Bistability, Bifurcation and Chaos in a Laser System

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1. Introduction.

We consider the one–dimensional discrete dynamical system $x_{n+1} = f(x_n)$, where

$$f(x) = Q - \frac{Ax}{1+x^2},\ \forall\, x \in \mathbb{R}.$$

This system arises in Laser Physics, as was pointed out by Bonifacio and Lugiato [B, H]. It provides a model for the evolution of a Fabry–Perot cavity containing a saturable absorber and driven by an external laser. Time is measured in steps of one cavity lifetime, Q is the appropriately normalised (non–dimensionalised) input field, and A is a parameter which depends on the specifics of the apparatus. In this application, A is always positive. It is known as the "bistability parameter"(see below). The value of x_n is the normalised field in the cavity at time n.

This system is a highly–simplified model of the physical system. A more elaborate standard model involves a 3–dimensional continuous dynamical system (formally identical to the well–known Lorenz system), and has been the subject of much study and simulation [B, C]. The simplified system is derived by taking an adiabatic limit in the standard model, and discretising. Theoretical and experimental work [M, HA, CA] shows that both the standard model and real systems admit very varied behaviour. The system may converge to a fixed point, possess two stable fixed points, converge to a limit cycle (pulse), or behave chaotically, depending on the parameters. There is considerable potential application for devices exhibiting these phenomena, in modulators, switching devices, optical memories, and so on.

It is interesting to ask to what extent the simple system mirrors the physical system. Heffernan [H] simulated it for a substantial sample of parameter values (A, Q), and observed the occurrence of one or more stable fixed points and stable periodic cycles. Sampling (A, Q) along a number of lines in the (Q, A)–plane, he observed repeated period–doubling, but this did not appear to lead to chaos. Instead, the doublets re–merged eventually. The system was also studied by Bier and Bountis [BB], with particular reference to this re–merging phenomenon.

The purpose of the present investigation was to determine rigourously the parameter ranges for instability, bistability, and especially chaos in the system. The approach used was analytical. Maple, Mathematica and purpose–written programs in C and MODULA-2 were employed for symbolic manipulation, emulation, curve– and surface–sketching, root–solving, and estimation. This approach enabled us to identify parameter values for which an extensive range of pathological behaviour is exhibited, including various kinds of chaos. Emulation proved consistent with these findings.

For clarity, we define the terms stability, bistability, multistability, stable pulsing, and chaos. This is necessary because these terms are used in various ways in the literature.

Let X be a topological space and $f : X \to X$ be continuous. We denote by f^n the n–th iterate of f, and by $\mathrm{Per}(f)$ the set of period points of f.

We say that f is *T–chaotic* on X if we have

(1) the set $\mathrm{Per}(f)$ is dense in X and
(2) for each nonempty open set $U \subset X$, there exists $n > 0$ such that $f^n(U) = X$.

This concept is due to O. Tamaschke (private communication). Condition (2) is redundant if X is a closed interval, as is easily seen.

We say that f is *D-chaotic* if we have

(1) $Per(f)$ is dense in X and
(2) (topological transitivity) for each nonempty open set $U \subset X$, $\bigcup_{n=1}^{\infty} f^n(U)$ is dense in X.

This concept is due to Devaney [D]. If f is D–chaotic, and X is a metric space, then f has the property of *sensitive dependence on initial conditions*: there exists $\delta > 0$ such that for each nonempty open set $U \subset X$ there exist points $x, y \in U$ and $n \in \mathbb{N}$ such that $\mathrm{dist}(f^n(x), f^n(y)) > \delta$ (cf. [BA]). In the literature, one sometimes finds this property alone, or even watered–down versions of it, taken as the definition of chaos.

Evidently, T–chaos is stronger than D–chaos, and stronger than all other chaos–concepts in use. So if a system is shown to be T–chaotic, that should satisfy everyone.

We observe that if some iterate f^n is T–chaotic, then so is f.

Now consider maps $f : \mathbb{R} \to \mathbb{R}$. If $f(K) \subset K$, then we say that f is T–chaotic (respectively, D–chaotic) *on* K if $f|K$ is T–chaotic (respectively, D–chaotic).

If some iterate f^n is D–chaotic in a set K, then it is readily seen that f itself is D–chaotic on $\bigcup_{j=0}^{n-1} f^j(K)$. Some authors would not regard f as chaotic unless it is chaotic on some nontrivial *interval* X. Now f might well be chaotic on some Cantor–type set, without being chaotic on any nontrivial interval. To distinguish these possibilities, we say that f exhibits *fractal chaos* if f is chaotic on some nonempty totally–disconnected perfect set X, and that f exhibits *interval chaos* if f is chaotic on some closed interval of positive length.

