Reversible maps in the group of quaternionic Möbius transformations

By ROMAN LÁVIČKA †

Mathematical Institute, Charles University, Prague, Czech Republic
e-mail: lavicka@karlin.mff.cuni.cz

ANTHONY G. O'FARRELL‡

National University of Ireland, Maynooth, Ireland
e-mail: anthonyg.ofarrell@gmail.com

AND

IAN SHORT‡

National University of Ireland, Maynooth, Ireland
e-mail: Ian.Short@nuim.ie

(*Received*)

Abstract

The reversible elements of a group are those elements that are conjugate to their own inverse. A reversible element is said to be reversible by an involution if it is conjugate to its own inverse by an involution. In this paper, we classify the reversible elements and the elements reversible by involutions in the group of quaternionic Möbius transformations.

1. *Introduction*

The *reversible* elements of a group G are those members g of G for which there exists $h \in G$ with $h^{-1}gh = g^{-1}$. Equivalently, $g \in G$ is reversible if and only if there are elements $h, k \in G$ with $g = hk$ and $g^{-1} = kh$. The element g is *reversible by an involution* if h can be chosen to be an involution. Equivalently, g is reversible by an involution if and only if there are involutions $h, k \in G$ with $g = hk$. The property of being reversible, the property of being reversible by an involution, and the property of being an involution, are conjugation invariants. Interest in reversibility has grown from the notion of time-reversibility in dynamical systems, and time-reversible systems are related to Hamiltonian dynamical systems. See [**15**] for a survey of the physical background of time-reversible systems. The reversible maps and maps reversible by involutions have been classified in many groups, too many to list here, and we reference only [**12**], [**13**], and [**16**], as samples of the rich literature. The purpose of this paper is to describe the reversible maps and the

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maps reversible by involutions in certain groups defined using quaternions, in particular, the group of quaternionic Möbius transformations.

Consider Möbius transformations of the extended complex plane, $\mathbb{C}_\infty$, of the form

$$z \mapsto \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{C}$, and $ad-bc \neq 0$ (subject to the usual conventions regarding the point ∞). These maps are conformal, and the full collection of them forms a group, denoted $\mathcal{M}_2^+$. Adjoin to $\mathcal{M}_2^+$ all maps of the form

$$z \mapsto \frac{a\bar{z}+b}{c\bar{z}+d},$$

where $a, b, c, d \in \mathbb{C}$, and $ad-bc \neq 0$, and another group $\mathcal{M}_2$ is formed, the 2-*dimensional Möbius group*. Let $\mathcal{M}_2^- = \mathcal{M}_2 \setminus \mathcal{M}_2^+$. The maps in $\mathcal{M}_2^-$ are anti-conformal. An *inversion* in $\mathbb{C}_\infty$ means either an inversion in a circle, or a Euclidean reflection. The group $\mathcal{M}_2$ contains all inversions. Sometimes the term 'reflection' is used instead of 'inversion': in this paper the term 'reflection' is reserved for Euclidean reflections. The inversions generate $\mathcal{M}_2$, and since they are anti-conformal, the group $\mathcal{M}_2^+$ consists of composites of even numbers of inversions, and $\mathcal{M}_2^-$ consists of composites of odd numbers of inversions. An *inversion* in $\mathbb{R}^N_\infty$ (the one-point compactification of $\mathbb{R}^N$) means either an inversion in an $(N-1)$-dimensional sphere, or a Euclidean reflection in an $(N-1)$-dimensional plane. The N-dimensional Möbius group $\mathcal{M}_N$ is the group of transformations of $\mathbb{R}^N_\infty$ generated by inversions in $\mathbb{R}^N_\infty$. The subgroup $\mathcal{M}_N^+$ consists of the conformal members of $\mathcal{M}_N$, and the subset $\mathcal{M}_N^-$ consists of the anti-conformal members of $\mathcal{M}_N$. It is well known that the field of quaternions, $\mathbb{H}$, can be identified with $\mathbb{R}^4$, but it is less well known that $\mathcal{M}_4^+$ can be identified with the group of bijections of $\mathbb{H}_\infty$ (the one-point compactification of $\mathbb{H}$) of the form

$$z \mapsto (az+b)(cz+d)^{-1}, \tag{1·1}$$

where $a, b, c, d \in \mathbb{H}$ (again, assuming certain conditions regarding the point ∞ that we later analyse). Moreover, the bijections of the form

$$z \mapsto (a\bar{z}+b)(c\bar{z}+d)^{-1}, \tag{1·2}$$

where $a, b, c, d \in \mathbb{H}$, make up the rest of $\mathcal{M}_4$. The Möbius group, which was defined geometrically with inversions, can now be studied with the algebra of quaternions. In this paper, we describe the reversible maps in $\mathcal{M}_4$ using the algebra of quaternions, and the geometry of the Möbius group acts as a guide.

There are many incarnations of the groups $\mathcal{M}_N$, hence many ways to calculate the reversible maps in $\mathcal{M}_N$. For example, $\mathcal{M}_N$ is the group of hyperbolic isometries of $(N+1)$-dimensional hyperbolic space, so an entirely geometric treatment of reversible maps can be given. Alternatively, $\mathcal{M}_N$ can be realised as a matrix group: the group of positive Lorentz transformations of $\mathbb{R}^{N+2}$, and the reversible maps can be found with linear algebra (see [**10**] for work on involutions in the Lorentz group). Nevertheless, the theory of quaternionic Möbius transformations, coupled with geometric intuition, has a beautiful coherence that will seem familiar to those acquainted with the usual $\mathcal{M}_2^+$ action on $\mathbb{C}_\infty$.

