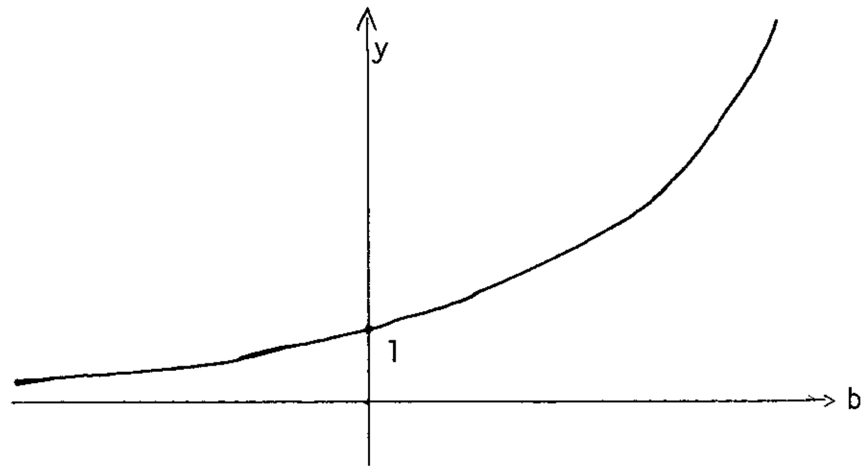
(3.3) The next step is to define the positive power a^{b*} for all positive bases a and all <u>real</u> exponents b. We do this by insisting that it be an <u>increasing</u> function of b for $a > 1$ and a <u>decreasing</u> function of b for $0 < a < 1$. Thus, for instance, $2^{\pi *}$ is larger than the numbers

$$2^{3*},\quad 2^{\frac{31}{10}*},\quad 2^{\frac{314}{100}*},\quad 2^{\frac{3141}{1000}*},\ \ldots$$

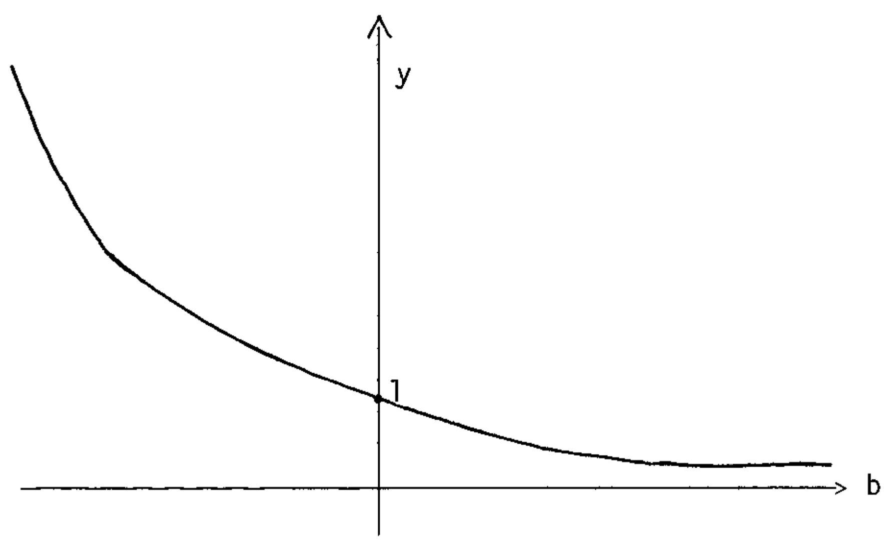
and smaller than the numbers

$$2^{4*},\quad 2^{\frac{32}{10}*},\quad 2^{\frac{315}{100}*},\quad 2^{\frac{3142}{1000}*}\ ,\ldots,\quad .$$

It is not altogether obvious that this definition actually defines one definite number a^{b*} for each $a > 0$ and each b, but it is true. Moreover, a^{b*} is a continuous function of a and b. For $a > 1$, the graph $y = a^{b*}$ looks like this:

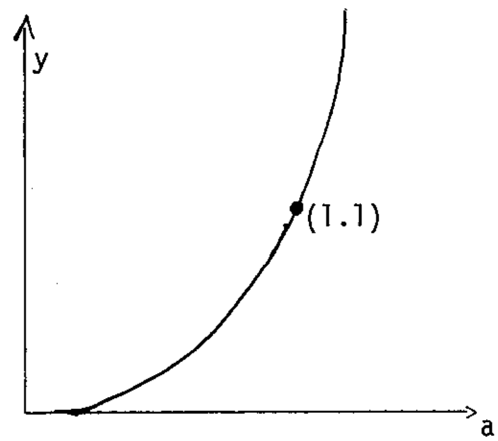


and for $a < 1$, it looks like this :

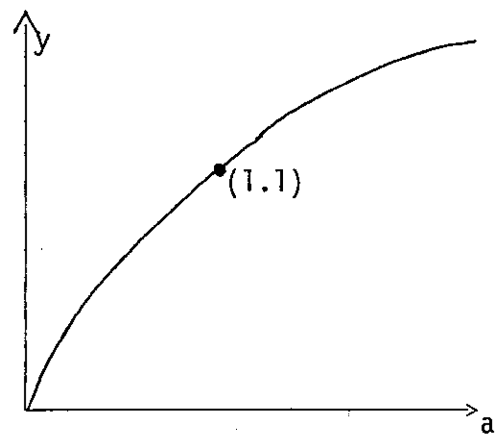


By the way, 1^{b*} is defined as 1 for all b.

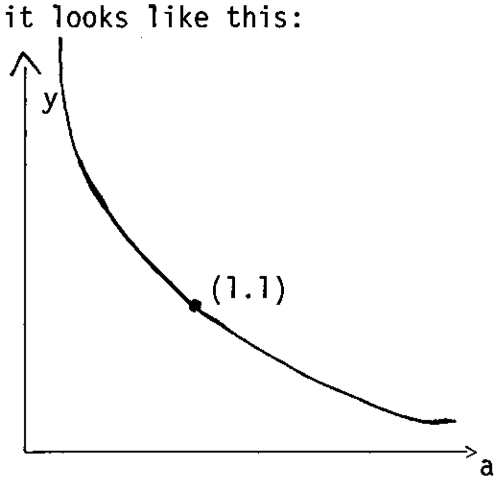
For $b > 1$, the graph $y = a^{b*}$ looks like this:



For $0 < b < 1$, it looks like this :



For $b < 0$, it looks like this:



(3.4) In practice, we work out a^{b}* by using logarithms.

For positive a and x, we define $\log_a x$ by the formula

$$a^{\log_a x}* = x,$$

i.e. $\log_a x$ is the power to which we must raise the base a to get x. Another way of putting it is that

$$\log_a a^{y}* = y ,$$

for all real y.

For example,

$$\log_3 27 = 3,$$

$$\log_5 \sqrt{5} = \tfrac{1}{2},$$

$$\log_2 \tfrac{1}{2} = -1.$$

The three laws of exponents yield three laws of logarithms:

$$\log_a bc = \log_a b + \log_a c,$$

$$\log_a b^{c*} = c \log_a b,$$

$$\log_c b = \frac{\log_a b}{\log_a c},$$

which hold for all positive a,b,c.

The third law shows that if logs to some particular base a are known, then logs to any other base c can be found. Two particular bases are popular. Base 10 logarithms are extensively tabulated, and base e logarithms are easy to work out. The number e = 2.718... is chosen to make

$$\frac{d}{dx} \log_e x = \frac{1}{x}.$$

(For other bases a,

$$\frac{d}{dx} \log_a x = \frac{\beta}{x}$$

for a constant $\beta \neq 1$; in fact, $\beta = \log_a e$.)

As a result,

$$\frac{d^n}{dx^n} \log_e x = \frac{(-1)^{n+1}(n-1)!}{x^n},$$

so the Taylor series of $\log_e x$ about 1 is

$$(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots \quad .$$

This series may be used to work out many significant figures of $\log_e x$ quickly, if x is close to 1, say

$$|x-1| < \frac{1}{10} .$$

If x is not close to 1, then we have to take the square root enough times to get it close, then work out the logarithm, and then multiply it by 2 as many times as we took the square root.