Now we restrict further, to smooth maps $f : \mathbb{R} \to \mathbb{R}$. For such maps, a fixed point p is *stable* or *attracting* (resp., *unstable* or *repelling*) if $|f'(p)| < 1$, (resp., $|f'(p)| > 1$). To a stable fixed point p, we associate its *basin of attraction*

$$\{x \in \mathbb{R} : f^n(x) \to p\}$$

and its *local stable set* or *immediate basin of attraction*, which is the connected component of p in the basin of attraction. To a repelling fixed point p, we associate its local unstable set, which is the largest open interval about p in which f is one–to–one and $|f(x) - p| > |x - p|$. In a similar way we talk about *attracting* or *repelling cycles* and their *local stable* or *unstable sets.*

We say that f exhibits *n–stability* if it has n attracting fixed points. *Stability* is 1–stability, and *bistability* is 2–stability. *Multistability* is n–stability for some n. We say that f exhibits *n–stable pulsing* if it has n attracting cycles. So 1–stable pulsing (or just *stable pulsing*) occurs if there is a unique attracting periodic cycle, and *bistable pulsing* occurs if there are two. It is important to distinguish, for instance, bistability (two attracting fixed points) from stable pulsing with a two–cycle (one attracting two–cycle). The parameter A is called the bistability parameter because it was realised that for some A–values (small ones) bistability cannot occur for any Q, whereas for others it can.

In physical systems there is noise, and in numerical simulation there is round–off error*. As a result, fractal chaos is not observable in the long–term behaviour of experiment or simulation if the complement of the fractal is attracted to periodic cycles. The effect on the transient behaviour may, however, be significant. Cavity lifetimes range up to a few nanoseconds, so only transient behaviour that survives for (at least) millions of time–steps will show up on an oscilloscope. But if it is proposed to use the device as a component in an optical computer or for communications, then it is essential that transients decay to insignificance in a few time–steps. In this case, the occurrence of extensive unstable invariant sets must be avoided. Fractal chaos implies the existence of repelling periodic points of arbitrarily–large order, and if the system is unlucky enough to start on or very near to one of these, then it could take thousands of steps to settle. A device incorporating such fractal chaos would be unreliable, slow (clock speeds of the order of microseconds), or both.

Our main finding is the existence of a major threshold $A = A_0$, somewhere between 3.5 and 3.9. We also find minor thresholds at $A = 8$ and $A = 10.98$, and a number of others. The parameter A is the main control in the system. For fixed values of A between $-A_0$ and A_0, the system exhibits stability or stable pulsing. Variation of the parameter Q produces a more–or–less complicated sequence of period–doubling bifurcations and re–mergings, as observed by Heffernan. For A outside this range there is an open set of Q's for which the system exhibits structurally–stable fractal chaos. There are particular values of Q at which homoclinic bifurcations occur, and at these the system appears to exhibit interval chaos. Bistability occurs in a relatively narrow range of Q's when A exceeds 8. Bistable pulsing with (for instance) a stable fixed point and a stable 2–cycle can occur. We show that n–stable pulsing with $n > 2$ is impossible. Fortunately, bistability is never accompanied by chaos, so to the extent that this model reflects reality, there are reasonable prospects for the practicality of optical switching devices based on this technology. However, if the "apparatus parameter" A is chosen to allow bistability, and the applied field Q is brought from zero to a value which gives bistability, then the resulting curve of systems includes chaotic systems.

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* Round–off does not quite emulate noise. For instance, round–off errors will not usually push a small positive number through zero.

2. Preliminaries.

We assume $A \neq 0$. Let

$$
\begin{aligned}
g(x) &= \frac{-x}{1+x^2} \text{ [Figure 1.]. Then} \\
g(x) &= -\frac{1}{2}\left\{\frac{1}{x+i} + \frac{1}{x-i}\right\}, \\
g^{(k)}(x) &= \frac{(-1)^{k+1}k!}{2}\left\{\frac{1}{(x+i)^{k+1}} + \frac{1}{(x-i)^{k+1}}\right\} \\
&= \frac{(-1)^{k+1}k!}{(x^2+1)^{k+1}}\left\{x^{k+1} - \binom{k+1}{k-1}x^{k-1} + \cdots\right\}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
g'(x) &= \frac{x^2-1}{(x^2+1)^2}, \\
g''(x) &= \frac{-2(x^3-3x)}{(x^2+1)^3}, \\
g'''(x) &= \frac{6(x^4-6x^2+1)}{(x^2+1)^4}.
\end{aligned}
$$

Figure 1: Graph of g

Now $f(x) = Q + Ag(x)$, so f has the same critical points, inflections, and Schwarzian derivative

$$S(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

as does g. The critical points are at $x = \pm 1$, and at these points $f(x) = Q \mp \frac{1}{2}A$. The nonzero inflections are at $\pm\sqrt{3}$, and at these points

$$f(x) = Q \mp \frac{A\sqrt{3}}{4}$$

and $f'(x) = A/8$.

The fixed points of f are given by

$$x^3 - Qx^2 + (A+1)x - Q = 0.$$

There is at least one, and at most 3. There cannot be more than 2 attracting fixed points, since there is a non–attracting fixed point between any two attracting fixed points. Thus n-stability occurs only in the forms of stability and bistability.

The Schwarzian is

$$\frac{-6}{(1-x^2)^2}.$$

Since it is negative, there is a severe restriction on the possible occurrence of attracting periodic cycles. If p is an attracting periodic point of period n, then there is a point $q = f^i(p)$ in the orbit of p such that the local stable set $W(f^n, q)$ of q for f^n is an unbounded interval, or contains one of the critical points ± 1 [D]. Thus, there can be at most 4 attracting cycles, a priori. We can actually say more.