This paper is a complete treatment of the reversible maps of quaternionic Möbius transformations. The account could have been significantly shortened by referencing classical works such as [**8**], but we choose to provide all details for two reasons. First, and most

importantly, the paper is more readable for our having provided background material. The underlying theory is sufficiently shallow that the account benefits from the clarity of additional information, without being too heavily burdened by the extra weight of detail. Second, although the geometric theory of 4-dimensional Möbius transformations is known, there are few concise accounts of the transition from $\mathcal{M}_4$, defined as the group generated by inversions, to the equivalent quaternionic construction. We are careful in the remainder of this introduction to explain which parts of this document are original, and which parts are included for completeness and clarity of exposition.

The structure of the paper is as follows. In §2, we give a brief introduction to quaternions. In §3, we describe the 4-dimensional orthogonal group in terms of quaternions. There is a brief digression in §3 to classify the reversible maps in the multiplicative group $\mathbb{H}\setminus\{0\}$. The material from the digression, although presented unusually in the context of reversible maps, is well known. In §4 and §5, we consider involutions and composites of involutions in the orthogonal group. The representation of the 4-dimensional orthogonal group in terms of quaternions can be found in [**8**], and the properties of the involutions in the orthogonal group are well known ([**13**] and [**14**]). Nevertheless, the orthogonal involutions do not appear to have been studied with quaternions, and the work of §4 and §5 is necessary for later sections.

In §6, the quaternionic representation of the 4-dimensional Möbius group is reconciled with the geometric definition as a group generated by inversions. Certain elements are then distinguished according to conjugacy. Most of this material can be found in [**11**]. In §7 and §8 we analyse the involutions and composites of involutions in the Möbius group. The main results are summarised in the next two theorems, which are original. The proofs can be found in §8.

THEOREM 1·1. *Each transformation in $\mathcal{M}_4$ is reversible by an involution.*

This result implies that all members of $\mathcal{M}_4$ are reversible. The situation in $\mathcal{M}_4^+$ is not as straightforward. The reversibility properties are summarised in Theorem 1·2. In that theorem, the term *loxodromic* is used to describe a Möbius transformation f in $\mathcal{M}_4^+$ that is conjugate to a map of the form $x \mapsto \lambda axb$, where $\lambda > 0$, $\lambda \neq 1$, and a and b are unit quaternions. Also in Theorem 1·2, the notion of a *real* part of a quaternion is used. This notion is defined formally in §2; it has an analogous meaning to the real part of a complex number.

THEOREM 1·2. *Each map $f \in \mathcal{M}_4^+$ that is not loxodromic is reversible by an involution. Given a loxodromic map $f \in \mathcal{M}_4^+$ that is conjugate to $x \mapsto \lambda axb$, where $\lambda > 0$, $\lambda \neq 1$, and $|a| = |b| = 1$; the following are equivalent:*

(i) *the map f is reversible;*
(ii) *the map f is reversible by an involution;*
(iii) *the absolute values of the real parts of a and b are equal.*

If f is not reversible then it nevertheless may be expressed as the composite of three involutions in $\mathcal{M}_4^+$.

We remark that, in contrast to Theorem 1·2, every element of $\mathcal{M}_2^+$ is reversible. This can easily be proven using the usual representation of maps in $\mathcal{M}_2^+$ using complex numbers. Also, every element of $\mathcal{M}_3^+$ is reversible. On the other hand, in $\mathcal{M}_1^+$ (the group of conformal real Möbius transformations), all loxodromic maps are reversible, and the only

maps that are not loxodromic, but are reversible, are involutions. A complete analysis of reversibility in all groups $\mathcal{M}_N^+$ will appear in [**18**].

In this paper we combine quaternion algebra with geometry, and in so doing we follow H. S. M. Coxeter ([**8**]), and P. G. Gormley ([**11**]—an extension of [**8**]). Others have recently studied quaternionic Möbius transformations for various purposes. See, for example, [**6**]. There has also recently been interest in representations of $\mathcal{M}_N$ in terms of Clifford algebras, following a sequence of papers written by L. V. Ahlfors in the 1980s ([**2**], [**3**], [**4**], and [**5**]). Ahlfors references K. Th. Vahlen as his source of inspiration (perhaps Vahlen was the first to work with quaternionic Möbius transformations). The theory of this paper can be generalised to N-dimensions using Ahlfors' techniques, but the additional algebraic complexity of the Clifford group would stretch the paper sufficiently that instead we recommend that one of the other models of hyperbolic space be used to determine the reversible maps.

2. *Quaternions*

Let $\mathbb{H}$ denote the non-commutative field of quaternions. A general element of $\mathbb{H}$ takes the form $s+iu_1+ju_2+ku_3$, where $s, u_1, u_2, u_3 \in \mathbb{R}$, and i, j, and k satisfy the relations

$$i^2 = j^2 = k^2 = -1, \quad ij = k, \quad jk = i, \quad ki = j.$$

We identify the quaternions with $\mathbb{R}^4$ via the correspondence $s + iu_1 + ju_2 + ku_3 \mapsto (s, u_1, u_2, u_3) \in \mathbb{R}^4$.

The *real part* of $x = s + iu_1 + ju_2 + ku_3$ is denoted $\mathrm{Re}[x]$ and it is equal to s. The *imaginary part* of x is $iu_1 + ju_2 + ku_3$. A quaternion x is *real* if $u_1 = u_2 = u_3 = 0$, and x is *purely imaginary* if $\mathrm{Re}[x] = 0$. We identify the real quaternions with $\mathbb{R}$ and we identify the quaternions of the form $s + it$, for $s, t \in \mathbb{R}$, with $\mathbb{C}$. It is sometimes convenient to reduce quaternionic calculations to calculations in $\mathbb{C}$ because of the commutativity in $\mathbb{C}$. We identify purely imaginary quaternions with triples of real numbers via the correspondence $iu_1 + ju_2 + ku_3 \mapsto (u_1, u_2, u_3)$. Then vector products of purely imaginary quaternions can be taken. Write quaternions x and y in terms of their real and imaginary parts as $x = s + u$ and $y = t + v$, where $s, t \in \mathbb{R}$, and u and v are purely imaginary quaternions. Let $u \cdot v$ denote the scalar product of u and v and let $u \times v$ denotes the vector product of u and v. Then

$$xy = (st - u \cdot v) + (sv + tu + u \times v). \tag{2·1}$$

The *conjugate* of x is the quaternion $\bar{x} = s - iu_1 - ju_2 - ku_3$. The conjugate of x is particularly useful as it is the multiplicative inverse of x, if x is of unit norm. Next are three useful lemmas; the proofs of the first two follow immediately from (2·1).