To work out square roots, we may use the standard algorithm, based on the formula

$$(10a + b)^2 = 100a^2 + 20ab + b^2.$$

To work out positive powers of positive numbers, we use the formula

$$a^{b*} = 10^{b \log_{10} a*} ,$$

or the formula

$$a^{b*} = e^{b \log_e a*} ,$$

depending on whether we are using tables or not.

(3.5) Modern scientific calculators give a^{b*} at the touch of a key, so this is how people usually work it out. If you are building a calculator, or are out in a field somewhere, or if you need more significant figures than tables or calculator will give, then you have to fall back on fundamentals, and use series.

(3.6) As an example, let us work out by hand a few figures of $\Pi^{\sqrt{2}*}$. We use

$$\Pi^{\sqrt{2}*} = e^{\sqrt{2} \log_e \Pi*}.$$

To work out $\log_e \Pi$, we start by taking square roots.

```
                          1.772454
                    | 3.14'15'92'7         |
 20       340       | 1                    |      35440
 ×7        ×7       | 2 14                 |         ×4
140      2380       | 1 89                 |     141760
+49       +49       |   25 15              |        +16
189      2429       |   24 29              |     141776
                    |      86 92           |
3540                |      70 84           |     354480
  ×2                |      16 08 70        |         ×5
7080                |      14 17 76        |    1772400
  +4                |       1 90 9400      |        +25
7084                |       1 77 2425      |    1772425
                    |         13 6975      | next is 4, by inspection.
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So $\sqrt{\Pi}$ = 1.772454.

Continuing in the same manner,

$$\sqrt{1.772454} = 1.331335,$$

$$\sqrt{1.331335} = 1.153834,$$

$$\sqrt{1.153834} = 1.074167,$$

so

$$\Pi^{\frac{1}{16}*} = 1.074167,$$

$$\log_e \Pi^{\frac{1}{16}*} = 0.074167 - \frac{(0.074167)^2}{2} + \frac{(0.074167)^3}{3} - \ldots$$

$$= 0.074167 - 0.002750 + 0.000136 - 0.000008$$

$$= 0.071545$$

$$\log_e \Pi = 16 \times 0.071545 = 1.14472$$

$$\sqrt{2}\,\log_e \Pi = 1.41421 \times 1.14472 = 1.61887.$$

To work out $e^{1.61887*}$, we use the power series

$$e^{x*} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots \quad .$$

For rapid results, we want small x, so we use

$$e^{1.61887*} = \left(e^{\frac{1.61887}{128}*}\right)^{128} = \left(e^{0.012647*}\right)^{2^7}.$$

We have

$$e^{0.012647*} = 1 + 0.012647 + \frac{(0.012647)^2}{2} + \ldots$$

$$= 1 + 0.012647 + 0.000080$$

$$= 1.012727,$$

and squaring seven times gives

$$\pi^{\sqrt{2}*} = e^{1.61887*} = (1.012727)^{2^7} = 5.047.$$

§4. Climax.

(4.1) The formula

$$a^{b*} = e^{b \log_e a*} \qquad (a > 0,\ b > 0)$$

provides the key to extending the definition of a^b to all complex a and b (except $a = 0$).