Theorem (2.1). *f has at most two attracting cycles.*

Proof. We give the argument for the case $A > 0$. The case $A < 0$ is similar and $A = 0$ is trivial.

Suppose we have 3 distinct attracting periodic points p_i with periods n_i for $1 \le i \le 3$. Let G_i^j be the immediate basin of attraction of $f^j(p_i)$ for f^{n_i} (for $0 \le j \le n_i - 1$). Let

$$E_i = \cup_{j=0}^{n_i-1} G_i^j.$$

There are 3 cases to consider.

Case 1: $1 \in E_1$, $-1 \in E_2$, and E_3 is unbounded. Suppose E_3 is unbounded to the right. Then $[s, \infty) \subseteq E_3$ for some s in the image of f. Hence $f(-1) \in E_3$ which contradicts the fact that $-1 \in E_2$. Suppose E_3 is unbounded to the left, then a similar argument shows that $f(1) \in E_3$ which contradicts the fact that $1 \in E_1$.

Case 2: $1 \in E_1$, E_2 is unbounded right, and E_3 is unbounded left. As in Case 1 we get that $f(1) \in E_2$ which is a contradiction.

Case 3: $-1 \in E_1$, E_2 is unbounded right, and E_3 is unbounded left. As in Case 1 we get that $f(-1) \in E_3$ which is a contradiction. ∎

We observe that this theorem may also be proved by appealing to what is known about complex dynamics. The theory of rational dynamical systems acting on the Riemann sphere (cf. [D, p. 281]) shows that each attracting cycle for f attracts a critical point. Since ± 1 are the only critical points of f, even on the sphere, it follows that there are at most 2 attracting cycles, as required.

Variation of Q has the effect of sliding the graph of f vertically, and considering this it is clear that there will be only one fixed point if the slope at the inflections is less than or equal to 1. This applies when $-1 \leq A \leq 8$. For $A < -1$ or $A > 8$, the existence of more than one fixed point depends on the value of Q. In any case, there is only one fixed point when $|Q|$ is very large, and it attracts the whole line.

The points where $f'(x) = 1$ are given by

$$x = \pm\sqrt{\frac{A - 2 \pm \sqrt{A^2 - 8A}}{2}}.$$

There are no such points unless $A \leq -1$ or $A \geq 8$. There are 4 if $A > 8$ and 2 if $A < -1$.

The points where $f'(x) = -1$ are given by

$$x = \pm\sqrt{\frac{-A - 2 \pm \sqrt{A^2 + 8A}}{2}}.$$

There are no such points if $-8 < A < 1$, 4 if $A < -8$ and 2 if $A > 1$.

It is clear from this preliminary analysis that the primary measure of the complexity of the system is A. We proceed to examine the various interesting ranges of A, and to see the effect of Q in each range. First we focus on cycles, and then we look at chaos.

3. Cycles.

In this section we examine the simple bifurcations of the system, and the occurrence of various kinds of cycles. Apart from determining the multistable pulsing behaviour of the system, this will give us some indication of where to look for chaos.

(3.1) Universal stability: $|A| \leq 1$.

When $|A| \leq 1$, we have $|f'(x)| < 1$ except perhaps at $x = 0$. Thus $|f(x) - f(y)| < |x - y|$ whenever $x \neq y$. Since f is bounded, it follows that for each $x_0 \in \mathbb{R}$, the system converges to the unique fixed point. Convergence is exponential except in the cases $(A, Q) = (1, 0), (-1, 0)$.

Figure 2: The fixed point when $A = 1$

Figure 2 shows the location of the fixed point as a function of Q, when $A = 1$.

To obtain exponential convergence at a fixed worst rate independent of Q, it is necessary to restrict to $|A|$ strictly less than 1.

(3.2) Period–doubling for $1 < A < 8$.

Let $1 < A < 8$. The range of f' then lies in the interval $[-A, A/8]$. As we vary Q from $-\infty$ to $+\infty$, the graph of f slides vertically, and (for generic A, i.e. for an open dense set of A's) period–doubling bifurcations occur when the points where $f'(x) = -1$ fall on the diagonal. This happens at $Q = \pm Q_1(A)$, where

$$Q_1(A) = \pm \left(b + \frac{Ab}{1+b^2} \right),$$

where

$$b = \sqrt{\frac{-2 - A + \sqrt{A(A+8)}}{2}}.$$

Figure 3: Onset of two–cycles

The graphs of $\pm Q_1(A)$ are shown in figure 3. As Q increases through $-Q_1$ or decreases through Q_1, the system bifurcates from a single universally–attracting fixed point to an attracting two-cycle. Heffernan's work suggested that there is no further bifurcation when $A = 2$, but that 4–cycles occur by $A = 3$ — we were unable to replicate his work exactly; his diagrams lack the symmetry under $(Q, x) \mapsto (-Q, -x)$ which theory indicates (cf. section (3.4) below) and have other imperfections, but nevertheless represent a considerable achievement, given the primitive equipment available to him. We observe that the onset of 4–cycles cannot begin before $f^{2'}$ can take the value -1, so a lower bound for the A–threshold for the onset of 4–cycles is