Lemma 2·1. *Let $x, y \in \mathbb{H}$. Then $x\bar{y} + y\bar{x} = 2(x \cdot y)$.*

Lemma 2·2. *An element $a \in \mathbb{H}$ satisfies $a^2 = -1$ if and only if a is a purely imaginary unit vector.*

Lemma 2·3. *Suppose that unit quaternions a and b satisfy $axb = x$ for each $x \in \mathbb{H}$. Then either $a = b = 1$ or $a = b = -1$.*

Proof. Set $x = 1$. Then $ab = 1$, so that $ax = xa$ for all x. Set $x = i$ and j in turn to see that a is real, therefore either $a = 1$, or $a = -1$, as required.

3. *The 4-dimensional orthogonal group*

As a preliminary step toward describing quaternionic Möbius transformations; we represent orthogonal maps (which *are* Möbius transformations) in terms of quaternions. Let O_4 denote the group of orthogonal maps of $\mathbb{R}^4$ and let SO_4 denote the subgroup of conformal maps in O_4. It is readily proven with linear algebra (see, for example, [**9**]) that elements of SO_4 are expressible as the direct sum of two 2-dimensional rotations (about 0) and elements of $\mathrm{O}_4 \setminus \mathrm{SO}_4$ are expressible as the direct sum of a 2-dimensional rotation (about 0) and a 2-dimensional reflection (in a hyperplane through 0). The material on orthogonal maps in this section can be found in [**8**].

Lemma 3·1. *The map*

$$\tau_y : \mathbb{R}^4 \to \mathbb{R}^4, \quad x \mapsto -y\bar{x}y,$$

where $|y| = 1$*, is the reflection in the Euclidean hyperplane* Π_y *perpendicular to* y*.*

Proof. It is immediate that τ_y is linear and that $\tau_y(y) = -y$. Using Lemma 2·1, one can check that $\tau_y(x) = x$ for each $x \in \Pi_y$.

The next theorem is [**8**, Theorem 8.1].

Theorem 3·2. *The group* O_4 *is the set of all maps of one of the forms* $x \mapsto axb$ *or* $x \mapsto a\bar{x}b$*, where* $|a| = |b| = 1$*. The subgroup* SO_4 *is the set of all maps of the form* $x \mapsto axb$*, where* $|a| = |b| = 1$*.*

Proof. Consider maps $\alpha(x) = axb$ and $\beta(x) = a\bar{x}b$, where $|a| = |b| = 1$. They are linear, and they are orthogonal since, for example, $|\alpha(x)| = |a||x||b| = |x|$. The first part of Theorem 3·2 follows because O_4 is generated by linear reflections, which, according to Lemma 3·1, assume the form τ_y, where $|y| = 1$. The second part of Theorem 3·2 holds because a member of O_4 is of the same form as α, rather than β, if and only if it is the composite of an even number of (anticonformal) reflections of the form τ_y.

Two maps $x \mapsto axb$ and $x \mapsto c\bar{x}d$, for $|a| = |b| = |c| = |d| = 1$, are never the same as the former is conformal and the latter is anti-conformal. Suppose maps $x \mapsto axb$ and $x \mapsto cxd$ are the same. Then, by Lemma 2·3, either $c = a$ and $d = b$, or $c = -a$ and $d = -b$. Similar comments apply to maps of the form $x \mapsto a\bar{x}b$.

Corollary 3·3. *A map* $\alpha \in \mathrm{SO}_4$ *fixes the real line pointwise if and only if the map* α *is of the form* $\alpha(x) = \bar{a}xa$*, where* $|a| = 1$*. All such maps form the group* SO_3 *of special orthogonal transformations of the set of purely imaginary quaternions (which is identified with* $\mathbb{R}^3$*).*

Proof. If the orthogonal map $\alpha(x) = bxa$, where $|a| = |b| = 1$, fixes $\mathbb{R}$ pointwise, then $\alpha(1) = 1$, therefore $b = \bar{a}$. With this observation, the corollary follows from Theorem 3·2.

A consequence of Corollary 3·3, and a digression from the classification of orthogonal maps in terms of quaternions, is a conjugacy classification in the multiplicative group of quaternions. (Proposition 3·4, below, is equivalent to [**8**, Lemma 2.2].)

Proposition 3·4. *Two non-zero elements* $a, b \in \mathbb{H}$ *are conjugate in the multiplicative group* $\mathbb{H} \setminus \{0\}$ *if and only if* $|a| = |b|$ *and* $\mathrm{Re}[a] = \mathrm{Re}[b]$*.*

Proof. Suppose $u^{-1}au = b$, for some $u \in \mathbb{H} \setminus \{0\}$. Then $|b| = |u|^{-1}|a||u| = |a|$. Let $v = u/|u|$, so that $\bar{v}av = b$. Therefore $\bar{v}\bar{a}v = \bar{b}$. Hence $2\mathrm{Re}[b] = b+\bar{b} = \bar{v}(a+\bar{a})v = 2\mathrm{Re}[a]$. For the converse, if $|a| = |b|$ and $\mathrm{Re}[a] = \mathrm{Re}[b]$ then there is a map $f \in \mathrm{SO}_3$ with $f(\mathrm{Im}[a]) = \mathrm{Im}[b]$ and, by Corollary 3·3, $\bar{v}av = b$ for some unit quaternion v.