First, we have to define e^{z*} and $\log_e w$ for arbitrary complex z and arbitrary nonzero complex w.

(4.2) We may define e^{z*} by the series

$$1 + z + \frac{z^2}{2} + \frac{z^6}{6} + \ldots ,$$

or, equivalently, by setting

$$e^{x+iy} = e^x \cos y + ie^x \sin y$$

for all real x and y. Of course, e^{z*} is no longer positive, in general.

We define $\log_e w$ by saying that z is a value of $\log_e w$ (denoted by $z \in \log_e w$) if

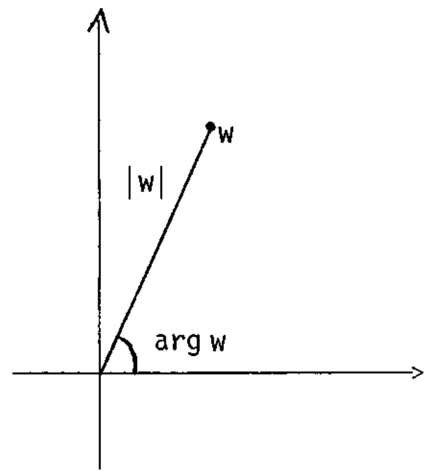
$$e^{z*} = w.$$

It turns out that e^{z*} takes every nonzero complex value, and is a periodic function, with pure imaginary period $2\pi i$:

$$e^{z+2\pi i*} = e^{z*}$$

It follows that $\log_e w$ is defined for all nonzero w, has infinitely many values, and the various values differ by whole multiples of $2\pi i$. There is an equation

$$\log_e w = \log_e|w| + i \arg w$$



(between multiple-valued functions, or <u>relations</u> as they are known nowadays), where $\log_e|w|$ is the real logarithm of the absolute value of w, and arg w is the <u>argument</u> of w. To see this, let

$$z = x + iy$$

be a value of $\log_e w$, let $|w| = \rho$, and let θ be a value of arg w. Then

$$w = \rho(\cos\theta + i\sin\theta),$$

$$e^{z*} = w,$$

$$e^{x*}(\cos y + i\sin y) = \rho(\cos\theta + i\sin\theta)$$

$$e^{x*} = \rho,$$

$$x = \log_e\rho, \text{ (the real value)}$$

$$\cos y = \cos\theta, \ \sin y = \sin\theta,$$

$$y = \theta + 2n\pi, \text{ for some integer } n.$$

Thus $x = \log_e|w|$ and $y \in \arg w$.

(4.2) Now we may define a^b for $a \neq 0$ and all b by the formula

$$a^b = e^{b \log_e a*},$$

i.e. z is a value of a^b if and only if $z = e^{bw*}$ for some value w of $\log_e a$.

(4.3) As an example, we work out the values of

$$(2+3i)^{1+i} = e^{(1+i)\log_e(2+3i)*}.$$

We have

$$\log_e(2+3i) = \log_e|2+3i| + i\arg(2+3i)$$

$$= \log_e\sqrt{13} + i\arctan\frac{3}{2} + 2n\pi i$$

$$= 1.28247 + 0.98279i + 2n\pi i,$$

$(1+i)\log(2+3i)$

$= (0.29968 - 2n\pi) + (2.26527 + 2n\pi)i,$

$(2+3i)^{1+i}$

$= e^{0.29968*}(e^{-2\pi*})^n e^{2.26527i*}$

$= 1.3494(0.001867)^n(\cos 2.26527 + i\sin 2.26527)$

$= (0.001867)^n(0.863 + 1.037i),$

where $n = 0,1,-1,2,-2,3, \ldots$.

All these values happen to share the same argument. This does not occur in general.

§5. Recapitulation.

(5.1) If b is an integer, then a^b is still singlevalued. For if w_1 and w_2 are two values of $\log_e a$, then they differ by $2m\pi i$, for some integer m, hence

$$e^{b\,w_1} = e^{b\,w_2}$$

since

$$e^{2mb\pi i} = 1.$$

Moreover, the value of a^b is the old one, obtained by multiplying a by itself b times.

(5.2) If b is a rational number $\frac{m}{n}$, in which $n > 0$ and m and n have no common factors, then a^b has precisely n values, for much the same reason. If c is one value, then the others are

$$c\ e^{2\pi rbi*}, \quad r = 1,2,\ldots,n-1.$$

These values are the n-th roots of a^m (or, equivalently, the m-th powers of the n-th roots of a), as before.