$$\inf\{A > 0 : \inf_Q \inf_x (f^2)'(x) \leq -1\}.$$

Now $(f^2)'(x) = f'(f(x))f'(x)$, hence it cannot be less than $-A \cdot (A/8)$. With $x = 0$ and $Q = \sqrt{3}$, $(f^2)'(x)$ attains this value, so

$$\inf_Q \inf_x (f^2)'(x) = -A^2/8,$$

and a crude lower bound for the A–threshold (for the onset of 4–cycles) is $A = \sqrt{8}$. The actual value of the threshold may be obtained by observing that it occurs at or before the least positive A–value for which $f^{2'} = -1$ occurs at a two–cycle. Computation shows that this value is approximately 2.9285, and that a 4–cycle does not occur before 2.92.

Definition. We define the threshold D_n to be the infimum of those positive A for which, for some Q, $f_{A,Q}$ has an n–cycle.

Theorem (3.1). *If f exhibits D–chaos on some infinite invariant set, then $A \geq D_{2^n}$ for each $n \geq 1$.*

Proof. In fact, $f_{A,Q}$ can have neither dense period points nor topological transitivity unless $A \geq D_{2^n}$. For the number of period m points of f is finite for each m, so if $\mathrm{Per}(f)$ is dense in an infinite set, then f must have points of arbitrarily high period. But equally, if f is topologically–transitive, then an argument like the proof of Sarkovskii's theorem shows that f has points of arbitrarily high period, and hence (by Sarkovskii's theorem) has points of period 2^n for each positive n. ∎

Computation using Maple shows that

$$\begin{array}{ccccc} 2.92 & \leq & D_4 & \leq & 2.93, \\ 3.4 & \leq & D_8 & \leq & 3.5, \\ 3.5 & \leq & D_{16} & \leq & 3.55. \end{array}$$

Thus D–chaos does not set in before $A = 3.5$.

We note in passing that application of the fundamental theorem of algebra to count period points for f shows that there is at most one 2–cycle, and there are at most two 3–cycles, three 4–cycles, thirty 8–cycles, and more generally

$$2^{2^n - n} - 2^{2^{n-1} - n}$$

2^n–cycles.

(3.3) Three–cycles for $A < 8$.

It is not difficult to select parameters $A < 8$ and Q for which there is a three-cycle. For instance, take $A = 2Q$ and let Q be the (unique) solution of

$$Q - \frac{2Q^2}{1+Q^2} = 1.$$

(In fact

$$\begin{aligned} Q &= 1 + \frac{2}{\sqrt[3]{3(9+\sqrt{57}}} + \sqrt[3]{\frac{9+\sqrt{57}}{9}} \\ &= 2.7692923542\ldots, \\ A &= 5.5385847084\ldots). \end{aligned}$$

Then $f(0) = Q$, $f(Q) = 1$, and $f(1) = 0$. Since $f'(1) = 0$, the 3–cycle is super–attracting. This system therefore has periodic points of all orders, by Sarkovskii's theorem [D], and exhibits sensitive dependence on initial conditions on an uncountable (possibly non-invariant) set, by the Li–Yorke theorem [L]. However, with noise added it will very rapidly converge to the 3–cycle, so this is an example of bad behaviour that is unobservable in the long–term. This result was confirmed by simulation.

Figure 4: Superattracting 3–cycle

Figure 4 shows the superattracting 3–cycle. If the parameters (A, Q) are varied slightly, then the superattracting property is lost (generically), but an attracting 3–cycle remains, so the bad behaviour of the system is robust.

More generally, it is easy to prove that three–cycles occur for each $A > 5.53\ldots$. In fact, let $A > 5.53\ldots$. The condition that $f_{A,Q}$ have a 3–cycle through the origin is that $f^2_{A,Q}(Q) = 0$. Excluding the trivial case $Q = 0$, this amounts to

$$Q - \frac{AQ}{1+Q^2} = \frac{A \pm \sqrt{A^2 - 4Q^2}}{2Q},$$

or

$$2Q^2\left(1 - \frac{A}{1+Q^2}\right) - A = \pm\sqrt{A^2 - 4Q^2}.$$

Letting $z = Q^2$ and squaring, we get

$$\left\{2z\left(1 - \frac{A}{1+z}\right) - A\right\}^2 = A^2 - 4z,$$

which gives a quartic in z. Solving using Mathematica, we get a positive real solution z provided

$$\frac{A^3}{4} - \frac{3A^2}{2} + A - 2 > 0.$$

Using subdivision, we find that this holds for $A > 5.53\ldots$.

Straight computational search for the onset of three–cycles shows that they start to occur at $A = 4.9442719$. So somewhere between $A = 3.5$ and $A = 4.95\ldots$, there is a threshold, above which (at least) sensitive dependence on initial conditions can occur. We return to this matter below, and obtain more exact information.

(3.4) Bistability for $A > 8$.