The reversible maps in $\mathbb{H} \setminus \{0\}$ can be classified using Proposition 3·4.

Corollary 3·5. *An element $a \in \mathbb{H} \setminus \{0\}$ is reversible if and only if $|a| = 1$.*

Proof. If $v^{-1}av = a^{-1}$ for some $v \neq 0$, then $|a| = 1$. The converse follows from Proposition 3·4, because $\mathrm{Re}[a] = \mathrm{Re}[a^{-1}]$ for a unit quaternion a.

In contrast to Corollary 3·5, check using (2·1) that -1 and 1 are the only involutions in $\mathbb{H} \setminus \{0\}$, therefore these are the only quaternions that are reversible by involutions.

Here is another corollary of Proposition 3·4, which we apply in Theorem 3·7, below.

Corollary 3·6. *Select $a, b \in \mathbb{H}$ of unit norm. Then $\mathrm{Re}[a] = \mathrm{Re}[b]$ if and only if there exist $u, v \in \mathbb{H}$ of unit norm such that $a = uv$ and $b = vu$.*

Proof. The condition involving u and v holds if and only if a and b are conjugate. The result then follows from Proposition 3·4.

A *rotation* in O_4 is an orthogonal map that can be written as the composite of two reflections. Such maps may be expressed as the direct sum of a 2-dimensional rotation about the origin and a 2-dimensional identity map. The final theorem in this section is [**8**, Theorem 5.2].

Theorem 3·7. *The set of rotations in O_4 is the set of maps of the form $\alpha(x) = axb$, where $|a| = |b| = 1$ and $\mathrm{Re}[a] = \mathrm{Re}[b]$.*

Proof. A rotation α is the composite of two reflections, say $\alpha = \sigma\tau$, where $\sigma(x) = -u\bar{x}u$ and $\tau(x) = -\bar{v}\bar{x}\bar{v}$, for unit quaternions u and v. Therefore $\alpha(x) = uvxvu$. Set $a = uv$ and $b = vu$. Then $|a| = |b| = 1$ and $\mathrm{Re}[a] = \mathrm{Re}[b]$, by Corollary 3·6. Conversely, suppose that $f(x) = axb$, where $|a| = |b| = 1$ and $\mathrm{Re}[a] = \mathrm{Re}[b]$. From Corollary 3·6, there are unit quaternions u and v with $a = uv$ and $b = vu$. Hence $\alpha = \sigma\tau$, where $\sigma(x) = -u\bar{x}u$ and $\tau(x) = -\bar{v}\bar{x}\bar{v}$ are both reflections in hyperplanes that contain the origin (by Lemma 3·1), so that α is a rotation.

4. *Involutions in the orthogonal group*

First, we give a geometric description of the involutions in O_4. Then we represent involutions by quaternion maps, using the results of the previous section. Recall that $\alpha \in \mathrm{O}_4$ is the direct sum of two 2-dimensional rotations about 0, if $\alpha \in \mathrm{SO}_4$, and α is the direct sum of a 2-dimensional rotation about 0 and a 2-dimensional reflection in a line through 0, if $\alpha \notin \mathrm{SO}_4$. In both cases, α is an involution if and only if the angle of each rotation is either 0 or π. Note that the direct sum of a 2-dimensional reflection and a 2-dimensional identity map is a 4-dimensional reflection, and such maps generate O_4. (The 4-dimensional reflections are the maps described in Lemma 2·1.) Now involutions in O_4 are examined with quaternions.

Proposition 4·1. *The involutions in SO_4 are the orthogonal maps of the form $x \mapsto x$, $x \mapsto -x$, and $x \mapsto axb$, where $a^2 = b^2 = -1$.*

Proof. The three given types of map are involutions, and they are members of SO_4, by Theorem 3·2. Conversely, a map $\alpha \in \mathrm{SO}_4$ is of the form $\alpha(x) = axb$, where $|a| = |b| = 1$, so if α is an involution then $a^2xb^2 = x$, for all $x \in \mathbb{H}$. By Lemma 2·3, either $a^2 = b^2 = 1$ or $a^2 = b^2 = -1$. In the former case, $a = \bar{a}$, hence $a = \pm 1$, therefore either $\alpha(x) = x$ or $\alpha(x) = -x$.

Using Lemma 2·2 and Theorem 3·7 we deduce that maps $x \mapsto axb$, where $a^2 = b^2 = -1$, are rotations (by π) with $\mathrm{Re}[a] = \mathrm{Re}[b] = 0$.

Proposition 4·2. *The anti-conformal involutions in* O_4 *are the maps of the form* $x \mapsto -a\bar{x}a$, $|a| = 1$ *(reflections), or maps of the form* $x \mapsto a\bar{x}a$, $|a| = 1$ *(direct sums of* 2*-dimensional reflections and* 2*-dimensional rotations by* π*).*

Proof. A map $\alpha \in \mathrm{O}_4 \setminus \mathrm{SO}_4$ has the form $\alpha(x) = a\bar{x}b$, where $|a| = |b| = 1$. This map is an involution if and only if $a\bar{b}x\bar{a}b = x$, for all $x \in \mathbb{H}$. By Lemma 2·3, this occurs if and only if either $a = -b$ or $a = b$.