(5.3) If a and b are positive, then the old positive power $c = a^{b*}$ is one of the values of a^b. The others are

$$c\ e^{2\pi bmi*}, \quad m = 1,-1,2,-2,3,-3,\ldots \ .$$

Unless b is rational, these numbers are all distinct, and lie on the circle with centre 0 and radius c. In fact, every point of this circle is the limit of a sequence of these values of a^b. This is not to say that every point of the circle _is_ a value of a^b. Most such points are not.

(5.4) If a is not positive or b is not rational, then the new concept of a^b is the only one available. This includes the case of irrational real powers of negative real numbers, such as

$$(-2)^{\sqrt{3}}, \quad (-1)^{\pi}, \quad (-\pi)^{\sqrt{2}},$$

and the case of pure imaginary powers of real numbers, such as

$$2^i, \quad (-1)^i, \quad 3^{-2i} \ .$$

You may enjoy working out the various values of these powers to, say, two decimal places, with the aid of a pocket calculator. It is also interesting to plot the corresponding points on the Argand diagram. If $a \neq 0$ and b is not rational, then there are three possible geometric patterns for the values of the power a^b. The general form is

$$\alpha\beta^n\left\{\cos(\gamma+n\delta) + i\sin(\gamma+n\delta)\right\}.$$

If $\beta \neq 1$ and $\delta \neq 0$, then the values are a sequence of sporadic points on an exponential spiral

$$r = \nu e^{\mu\theta}.$$

If $\beta = 1$, then the points are dense on a circle. If $\delta = 0$, they are all on a straight ray from the origin.

(5.5) As a last example, take i^i. The values of $\log_e i$ are

$$(2n+\tfrac{1}{2})\Pi i, \quad n \text{ integral}.$$

Thus the values of i^i are

$$e^{-(2n+\frac{1}{2})\Pi *},$$

and are all positive real numbers!

§6. Coda.

(6.1) The Riemann zeta function is defined for complex numbers s with positive real part by

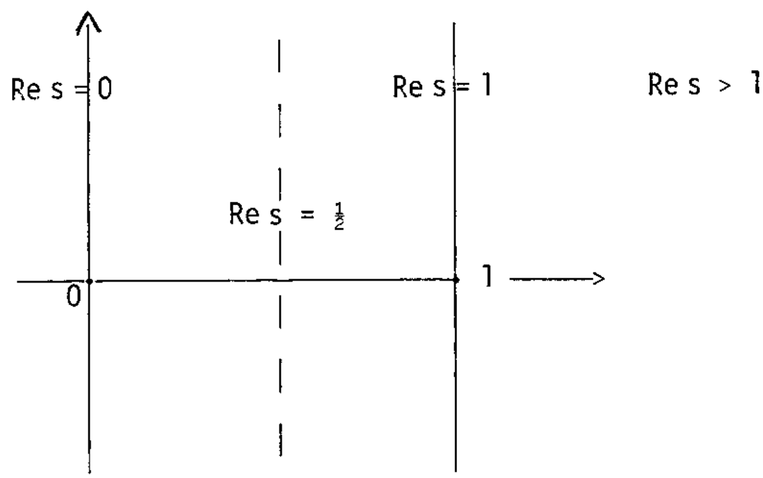
$$\zeta(s) = \frac{1}{s-1} + \tfrac{1}{2} - s\sum_{n=1}^{\infty}\int_1^0 \frac{t-\frac{1}{2}}{(n+t)^{s+1}}\,dt.$$

Here, $(n+t)^{s+1} = e^{(s+1)\log_e(n+t)*}$ is the value obtained by taking the real value of $\log_e(n+t)$. For $\operatorname{Re} s > 1$ there are two other expressions for $\zeta(s)$, a sum and a product:

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s},$$

$$\zeta(s) = \prod_{p \text{ prime}}\left\{\frac{1}{1-\frac{1}{p^s}}\right\}.$$

From these formulae you may find it possible to believe that $\zeta(s)$ carries a lot of information about the factorization of whole numbers. In fact, it turns out that the location of the zeros of $\zeta(s)$ (i.e. those numbers z such that $\zeta(z) = 0$) is of great importance for number theory. The Riemann Hypothesis is that all the zeros of $\zeta(s)$ in the half-plane



Re s > 0 lie on the line Re s = ½. The problem is to decide if this is true or not.