Fix $A > 8$. As Q varies, six simple bifurcations will occur, at the 4 values of Q which make the diagonal tangent to the graph of f, and at the 2 values for which the diagonal cuts the graph at a point where the slope is -1. In sequence, as Q increases, these bifurcations are either of types saddle-node, saddle-node, period-doubling, period-doubling, saddle-node, and saddle-node (in that order), or are of types saddle–node, period–doubling, saddle-node, saddle-node, period–doubling, and saddle-node (in that order). Which sequence occurs depends on whether the tangent line to the graph $y = Ag(x)$ at the positive point where the slope is 1 passes below or above the positive point where $Ag' = -1$. The threshold between the two patterns occurs at $A = 10.986548$ This was determined by using subdivision to solve the equation

$$u - l = \frac{Au}{1+u^2} - \frac{Al}{1+l^2},$$

where

$$u = \sqrt{\frac{A - 2 + \sqrt{A(A-8)}}{2}},$$

$$v = \sqrt{\frac{-A - 2 + \sqrt{A(A+8)}}{2}}.$$

(For $8 < A < 10.98\ldots$ the tangent passes *below* the point and the *first* sequence of bifurcations occurs). The first three bifurcations in the sequence involve negative Q and x and the rest positive Q and x. In view of the formula

$$f_{A,Q}(x) = -f_{A,-Q}(-x),$$

the map $m(x) = -x$ is a topological equivalence between $f_{A,Q}$ and $f_{A,-Q}$, so it is sufficient to describe the dynamics for either non–positive or non–negative Q.

Let $8 < A < 10.98\ldots$. Let $0 < Q_1(A) < Q_2(A) < Q_3(A)$ be the positive simple bifurcation parameters. For Q close to Q_1 (and $Q < Q_1$) or Q close to $-Q_1$ (and $Q > -Q_1$) the system has an attracting 2–cycle. For $Q > Q_3$, there is a universally–attracting fixed point. For $Q_1 < Q < Q_2$, there is a single universally attracting fixed point,

Figure 5: Squeeze Illustration: $A = 11$, $Q = 6$.

Now let $A > 10.98\ldots$. Let $0 < Q_1(A) < Q_2(A) < Q_3(A)$ be the positive simple bifurcation parameters. When $Q > Q_1$, the largest fixed point is attracting. As Q grows towards Q_1, the system bifurcates repeatedly, acquiring longer and longer attracting cycles. The lengthening cycles can be understood by observing that the trajectories have to "squeeze through a narrower and narrower gap"as Q approaches Q_1 from below. For example, experiments with $A = 20$ illustrate how the global features of the map complicate the simple saddle–node bifurcation at 8.7041. There is a stable attracting fixed point for $Q_2 > Q > 8.7041$. As Q approaches 8.7041 from below, there is an elaborate series of bifurcations, and no remerging. The periods grow arbitrarily long, and then are suddenly cut off when the new pair of fixed points appears.

We shall see below that this "squeeze" phenomenon is another source of chaos.

In all cases when $A > 8$, the most interesting behaviour occurs in the range $Q_2 < Q < Q_3$. There are then three fixed points, of which the largest is attracting the second is repelling, and the least is attracting. In other words, we have optical bistability . The onset of optical bistability was observed experimentally by Szöke et al. [S]. This open interval in which bistability occurs contains the Q–value for which the critical point 1 is fixed, which is $Q = 1 + A/2$. For large A, the curve approximates to a parabola near the critical point, and the endpoints Q_2 and Q_3 are roughly equidistant from this value.

It is easy to estimate that

$$Q_2, Q_3 \approx 1 + \frac{A}{2} \pm \frac{2}{A}.$$

For example, with $A = 20$, the bistability interval for Q is only about 0.2 units long.

Figure 6: Simple bifurcations for $A > 8$

It is interesting to note that when $A > 10.98\ldots$, the bifurcation that ends the bistability region on the left is a period–doubling bifurcation. Thus for Q slightly less than Q_2, the system exhibits bistable pulsing with one stable fixed point and one stable 2–cycle.

Theorem (3.2). *Let $A > 0$ and $Q \in \mathbb{R}$. If $f_{A,Q}$ exhibits bistability, then it does not exhibit chaos on any infinite set.*

Proof. Without loss in generality, we assume that $Q > 0$.

The fixed points p_1, p_2, p_3 are positive, and we may order them so that $p_1 < p_2 < p_3$. Evidently, $p_2 > 1$. Also, p_1 and p_3 are attracting and p_2 is repelling. It is readil;y seen that the local stable set of p_3 is $(p_2, +\infty)$.

Let q_2 be the unique point with $0 < q_2 < p_1$ and $f(q_2) = f(p_2)$. Choose q_1 such that

$$(q_2, p_2) \cap f^{-1}(p_1) = \{p_1, q_1\}.$$

Fix $x_0 \in (-\infty, p_2)$, and consider $x_n = f^n(x_0)$. If $x_0 < q_2$, then $x_1 > p_2$ and $x_n \to p_3$. Thus $(-\infty, q_2)$ is attracted to p_3. If $q_2 < x_0 < p_2$, then $x_n \to p_1$ or there exists m such that $x_m \in (p_1, q_1)$. Now the local stable set of p_1 must have a critical point (cf. Section 1 above), hence has 1, hence contains the open interval between q_1 and p_1. Thus in either case $x_n \to p_1$.