5. *Composites of involutions in the orthogonal group*

It is well known that each element of O_N can be expressed as the composite of N reflections, and there are elements of O_N that cannot be expressed as the composite of any fewer reflections (see, for example, [**9**]). In this section we show that each map in O_4 is the composite of two involutions, in other words, each map is reversible by an involution. A similar statement is proven in the subgroup SO_4. It is easy to verify these claims geometrically. Each 2-dimensional rotation about 0 is the composite of two 2-dimensional reflections in lines through 0. Using the decomposition of a 4-dimensional orthogonal map α described at the beginning of the previous two sections, the map α can be written as the composite of two involutions. Explicitly in case $\alpha \in \mathrm{SO}_4$: there exist two 2-dimensional rotations σ and τ with $\alpha = \sigma \oplus \tau$. Let $\sigma = \sigma_1\sigma_2$ and $\tau = \tau_1\tau_2$, for 2-dimensional reflections σ_1, σ_2, τ_1, and τ_2; then $\alpha = (\sigma_1 \oplus \tau_1)(\sigma_2 \oplus \tau_2)$. Notice that $\sigma_1 \oplus \tau_1 = (\sigma_1 \oplus \iota)(\iota \oplus \tau_1)$, where ι is the identity map, hence $\sigma_1 \oplus \tau_1$ is the composite of two 4-dimensional reflections, therefore it is a rotation (by π). Similarly for $\sigma_2 \oplus \tau_2$. Now composites of involutions in O_4 and SO_4 are examined with quaternions.

Theorem 5·1, below, is a special case of [**14**, Theorem B]. Theorem 5·2, also below, is a special case of the theorem from §7 of [**14**].

Theorem 5·1. *For any* $\alpha \in \mathrm{SO}_4$ *there are involutions* σ *and* τ *in* SO_4 *with* $\alpha = \sigma\tau$.

Proof. Let $\alpha(x) = axb$, for $a, b \in \mathbb{H}$, where $|a| = |b| = 1$. Choose a purely imaginary unit vector c that is perpendicular to both a and b. By (2·1), the vectors $d = ac$ and $e = cb$ are also purely imaginary. Define $\sigma(x) = dxe$ and $\tau(x) = cxc$. Bearing in mind Lemma 2·2, check that these maps σ and τ have the required properties.

Theorem 5·2. *Each map* $\alpha \in \mathrm{O}_4$ *is expressible as the composite of two involutions.*

Proof. It remains only to consider the case when $\alpha(x) = a\bar{x}b$, for $|a| = |b| = 1$. The proof resembles the proof of Theorem 5·1, but with $\tau(x) = c\bar{x}c$.

The next proposition is used in §8·2: it shows that a map f in SO_4 is reversible by an anti-conformal map if and only if either f or $-f$ is a rotation.

PROPOSITION 5·3. *Let $\alpha \in \mathrm{SO}_4$ be given by the formula $\alpha(x) = axb$, where $|a| = |b| = 1$. There is a map $\beta \in \mathrm{O}_4 \setminus \mathrm{SO}_4$ with $\beta^{-1}\alpha\beta = \alpha^{-1}$ if and only if $|\mathrm{Re}[a]| = |\mathrm{Re}[b]|$.*

Proof. If $|\mathrm{Re}[a]| = |\mathrm{Re}[b]|$ then either $\alpha = \sigma\tau$ or $\alpha = -\sigma\tau$ for reflections σ and τ, by Theorem 3·7. In this case choose $\beta = \sigma$. Conversely, let $\beta(x) = c\bar{x}d$, for unit quaternions c and d, and suppose that $\beta^{-1}\alpha\beta = \alpha^{-1}$. Then for all $x \in \mathbb{H}$,

$$d\bar{b}\bar{d}x\bar{c}\bar{a}c = \beta^{-1}\alpha\beta(x) = \alpha^{-1}(x) = \bar{a}x\bar{b}.$$

By Lemma 2·3, either $d\bar{b}\bar{d} = \bar{a}$, in which case $\mathrm{Re}[a] = \mathrm{Re}[b]$ (because the map $x \mapsto dx\bar{d}$ preserves the real line pointwise); or $d\bar{b}\bar{d} = -\bar{a}$, in which case $\mathrm{Re}[a] = -\mathrm{Re}[b]$.

6. *The 4-dimensional Möbius group*

The Möbius group $\mathcal{M}_4$ is the group of transformations of $\mathbb{R}^4_\infty$ generated by inversions in $\mathbb{R}^4_\infty$. The subgroup $\mathcal{M}_4^+$ of $\mathcal{M}_4$ consists of those transformations that are the composite of even numbers of inversions. In §6·1 it is shown that $\mathcal{M}_4$ has a simple algebraic representation in terms of quaternions. Then in §6·3 the maps in $\mathcal{M}_4$ are distinguished according to their different dynamics.

6·1. *Quaternionic representation of Möbius transformations*

Let $\mathbb{H}_\infty$ denote $\mathbb{H} \cup \{\infty\}$. Consider the map $f : \mathbb{H}_\infty \to \mathbb{H}_\infty$ which is given by the quaternion formula

$$f(x) = (ax + b)(cx + d)^{-1}, \quad \Delta(f) \neq 0, \tag{6·1}$$

where $\Delta(f) = |a|^2|d|^2 + |b|^2|c|^2 - 2\mathrm{Re}[a\bar{c}d\bar{b}]$, and we adopt the usual special rules regarding the point ∞. (Namely, if $c = 0$ then $f(\infty) = \infty$ and f is determined by (6·1) for all $x \in \mathbb{H}$; and if $c \neq 0$ then $f(-c^{-1}d) = \infty$, $f(\infty) = ac^{-1}$, and f is determined by (6·1) for all $x \in \mathbb{H} \setminus \{-c^{-1}d\}$.) If $c \neq 0$ then $f = f_1 f_2 f_3 f_4$, where

$$f_1(x) = x + ac^{-1}, \quad f_2(x) = -(b - ac^{-1}d)xc^{-1}, \quad f_3(x) = -x^{-1}, \quad f_4(x) = x + c^{-1}d.$$