Thus the basin of attraction of p_1 is (q_2, p_2), and the basin of attraction of p_3 is $(-\infty, q_2) \cup (p_2, +\infty)$. So there is no chaos. ∎

4. Chaos: The Major Threshold.

We have seen that the system behaves simply when $0 < A < 3.5$, and that it exhibits sensitive dependence on initial conditions on an uncountable (possibly non-invariant) set at $A = 4.95\ldots$.

Definition. The *major threshold* A_0 is the infimum of those positive A such that $f_{A,Q}$ exhibits D–chaos on some infinite invariant set for some Q.

To search for A_0, we consider snap–back repellors, defined as follows.

Definition. (a) A point p is called a *snap–back repellor of period* r for f if p is a repelling periodic point of period r and there exists a point x in the local unstable set of the cycle of p and a natural number k such that $f^k(x) = p$. Such a point x is called a *homoclinic point.* If $(f^k)'(x) \neq 0$, then p is said to be a *non–degenerate* snap–back repellor. Otherwise, there is a critical point c in the forward orbit of x. In that case, if m is the least number for which $f^m(c) = p$, then we call p an *m–step degenerate* snap–back repellor.

Definition. For $r \geq 1$, the threshold A_r is the infimum of those positive A such that for some Q $f_{A,Q}$ has a degenerate snap–back repellor of period r.

Theorem (4.1). *For each $n \geq 1$, we have $D_{2^n} \leq A_0 \leq A_{2^n}$.*

Remarks. 1. We shall see that $A_4 = 3.8\ldots$, and we have seen that $D_{16} = 3.5\ldots$, so this theorem shows that

$$3.5 \leq A_0 \leq 3.9.$$

2. We **conjecture** that

$$\sup_{n\geq 1} D_{2^n} = \inf_{n\geq 1} A_{2^n},$$

and if this is correct then more exact estimation of A_0 is a matter of mere computation.

Theorem (4.1) is a consequence of Theorem (3.1) and the following two lemmas.

Lemma (4.2). *Let $f : \mathbb{R} \to \mathbb{R}$ be smooth. If f has a non-degenerate snap–back repellor, then some iterate of f exhibits T–chaos on some nonempty perfect set, and hence f itself exhibits D–chaos on some nonempty perfect set.*

Proof. This is essentially proved in [D] (1.16). The hypotheses allow us to find disjoint closed intervals I and J and some $n \in \mathbb{N}$ such that

$$I \cup J \subset f(I) \cap f(J),$$
$$|f^{n\prime}| > 1 \text{ on } I \cup J.$$

Defining

$$K = \{x \in \mathbb{R} : f^{nk} \in I \cup J,\ \forall k \geq 0\},$$

it is readily seen that $f^n|K$ is topologically conjugate to the shift on $2^{\mathbb{N}}$, and hence is T–chaotic. Thus f is D–chaotic. ∎

Lemma (4.3). *For all but a finite number of A, if f_{A,Q_0} has a degenerate k–step snap–back repellor, then there is an open interval $I = (Q_0, Q_0 + \epsilon)$ or $I = (Q_0 - \epsilon, Q_0)$ (with $\epsilon > 0$) such that $f_{A,Q}$ has a non-degenerate k–step snap–back repellor for each $Q \in I$.*

Remark. In fact this is a special case of a more general statement about the behaviour of generic 1–parameter families of maps near degenerate snap–back repellors.

Proof. We give the proof for the period 1 case. The general case is similar.

Fix A and Q_0 such that f_{A,Q_0} has a degenerate k–step snap–back repellor. Choose a point c and a point p_0 such that

$$\begin{aligned} &f^k_{A,Q_0}(c) = p_0 = f_{A,Q_0}(p_0), \\ &(f^k_{A,Q_0})'(c) = 0, \\ &|f'_{A,Q_0}(p_0)| > 1, \end{aligned}$$

and c belongs to the local unstable set of the repelling fixed point p_0. Consider the system of equations

$$\left.\begin{aligned} f^k_{A,Q}(x) &= p \\ f_{A,Q}(p) &= p \end{aligned}\right\} \qquad (*)$$

for (x, p, Q) near (c, p_0, Q_0). At (c, p_0, Q_0) the Jacobian is

$$\det \begin{pmatrix} \frac{\partial f^k_Q(c)}{\partial Q} & -1 \\ 1 & f'_Q(p) - 1 \end{pmatrix}$$

evaluated at c, p_0, Q_0. Hence the Jacobian is

$$1 + (f'_{Q_0}(p_0) - 1)\frac{\partial f^k_Q(c)}{\partial Q}\bigg|_{Q=Q_0}$$

If this is zero, then we have 4 functionally independent polynomial equations relating A, Q_0, p_0, c, and this can only happen for a finite number of A. So we may assume that this does not happen, and then we may solve the system $(*)$ above to obtain Q as a function of x near $x = c$.

A straightforward computation shows that $\frac{dQ}{dx}(c) = 0$. If $\frac{d^2Q}{dx^2}(c) = 0$, then we have again 4 functionally independent polynomial equations relating A, Q_0, p_0, c, which can only happen for a finite number of A. Excluding these exceptional A, we see that the system $(*)$ above has 2 solutions for x as a function of Q when Q lies in an interval $(Q_0 - \epsilon, Q_0)$ or $(Q_0, Q_0 + \epsilon)$, for some $\epsilon > 0$, depending on the sign of $\frac{d^2Q}{dx^2}(c)$. By continuity, either of these relations yields a non-degenerate snap–back repellor for $f_{A,Q}$.
∎

We now search for degenerate snap–back repellors. We begin by estimating A_1. We have the following.

Theorem (4.4). *The threshold A_1 for the occurrence of period 1 degenerate snap–back repellors is the solution for A of*

$$(**)\begin{cases} f^2_{A,Q}(1) = f^3_{A,Q}(1), \\ f_{A,Q}(-1) = 1. \end{cases}$$

Proof. Because of the topological equivalence $x \mapsto -x$ between $f_{A,Q}$ and $f_{A,-Q}$, remarked on above, we can restrict to the case where 1 lies on the orbit of the homoclinic point. So we consider the possibility that for some $n \geq 1$

(1) $p = f^n(1)$ is a repelling fixed point and
(2) 1 lies in the orbit under f of some point belonging to the local unstable set of p.

This cannot occur for $n = 1$, so the first interesting case is $n = 2$. Since we are assuming $A < 8$, f' never reaches 1, so $f'(p) < -1$. The conditions then become:

$$f^3(1) = f^2(1) \tag{3}$$
$$f'(f^2(1)) < -1 \tag{4}$$
$$f(-1) \geq 1 \tag{5}$$

We note that Q must be negative. (This is seen as follows: If $Q = 0$, then $0 = p = f^2(1)$, so $A = 0$ and the map is constant, so p is not repelling. If $0 < Q \leq A/2$, then $f(1) \leq 0$, so $f^2(1) \geq Q > p$. If $A/2 < Q < 1 + A/2$, then $0 < f(1) < p$, so $f^2(1) > p$. If $Q = 1 + A/2$, then 1 is a stable fixed point. If $Q > 1 + A/2$, then $f^3(1) > f^2(1) > f(1) > 1$.)

If we let $W^u(f,p)$ denote the local unstable set of p, then (5) implies that $W^u(f,p) = (-1,1)$. (This is seen by noting first that the endpoints e of $W^u(f,p)$ satisfy one of the conditions:

$$f(e) = e \tag{6}$$
$$f(e) - p = p - e \tag{7}$$
$$f'(e) = 0 \text{ and } e \text{ is a local extremum,} \tag{8}$$
$$e = \pm\infty \tag{9}$$

Analysis shows that (8) is the relevant condition in our situation and so the endpoints are ± 1.)

For $0 < A < 20$, inspection of the graph using Mathematica makes it clear that

$\exists! Q_*(A)$ such that $-\infty < Q < 0$ and $f^3(1) = f^2(1)$.

Figure 7: $Q_*(A) : f^3(1) = f^2(1)$

So the question is, for which A does $Q = Q_*(A)$ satisfy conditions (4) and (5)? These conditions can be expressed

$$(4') \qquad -\sqrt{\frac{\sqrt{A^2+8A}-A-2}{2}} < f^2_{A,Q_*}(1)$$

$$(5'). \qquad A \geq 2 - 2Q_*(A)$$

It turns out that (4′) is satisfied already when $A = 2$, so (5′) becomes the determining condition, i.e. we have equality in (5) at the threshold, so the homoclinic bifurcation is doubly–degenerate (the fixed point is on the orbit of both critical points).

So far, we have just examined 2–step degenerate snap–back repellors of period 1. Could a k–step degenerate snap–back repellor of period 1 occur for some larger k and some A smaller than the solution of $(**)$? This is ruled out by the following lemma, thus completing the proof of the theorem.

Lemma 4.5. *Let $1 < A < 8$. Suppose that there exist Q and n such that $f^n_{A,Q}(1)$ is a repelling fixed point and 1 lies in the closure of the local unstable set of f. Then there exists Q' such that $f^2_{A,Q'}(1)$ is a repelling fixed point and 1 lies in the closure of the local unstable set of $f^2_{A,Q'}(1)$.*

Proof. Suppose there exist Q_0 and $n > 2$ such that $f^n_{A,Q_0}(1)$ is a repelling fixed point for f_{A,Q_0}. Then $p_0 = f^n_{A,Q_0}(1)$ lies in $(-1,1)$ and $f'_{A,Q_0}(p_0) < -1$. It is readily seen that we cannot have $f^{n-1}(1) = 0$, so there are two cases to consider.

Case 1, in which $f^{n-1}(1) < 0$. Let $a = f^{n-1}(1)$. Then $a < -1$, since $f(a) = p$ and $a \neq p$. Thus $f(1) = \inf f \leq a$.. Thus if $x \neq f(1)$ has $f(x) = f^2(1)$, then $-1 < x < 0$, $p = f(a) > f^2(1)$, $p < x$, $p < 0$, $f'(x) < f'(p) < -1$, $x > p > f(x)$.