Notice that $b - ac^{-1}d \neq 0$ since

$$|b - ac^{-1}d|^2 = \Delta(f)/|c|^2.$$

This equation and the decomposition $f = f_1 f_2 f_3 f_4$ can be combined to show that f is constant if $\Delta(f) = 0$. The maps f_1 and f_4 are translations, f_2 is a special orthogonal map followed by a scaling, and f_3 is inversion in the unit sphere followed by reflection in the plane with equation $\mathrm{Re}[x] = 0$. On the other hand, if $c = 0$ then neither a nor d can be 0, since $\Delta(f) \neq 0$. Therefore $f = f_5 f_6$, where

$$f_5(x) = axd^{-1}, \quad f_6(x) = x + a^{-1}b.$$

The map f_5 is a special orthogonal map followed by a scaling and f_6 is a translation. All these maps f_1, f_2, f_3, f_4, f_5, and f_6 lie in $\mathcal{M}_4^+$, hence $f \in \mathcal{M}_4^+$. To see the converse, that each map in $\mathcal{M}_4^+$ is of the form (6·1), first observe that each member of $\mathcal{M}_4^+$ can be expressed as the composite of translations, scalings, orthogonal maps, and maps of the same form as f_3. (This statement is well known. See, for example, [**17**, Theorem 4.4.4].) Now, all translation, scalings, and orthogonal maps can be represented in the form (6·1), and it is straightforward to verify that the composite of two maps $f(x) = (ax + b)(cx + d)^{-1}$ and $g(x) = (a'x + b')(c'x + d')^{-1}$, with coefficients in $\mathbb{H}$ and $\Delta(f), \Delta(g) \neq 0$, is a third map of the form $h(x) = (a''x + b'')(c''x + d'')^{-1}$, also with coefficients in $\mathbb{H}$. (That

$\Delta(h) \neq 0$ follows because f and g are both bijections of $\mathbb{H}_\infty$, therefore h is also, whereas h is constant if $\Delta(h) = 0$.) Therefore $\mathcal{M}_4^+$ can be described as the group of maps of the form (6·1).

The group $\mathcal{M}_4^+$ is a subgroup of index 2 in $\mathcal{M}_4$. The group $\mathcal{M}_4$ is generated by $\mathcal{M}_4^+$ and the reflection $x \mapsto \bar{x}$, therefore the next theorem is proven.

THEOREM 6·1. *In terms of quaternions, the group $\mathcal{M}_4^+$ consists of all maps of the form*

$$x \mapsto (ax + b)(cx + d)^{-1}, \quad \Delta(f) \neq 0, \tag{6·2}$$

with the usual conventions concerning the point ∞. The group $\mathcal{M}_4$ consists of all maps of the above form, as well as maps of the form

$$x \mapsto (a\bar{x} + b)(c\bar{x} + d)^{-1}, \quad \Delta(f) \neq 0, \tag{6·3}$$

with the usual conventions concerning the point ∞.

It is straightforward to check that the map f from (6·1) is the same map as the map $g(x) = (a'x + b')(c'x + d')^{-1}$, where $\Delta(g) \neq 0$, if and only if there is a real number $\lambda \neq 0$ with $a = \lambda a'$, $b = \lambda b'$, $c = \lambda c'$, and $d = \lambda d'$. Similar comments apply in $\mathcal{M}_4^-$. These facts are not used in the sequel.

6·2. *Conjugacy classes in the conformal Möbius group*

Each map in $\mathcal{M}_4^+$ is freely homotopic to the identity map and therefore has a fixed point, by the Lefschetz Fixed-Point Theorem. Möbius maps are distinguished according to the their fixed points. A map $f \in \mathcal{M}_4^+$ is

(i) *parabolic*, if it has only one fixed point;
(ii) *loxodromic*, if it has two fixed points p and q and, for each $x \in \mathbb{H}_\infty \setminus \{p, q\}$, the sequence $f^n(x)$ converges to p;
(iii) *elliptic*, otherwise.

This trichotomy is invariant under conjugation, although it is not as fine a classification of Möbius maps as a conjugacy classification. (That is, there are parabolic maps that are not conjugate, and likewise for elliptic and loxodromic maps.) For any given Möbius map f, we identify a map g conjugate to f which is of a simple algebraic form, and the structure of g will depend on whether f is parabolic, loxodromic, or elliptic.

LEMMA 6·2. *A map $f \in \mathcal{M}_4^+$ with at least two fixed points is conjugate to a map of the form $x \mapsto \lambda axb$, where $\lambda > 0$, and a and b are unit quaternions. If $\lambda = 1$ then f is elliptic, otherwise f is loxodromic.*

Proof. Let p and q be two fixed points of f, where $p \neq \infty$. Define $h \in \mathcal{M}_4^+$ as follows. If $q \neq \infty$ then $h(x) = (x - p)(x - q)^{-1}$ and if $q = \infty$ then $h(x) = x - p$. The map g which is equal to hfh^{-1} fixes 0 and ∞. By Theorem 6·1, $g(x) = \lambda axb$ for $\lambda > 0$ and unit quaternions a and b. That is, g is an orthogonal map followed by a scaling. If $\lambda \neq 1$ then g converges locally uniformly to either 0 (if $\lambda < 1$), or ∞ (if $\lambda > 1$), on $\mathbb{H} \setminus \{0\}$. Therefore g (and f) are loxodromic. If $\lambda = 1$ then g is an orthogonal map, and the dynamics of g differ from the dynamics of a loxodromic map.

It remains to consider the case when $f \in \mathcal{M}_4^+$ has one fixed point. Let ι denote the identity map of $\mathbb{H}_\infty$.