Now consider the effect of increasing Q. The point $f(1)$ increases, at least until it reaches -1. While this goes on, $f^2(1)$ increases, and the other place x where $f(x) = f^2(1)$ decreases towards -1. The fixed point p increases, and since it starts off greater than -1, it must cross x for some value of Q. The slope of f at this x is between the original slopes $f'(p)$ and $f'(x)$, and hence is less than -1. Thus there is a Q larger than Q_0 such that $f^2(1)$ is a repelling fixed point. Finally, we note that for this Q we have $f(1) < -1$ and $f(-1) > 1$ (the latter since $Q > Q_0$). So we are done in this case.

Case 2, in which $f^{n-1}(1) > 0$. Let $a = f^{n-1}(1)$. Then since $-f_{A,Q_0}(-x) = f_{A,-Q_0}(x)$ we see that $-p_0$ is a repelling fixed point for $f_{A,-Q_0}$, and is the image under $f_{A,-Q_0}$ of $-a \neq -p_0$, and that the image of $f_{A,-Q_0}$ contains $[-1,1]$, so we are back to Case 1. ∎

Remarks. 1. We wrote a C program to estimate A_1. The algorithm was:

> Search in [2,6] for $A_1 = \inf A$ such that (4′) and (5′) hold, where $Q_*(A)$ is computed by solving $f^3(1) = f^2(1)$ using bisection on the interval $[-A, 0]$.

This short program runs quickly and gives $A_1 = 4.875130$.

2. At the threshold $A = A_1$, the corresponding value of $Q_*(A)$ is -1.437565 and the fixed point is at -0.258056. A plot of the bifurcation values $Q_*(A)$ is shown in figure 7. We note that calculation indicates that $Q_*(A)$ decreases monotonically to -2 as $A \uparrow \infty$.

3. At the threshold values $Q_*(A)$, for $A > 4.875130$, emulation of the system showed apparent interval chaos. O'Reilly has demonstrated that interval chaos does indeed occur for these doubly–degenerate systems, and the details will appear elsewhere.

Finally, we considered the occurrence of degenerate snap–back repellors of period greater than 1. We wrote Maple routines [CG] to search for A and Q satisfying the conditions:

$$\begin{cases} f^{n+m}(1) = f^n(1), \\ |(f^m)'(f^n(1))| > 1, \\ 1 \in \bigcup_j f^j(f^n(1) - \epsilon, f^n(1) + \epsilon)), \ \forall \epsilon > 0, \end{cases}$$

Details of the Maple scripts used are available on request from aof@maths.may.ie. It was found that $n = m = 2$ produced only a very modest upper bound of 4.9 for A_2, but that $n = m = 4$ gave a significant improvement. A degenerate snap–back repellor of period 4 occurs for $A = 3.9$. Thus $A_4 \leq 3.9$ and so 3.9 is also an upper bound for A_0.

Figure 8 shows an overall sketch of the behaviour of the system as (A, Q) is varied.

Figure 8: Overview

The hatched region is one of those in which fractal chaos has been demonstrated. This chaos is caused by a period 1 snap–back repellor. There are many additional bands of fractal chaos, due to higher–period snap–back repellors. When A exceeds 10.98, a new source of chaos arises from the "squeeze" phenomenon. Consider values of Q a little less than Q_1 (cf. Figure 5). Graphical analysis shows the forward orbit of the critical point -1 passing through the narrow passage between the diagonal and the last point on the graph with $f' = 1$, and its points bunch closely. The backward orbit of the (repelling) fixed point passes through the same gap, and bunches also. Continuity shows that for any large m, there exists a value of Q such that -1 is mapped to the fixed point after m steps. Thus the tangent–node bifurcation is an accumulation point of homoclinic bifurcations.

Summary:

The system

$$x_{n+1} = f(x_n)$$

$$f(x) = Q - \frac{Ax}{1+x^2}$$

was studied for positive A, to see what range of behaviour it exhibits. Related laser systems have shown bistability and chaos. It was found that the "bistability parameter" A is the major control. This parameter passes through a major threshold A_0 estimated between 3.5 and 3.9. The system behaves predictably for $0 < A < A_0$, exhibiting stable pulsing and a more–or–less complicated finite sequence of simple bifurcations and remergings as Q is varied. For $A > A_0$, homoclinic bifurcations occur as Q is varied, and the system exhibits structurally–stable fractal chaos and computation indicates unstable interval chaos at threshold Q–values. For $A > 8$, bistability occurs. There is a further threshold at $A = 10.98\ldots$, above which bistable pulsing is possible. For large A, bistability occurs when Q differs from $1 + A/2$ by an amount that is asymptotic to $2/A$, so that the bistability region narrows as A increases. In the bistable case, the local stable set of the lower fixed point is very short, and gets even shorter as A grows. Thus there is a marked lack of symmetry in the system, considered as a two–state system: it is much more likely to be found in one state than the other. Bistability is never associated with chaos. However, if (A, Q_0) gives bistability, then there exists Q between 0 and Q_0 for which (A, Q) gives chaos. It would be interesting to see experimental investigation of a Fabry–Perot system, tuned to say $A = 6$ or so. The system never exhibits n–stable pulsing with $n > 2$.

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