Lemma 6·3. *A parabolic map in $\mathcal{M}_4^+$ is conjugate to a map of the form $x \mapsto \bar{a}xa + 1$, where a is a unit quaternion.*

Proof. Each map in $\mathcal{M}_4^+$ is conjugate to a map f that fixes ∞. By Theorem 6·1, f is of the form $f(x) = \lambda\alpha(x) + c$, for $\lambda > 0$, $\alpha \in \mathrm{SO}_4$, and $c \in \mathbb{H}$. Since $f(x) = x$ has no solutions in $\mathbb{H}$, neither does the equation $(\lambda\alpha - \iota)(x) = -c$, therefore $(\lambda\alpha - \iota)$ is not invertible. This means that $\lambda = 1$ and there exists a unit quaternion y with $\alpha(y) = y$. Through conjugation, assume that $y = 1$. Suppose $\alpha(x) = bxa$, where $|a| = |b| = 1$. Then $\alpha(1) = 1$ implies that $b = \bar{a}$. A unit vector in $\mathbb{C}$ can be found with real part equal to $\mathrm{Re}[a]$; thus by Proposition 3·4, a further conjugation can be applied so that f is assumed to be of the form

$$f(x) = \bar{a}xa + d,$$

where $d = d_1 + jd_2$; $a, d_1, d_2 \in \mathbb{C}$; and $|a| = 1$. If $a = 1$ or $a = -1$, then choose $\beta \in \mathrm{SO}_4$ with $\beta(1) = d/|d|$ and define $\phi = |d|\beta$. Otherwise, define $\phi(x) = pxp + q$, where $p^2 = d_1$ and $q = jd_2/(1 - a^2)$. The complex number d_1 is not 0, because when $d_1 = 0$, conjugation of f by $x \mapsto x + q$ yields an orthogonal map (with two fixed points). Check that $\phi^{-1}f\phi(x) = \bar{a}xa + 1$, which is the required result. (To perform this check, it is useful to note that $a, d_1, d_2, p \in \mathbb{C}$, therefore these four complex numbers commute. Also note that $\bar{a}j = ja$.)

6·3. *Conjugacy classes in the Möbius group*

Transformations in the group $\mathcal{M}_4$ are classified as parabolic, loxodromic, or elliptic according to the criteria given at the beginning of §6·2. There are, however, anti-conformal Möbius transformations without fixed points. For example, $x \mapsto -(\bar{x})^{-1}$ cannot have a fixed point because $x = -(\bar{x})^{-1}$ if and only if $|x|^2 = -1$, which is impossible. These maps without fixed points must be elliptic. They are classified fully in Lemma 6·5, but before that lemma, two elementary properties of the map $\gamma(x) = (\bar{x})^{-1}$, which is inversion in the 4-dimensional unit sphere, are recorded.

Lemma 6·4. *The map γ commutes with all orthogonal maps.*

Proof. Let $\alpha(x) = axb$, where $|a| = |b| = 1$. Then

$$\gamma\alpha(x) = \left(\overline{axb}\right)^{-1} = \left(\bar{b}\bar{x}\bar{a}\right)^{-1} = \alpha\gamma(x)$$

for all $x \in \mathbb{H}$.

Let λ denote the map $x \mapsto \lambda x$, for some $\lambda > 0$. Notice that $\gamma\lambda = \lambda^{-1}\gamma$.

Lemma 6·5. *If f is an anti-conformal Möbius map without a fixed point then f is conjugate to $\alpha\gamma$ for some $\alpha \in \mathrm{SO}_4$.*

Proof. The map f^2 has a fixed point p. Define $q = f(p)$, then $f(q) = p$. By conjugation, assume that $p = 0$ and $q = \infty$. Then $f\gamma$ fixes 0 and ∞, hence $f\gamma(x) = \lambda\alpha(x)$ for some $\lambda > 0$ and $\alpha \in \mathrm{SO}_4$. Let $h(x) = \sqrt{\lambda}x$. Then $h^{-1}fh = \alpha\gamma$, as required.

The remaining members of $\mathcal{M}_4^-$ have a fixed point. The proof of the next lemma, which is omitted, resembles the proofs of Lemma 6·2 and Lemma 6·3.

Lemma 6·6. *If $f \in \mathcal{M}_4^-$ has a fixed point, then f is conjugate to a map of one of*

the forms $x \mapsto a\bar{x}b$, $x \mapsto \lambda a\bar{x}b$, *or* $x \mapsto \bar{a}\bar{x}a + 1$, *for unit quaternions* a *and* b, *and* $\lambda \in (0, \infty)\setminus\{1\}$, *depending on whether* f *is elliptic, loxodromic, or parabolic, respectively.*

7. *Involutions in the Möbius group*

By applying suitable conjugations to achieve the simpler algebraic forms described in Lemma 6·2, Lemma 6·3, and Lemma 6·6, we see that neither loxodromic nor parabolic Möbius maps have finite order, therefore there are no loxodromic or parabolic involutions.

Elliptic Möbius maps that are conformal are conjugate to orthogonal maps, and the involutive orthogonal maps were classified in §4. An elliptic Möbius map that is not conformal either has a fixed point, in which case it is conjugate to an orthogonal map, or it does not have a fixed point. If it does not have a fixed point then it is conjugate to $\alpha\gamma$, where $\alpha \in \mathrm{SO}_4$ and γ is inversion in the 4-dimensional unit sphere, by Lemma 6·5. Because α commutes with γ, the map $\alpha\gamma$ is involutive if and only if α is involutive.

8. *Composites of involutions in the Möbius group*

We prove Theorem 1·1 and Theorem 1·2 in this section. The analysis is split between elliptic, loxodromic, and parabolic transformations.

8·1. *Elliptic transformations*

Recall from §6·2 and §6·3 that an elliptic Möbius map with a fixed point is conjugate to an orthogonal map. Recall from §5 that each member of SO_4 is expressible as the composite of two involutions, and each member of O_4 is expressible as the composite of two involutions. Those Möbius maps without fixed points are conjugate to maps of the form $\alpha\gamma$, where $\alpha \in \mathrm{O}_4$ and $\gamma(x) = (\bar{x})^{-1}$. Choose σ and τ to be involutions in O_4 such that $\alpha = \sigma\tau$, then $\alpha\gamma = \sigma(\tau\gamma)$ is a decomposition into two involutions.

In summary, the elliptic members of $\mathcal{M}_4^+$ and $\mathcal{M}_4$ are reversible by involutions.

8·2. *Loxodromic transformations*

Throughout this section, $\gamma \in \mathcal{M}_4^+$ is the map given by the equation $\gamma(x) = (\bar{x})^{-1}$. Lemma 6·4 and the comment following that lemma are applied frequently in this section.

THEOREM 8·1. *Each loxodromic element of* $\mathcal{M}_4$ *is expressible as the composite of two involutions.*

Proof. Through conjugation, assume that a loxodromic map $f \in \mathcal{M}_4$ is of the form $f = \lambda\alpha$, where $\lambda > 0$, $\lambda \neq 1$, and $\alpha \in \mathrm{O}_4$. By Theorem 5·2, choose involutions $\sigma, \tau \in \mathrm{O}_4$ with $\alpha = \sigma\tau$. Then

$$f = (\lambda\gamma\sigma)(\gamma\tau)$$

is the required decomposition.

The next proposition concerns the implication (i) ⇒ (iii) from Theorem 1·2.

PROPOSITION 8·2. *Let* $f \in \mathcal{M}_4^+$ *be given by the formula* $f(x) = \lambda axb$, *where* $\lambda > 0$, $\lambda \neq 1$, $a, b \in \mathbb{H}$, *and* $|a| = |b| = 1$. *If* f *is reversible, then* $|\mathrm{Re}[a]| = |\mathrm{Re}[b]|$.

Proof. Let α denote the orthogonal map $x \mapsto axb$. Since f is reversible, there is $g \in \mathcal{M}_4^+$ with $g^{-1}fg = f^{-1}$. Hence $f(g(0)) = g(0)$ and $f(g(\infty)) = g(\infty)$. Since 0 and ∞ are the only fixed points of f, either $g(0) = 0$ and $g(\infty) = \infty$, or $g(0) = \infty$. In the former case,

g is an orthogonal map followed by a scaling, that is, $g(x) = \mu cxd$, where $\mu > 0$ and $|c| = |d| = 1$. Therefore

$$\lambda = |g^{-1}fg(1)| = |f^{-1}(1)| = \lambda^{-1},$$

which is impossible as $\lambda \neq 1$. In the latter case, the map $h = \gamma g$ fixes 0 and ∞, therefore $h = \mu\beta$ for some $\mu > 0$ and $\beta \in \mathrm{O}_4 \setminus \mathrm{SO}_4$. The equation $g^{-1}fg = f^{-1}$ then reduces to $\beta^{-1}\alpha\beta = \alpha^{-1}$. The result now follows from Proposition 5·3.

The next theorem concerns the double implication (ii) $\Leftrightarrow$ (iii) from Theorem 1·2. The truth of the implication (ii) $\Rightarrow$ (i) follows from the definitions, hence the three statements (i), (ii), and (iii) from Theorem 1·2 are seen to be equivalent once Theorem 8·3 is proven.

Theorem 8·3. *The loxodromic Möbius map f, defined by $f(x) = \lambda axb$, where $\lambda > 0$, $\lambda \neq 1$, $a, b \in \mathbb{H}$, and $|a| = |b| = 1$, is expressible as the composite of two involutions in $\mathcal{M}_4^+$ if and only if $|\mathrm{Re}[a]| = |\mathrm{Re}[b]|$.*

Proof. If f is expressible as the composite of two involutions, then it is reversible, and the equality $|\mathrm{Re}[a]| = |\mathrm{Re}[b]|$ follows from Proposition 8·2. Conversely, if $|\mathrm{Re}[a]| = |\mathrm{Re}[b]|$ then by Theorem 3·7, there are reflections σ and τ such that either $f = \lambda\sigma\tau$ or $f = -\lambda\sigma\tau$. Let $g = \lambda\gamma\sigma$ and $h = \gamma\tau$ in the first case, and replace g with $-g$ in the second case, then both g and h are involutions in $\mathcal{M}_4^+$ and $f = gh$.

Theorem 8·4. *Each loxodromic element of $\mathcal{M}_4^+$ is the composite of three involutions.*

Proof. It suffices to show that $f = \lambda\alpha$, where $\lambda > 0$ and $\alpha \in \mathrm{SO}_4$, is the composite of three involutions. Let $\alpha = \sigma\tau$, for involutions $\sigma, \tau \in \mathrm{SO}_4$. By Proposition 4·1, either $\sigma(x) = axb$, where $a^2 = b^2 = -1$; $\sigma(x) = x$; or $\sigma(x) = -x$. In the first case, by Theorem 3·7, choose reflections σ_1 and σ_2 with $\sigma = \sigma_1\sigma_2$. In the second and third cases, choose any reflection σ_1, and define $\sigma_2 = \sigma_1$, in the second case, and define $\sigma_2 = -\sigma_1$, in the third case. Then

$$f = (\lambda\gamma\sigma_1)(\gamma\sigma_2)\tau$$

is the required decomposition of f into three involutions.

8·3. *Parabolic transformations*

Theorem 8·5. *Each parabolic map in $\mathcal{M}_4^+$ is the composite of two involutions.*

Proof. By conjugating, assume that a parabolic map f has the form $f(x) = \bar{a}xa + 1$, for some unit quaternion a. Choose a purely imaginary unit quaternion c that is perpendicular to a, then the vector $b = -ca$ is also purely imaginary. Using Lemma 2·2, check that $a = cb$, $\bar{a} = bc$, and $f = \sigma\tau$, where $\sigma(x) = bxb + 1$ and $\tau(x) = cxc$ are involutions.

Theorem 8·6. *Each parabolic map in $\mathcal{M}_4$ is the composite of two involutions.*

Proof. A parabolic map $g \in \mathcal{M}_4 \setminus \mathcal{M}_4^+$ has the form $g = f\phi$, where f is a parabolic map in $\mathcal{M}_4^+$ and $\phi(x) = \bar{x}$. Recall the decomposition $f = \sigma\tau$ from Theorem 8·5. Then $g = \sigma(\tau\phi)$ is also a decomposition into two involutions.

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